Leavitt path algebras: Something for everyone

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Overview

Brief history, and motivating examples

One of the first theorems you saw as an undergraduate student:

Dimension Theorem for Vector Spaces. Every nonzero vector space V has a basis. Moreover, if $\mathcal{B}$ and $\mathcal{B}'$ are two bases for V, then $|\mathcal{B}| = |\mathcal{B}'|$.

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Note: V has a basis $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ $\Leftrightarrow$ $V \cong \oplus_{i=1}^n \mathbb{R}$ as vector spaces. So:

One result of Dimension Theorem, Rephrased:

$$\oplus_{i=1}^n \mathbb{R} \cong \oplus_{i=1}^m \mathbb{R} \;\Leftrightarrow\; m = n.$$

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If $0 \neq v \in V$, then $\{v\}$ is linearly independent.

If $kv = 0$, need to show $k = 0$. But $k \neq 0 \Rightarrow \frac{1}{k}kv = 0 \Rightarrow v = 0$, contradiction.

Similar idea (multiply by the inverse of a nonzero element of K) shows that a maximal linearly independent subset of V actually spans V.

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For a module P over a ring R we can still talk about a *basis* for P. (Note: in general, not all modules *have* bases; those that do are called *free* R-modules.)

$\oplus_{i=1}^n R$ always has a basis having n elements, e.g.,

$$\{e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, 0, ..., 1)\}$$

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Question: Is the Dimension Theorem true for rings in general? That is, if R is a ring, and $\oplus_{i=1}^{n} R \cong \oplus_{i=1}^{m} R$ as R-modules, must $m = n$?

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(But the answer is YES for the rings $\mathbb{Z}$, $\mathrm{M}_2(\mathbb{R})$, $C(\mathbb{R})$)

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Think of V as $\oplus_{i=1}^{\infty}\mathbb{R}$. Then think of S as $\mathrm{RFM}(\mathbb{R})$.

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More formally: Let

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

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Similarly, we should be able to 'go back' from pairs of matrices to a single matrix, by interweaving the columns.

That is, more formally, $(M_1, M_2) \mapsto M_1X_1 + M_2X_2$ is a reasonable way to associate a pair of matrices with a single one.

Brief history, and motivating examples

Here's what's really going on. These equations are easy to verify:

$$Y_1X_1 + Y_2X_2 = I,$$

$$X_1Y_1 = I = X_2Y_2, \quad \text{and} \quad X_1Y_2 = 0 = X_2Y_1.$$

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Using these, we get inverse maps:

$S \to S \oplus S$ via $M \mapsto (MY_1, MY_2)$, and

$S \oplus S \to S$ via $(M_1, M_2) \mapsto M_1X_1 + M_2X_2$.

For example:

$$M \mapsto (MY_1, MY_2) \mapsto MY_1X_1 + MY_2X_2 = M \cdot I = M.$$

Brief history, and motivating examples

Using exactly the same idea, let R be ANY ring which contains four elements y_1, y_2, x_1, x_2 satisfying

$$y_1x_1 + y_2x_2 = 1_R,$$

$$x_1y_1 = 1_R = x_2y_2, \quad \text{and} \quad x_1y_2 = 0 = x_2y_1.$$

Then $R \cong R \oplus R$.

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Actually, when $R \cong R \oplus R$ as R-modules, then $\oplus_{i=1}^{m} R \cong \oplus_{i=1}^{n} R$ for *all* $m, n \in \mathbb{N}$.

Leavitt algebras

Natural question:

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Theorem

(William G. Leavitt, Trans. Amer. Math. Soc., 1962)

For every $m < n \in \mathbb{N}$ and field K there exists a K-algebra $R = L_K(m,n)$ with $\oplus_{i=1}^m R \cong \oplus_{i=1}^n R$, and all isomorphisms between free left R-modules result precisely from this one. Moreover, $L_K(m,n)$ is universal with this property.

Leavitt algebras

The $m = 1$ situation of Leavitt's Theorem is now somewhat familiar. Similar to the $n = 2$ case that we saw above, $R \cong R^n$ if and only if there exist

$$x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in R$$

for which

$$\sum_{i=1}^{n} y_i x_i = 1_R \quad \text{and} \quad x_i y_j = \delta_{i,j} 1_R.$$

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$L_K(1, n)$ is the quotient

$$K < X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n > \ / \ < (\sum_{i=1}^{n} Y_i X_i) - 1_K;\ X_i Y_j - \delta_{i,j} 1_K >$$

Note: $\mathrm{RFM}(K)$ is much bigger than $L_K(1, 2)$.

Leavitt algebras

As a result, we have: Let S denote $L_K(1,n)$. Then

$$S^a \cong S^b \iff a \equiv b \bmod (n-1).$$

In particular, $S \cong S^n$.

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Remember, a ring R being *simple* means:

$\forall\ 0 \neq r \in R, \exists\ \alpha_i, \beta_i \in R$ with $\sum_{i=1}^{n} \alpha_i r \beta_i = 1_R$.

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Actually, $L_K(1,n)$ is REALLY simple:

$\forall\ 0 \neq r \in L_K(1,n), \exists\ \alpha, \beta \in L_K(1,n)$ with $\alpha r \beta = 1_{L_K(1,n)}$.

Building rings from combinatorial objects

Here's a familiar idea. Consider the set $T = \{x^0, x^1, x^2, ...\}$. Define multiplication on T in the usual way: $x^i \cdot x^j = x^{i+j}$. Consider formal symbols of the form

$$k_1 t_1 + k_2 t_2 + \cdots + k_n t_n$$

where $t_i \in T$, and $k_i \in \mathbb{R}$. Denote this set of symbols by KT. We can add and multiply two symbols of this form, as usual, e.g. $(kt)(k't') = kk'(t \cdot t')$.

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Also, e.g. if we impose the relation $x^n = x^0$ on T, call the new semigroup $\overline{T}$, then $\overline{T} = \{x^0, x^1, x^2, ..., x^{n-1}\}$, and

$$\mathbb{R}\overline{T} \cong \mathbb{R}[x]/\langle x^n - 1\rangle$$

Building rings from combinatorial objects

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For instance:

matrix rings, group rings, multivariable polynomial rings, etc ...

General path algebras

Let E be a directed graph. (We will assume E is finite for this talk, but analysis can be done in general.) $E = (E^0, E^1, r, s)$

$$\overset{s(e)}{\bullet} \xrightarrow{\ e\ } \overset{r(e)}{\bullet}$$

The *path algebra of E with coefficients in K* is the K-algebra KS

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In particular, in KE,

$$\text{for each edge } e, \quad s(e) \cdot e = e = e \cdot r(e)$$

$$\text{for each vertex } v, \quad v \cdot v = v$$

$$1_{KE} = \sum_{v \in E^0} v.$$

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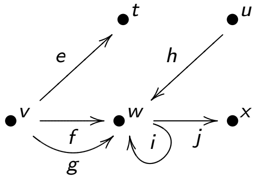

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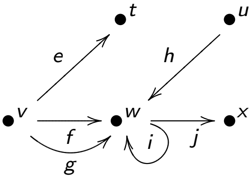

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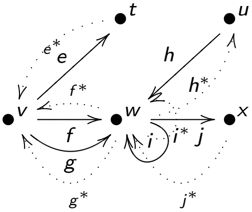

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(CK1) $e^*e = r(e)$; and $f^*e = 0$ for $f \neq e$ (for all edges e, f in E).

(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for each vertex v in E.

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Definition

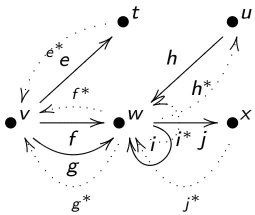
The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \; / \; < (CK1), (CK2) >$$

Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

$\widehat{E}$ =



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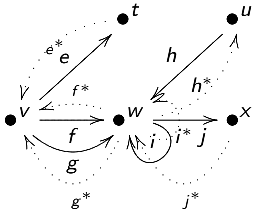

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$ff^* = \ldots$ (no simplification) Note: $(ff^*)^2 = f(f^*f)f^* = ff^*$

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$$E = \bullet^{v} \circlearrowright x$$

Then $L_K(E) \cong K[x, x^{-1}]$.

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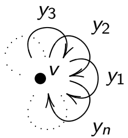
$E = R_n =$ 

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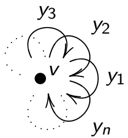
$E = R_n =$ 

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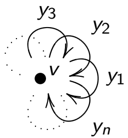

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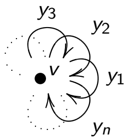

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$\sum_{i=1}^{n} y_i x_i = 1_{L_K(1,n)}$, and $x_i y_j = \delta_{i,j} 1_{L_K(1,n)}$,

while $L_K(R_n)$ has these SAME generators and relations, where we identify y_i^* with x_i.

Historical note, part 1

1962: Leavitt gives construction of $L_K(1, n)$.

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1980's: Cuntz, Krieger, and others generalize the $\mathcal{O}_n$ construction to directed graphs, and produce the *graph* C**-algebras* $C^*(E)$.

Historical note, part 1

1962: Leavitt gives construction of $L_K(1, n)$.

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1980's: Cuntz, Krieger, and others generalize the $\mathcal{O}_n$ construction to directed graphs, and produce the *graph C*-algebras* $C^*(E)$.

June 2004: Various algebraists attend the CBMS lecture series

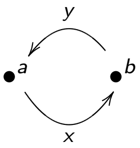
"Graph C^*-algebras: algebras we can see",

held at University of Iowa, given by Iain Raeburn.

Algebraic analogs of graph C*-algebras are defined and investigated starting Fall 2004.

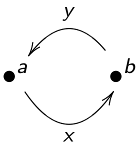
Some graph definitions

1. A *cycle*

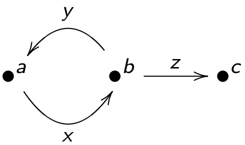


Some graph definitions

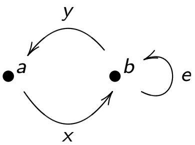
1. A *cycle*



2. An *exit* for a cycle.



or



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3b. *connects to* a cycle.

$$\begin{tikzcd} \bullet^a \arrow[r, bend right, "x"'] & \bullet^b \arrow[l, bend right, "y"'] & \bullet^c \arrow[l, "f"] \end{tikzcd}$$

Simplicity of Leavitt path algebras

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For which graphs E and fields K is $L_K(E)$ simple?

Note $L_K(E)$ is simple for

$$E = \bullet \longrightarrow \bullet \cdots\cdots\!\!> \qquad \longrightarrow \bullet \quad \text{since } L_K(E) \cong M_n(K)$$

and for

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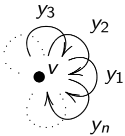

since $L_K(E) \cong L_K(1,n)$

but not simple for

$$E = R_1 = \bullet^v \circlearrowright x \quad \text{since } L_K(E) \cong K[x, x^{-1}]$$

Simplicity of Leavitt path algebras

Theorem

(A -, Aranda Pino, 2005) $L_K(E)$ is simple if and only if:

1. *Every vertex connects to every cycle and to every sink in E, and*
2. *Every cycle in E has an exit.*

Simplicity of Leavitt path algebras

Theorem

(A -, Aranda Pino, 2005) $L_K(E)$ is simple if and only if:

1. *Every vertex connects to every cycle and to every sink in E, and*
2. *Every cycle in E has an exit.*

Note: No role played by K.

Other ring-theoretic properties of Leavitt path algebras

We know precisely the graphs E for which $L_K(E)$ has various other properties, e.g.:

1. one-sided chain conditions
2. prime
3. von Neumann regular
4. two-sided chain conditions
5. primitive

Many more.

The monoid $\mathcal{V}(R)$

Recall: P is a *finitely generated projective* R-module in case $P \oplus Q \cong R^n$ for some Q, some $n \in \mathbb{N}$.

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Additional examples: Rf where f is idempotent (i.e., $f^2 = f$), since $Rf \oplus R(1-f) = R^1$.

So, for example, in $R = \mathrm{M}_2(\mathbb{R})$, $P = \mathrm{M}_2(\mathbb{R})e_{1,1} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$

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So $L_K(E)$ contains projective modules of the form $L_K(E)ee^*$ for each edge e of E.

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$\mathcal{V}(R)$ denotes the isomorphism classes of finitely generated projective (left) R-modules. With operation $\oplus$, this becomes an abelian monoid. Note R itself plays a special role in $\mathcal{V}(R)$.

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Remark: Given a ring R, it is in general not easy to compute $\mathcal{V}(R)$.

The monoid M_E

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Associate to E the abelian monoid $(M_E, +)$:

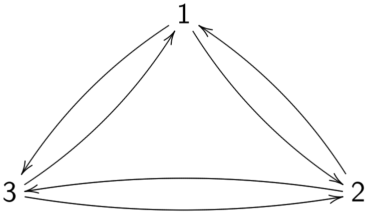
$$M_E = \{\sum_{v \in E^0} n_v a_v\}$$

with $n_v \in \mathbb{Z}^+$ for all $v \in E^0$.

Relations in M_E are given by: $a_v = \sum_{e \in s^{-1}(v)} a_{r(e)}$.

The monoid M_E

Example. Let F be the graph



So M_F consists of elements $\{n_1 a_1 + n_2 a_2 + n_3 a_3\}$ $(n_i \in \mathbb{Z}^+)$,

subject to: $a_1 = a_2 + a_3$; $a_2 = a_1 + a_3$; $a_3 = a_1 + a_2$.

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$$\begin{tikzcd} & 1 \arrow[dl, bend right] \arrow[dr, bend left] & \\ 3 \arrow[ur, bend right] \arrow[rr, bend right] & & 2 \arrow[ul, bend right] \arrow[ll, bend right] \end{tikzcd}$$

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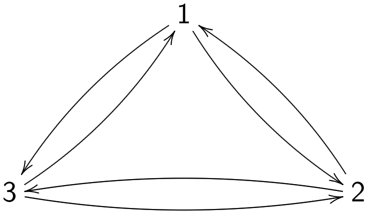

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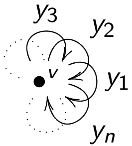
It's not hard to get: $\quad M_F = \{0,\ a_1,\ a_2,\ a_3,\ a_1 + a_2 + a_3\}$.

In particular, $M_F \setminus \{0\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

The monoid $\mathcal{V}(L_K(E))$

Example:

$$E = R_n =$$



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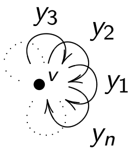

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So here, $M_E = \{0, a_v, 2a_v, ..., (n-1)a_v\}$.

In particular, $M_E \setminus \{0\} \cong \mathbb{Z}_{n-1}$.

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Theorem

(P. Ara, M.A. Moreno, E. Pardo, 2007)
For any row-finite directed graph E,

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Moreover, $L_K(E)$ is universal with this property.

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Moreover, $L_K(E)$ is universal with this property.

One (very nontrivial) consequence: Let S denote $L_K(1,n)$. Then

$$\mathcal{V}(S) = \{0,\ S,\ S^2, ..., S^{n-1}\}.$$

Historical Note, Part 2

So we can think of Leavitt path algebras in two ways:

1) the "quotient of a path algebra" approach, and

2) the "universal algebra which supports M_E as its $\mathcal{V}$-monoid" approach.

These were developed in parallel.

The two approaches together have complemented each other in the development of the subject.

Purely infinite simplicity

Here's a property (most likely unfamiliar to most of you ...)

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We call a unital simple ring R *purely infinite simple* if R is not a division ring, and for every $r \neq 0$ in R there exists α, β in R for which

$$\alpha r \beta = 1_R.$$

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Moreover, in this situation, we can easily calculate $\mathcal{V}(L_K(E))$ using the Smith normal form of the matrix $I - A_E$.

Eight Year Update

In addition to "expected" types of results, over the past eight years Leavitt path algebras have played an interesting / important role in resolving various questions outside the subject per se.

1. Kaplansky's question on prime non-primitive von Neumann regular algebras.
2. The realization question for von Neumann regular rings.
3. Constructing simple Lie algebras.
4. Connections to various C^*-algebras.
5. Constructing algebras with prescribed sets of prime / primitive ideals

Matrices over Leavitt algebras

One such connection:

Let $R = L_{\mathbb{C}}(1, n)$. So ${}_RR \cong {}_RR^n$.

So this gives in particular $R \cong \mathrm{M}_n(R)$ as rings.

Which then (for free) gives some additional isomorphisms, e.g.

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Also, ${}_RR \cong {}_RR^n \cong {}_RR^{2n-1} \cong {}_RR^{3n-2} \cong \ldots$, which also in turn yield ring isomorphisms

$$R \cong \mathrm{M}_n(R) \cong \mathrm{M}_{2n-1}(R) \cong \mathrm{M}_{3n-2}(R) \cong \ldots$$

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Answer: In general, yes.

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For instance, if $R = L(1,4)$, then it's not hard to show that $R \cong \mathrm{M}_2(R)$ as rings (even though $R \ncong {}_R R^2$ as modules).
Idea: 2 and 4 are nicely related, so these eight matrices inside $\mathrm{M}_2(L(1,4))$ "work":

$$X_1 = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} x_3 & 0 \\ x_4 & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix},\ X_4 = \begin{pmatrix} 0 & x_3 \\ 0 & x_4 \end{pmatrix}$$

together with their duals

$$Y_1 = \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix},\ Y_2 = \begin{pmatrix} y_3 & y_4 \\ 0 & 0 \end{pmatrix},\ Y_3 = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix},\ Y_4 = \begin{pmatrix} 0 & 0 \\ y_3 & y_4 \end{pmatrix}$$

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In general, using this same idea, we can show that:

if $d | n^t$ for some $t \in \mathbb{N}$, then $L(1,n) \cong \mathrm{M}_d(L(1,n))$.

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On the other hand ...

If $R = L(1,n)$, then the "type" of R is $n-1$. (Think: "smallest difference"). Bill Leavitt showed the following in his 1962 paper:

The type of $\mathrm{M}_d(L(1,n))$ is $\frac{n-1}{g.c.d.(d,n-1)}$.

In particular, if $g.c.d.(d,n-1) > 1$, then $L(1,n) \ncong \mathrm{M}_d(L(1,n))$.

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In particular, if $g.c.d.(d,n-1) > 1$, then $L(1,n) \not\cong \mathrm{M}_d(L(1,n))$.

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(Note: $d | n^t \Rightarrow g.c.d.(d,n-1) = 1$.)

Matrices over Leavitt algebras

Smallest interesting pair: Is $L(1,5) \cong \mathrm{M}_3(L(1,5))$?

We are led "naturally" to consider these five matrices (and their duals) in $\mathrm{M}_3(L(1,5))$:

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ x_5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & {x_1}^2 \\ 0 & 0 & x_2x_1 \\ 0 & 0 & x_3x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_4x_1 \\ 0 & 0 & x_5x_1 \\ 0 & 0 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_3 \\ 0 & 0 & x_4 \\ 0 & 0 & x_5 \end{pmatrix}$$

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Everything went along swimmingly... But we couldn't see how to generate the matrix units $e_{1,3}$ and $e_{3,1}$ inside $\mathrm{M}_3(L(1,5))$ using these ten matrices.

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Instead, this set (together with duals) works:

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(A-, Ánh, Pardo; Crelle's J. 2008) For any field K,

$$L_K(1,n) \cong \mathrm{M}_d(L_K(1,n)) \;\Leftrightarrow\; g.c.d.(d, n-1) = 1.$$

Indeed, more generally,

$$\mathrm{M}_d(L_K(1,n)) \cong \mathrm{M}_{d'}(L_K(1,n')) \;\Leftrightarrow\; n = n' \;\text{ and }\; g.c.d.(d, n-1) = g.c.d.(d', n-1).$$

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Moreover, we can write down the isomorphisms explicitly.

Along the way, some elementary (but apparently new) number theory ideas come into play.

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Here's what made this second set of matrices work. Using this partition in the particular case $n = 5, d = 3$, then the partition of $\{1, 2, 3, 4, 5\}$ turns out to be the two sets

$$\{1, 4\} \quad \text{and} \quad \{2, 3, 5\}.$$

The matrices that "worked" are ones where we fill in the last columns with terms of the form $x_i x_1^j$ in such a way that i is in the same subset as the row number of that entry.

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Here's what made this second set of matrices work. Using this partition in the particular case $n = 5, d = 3$, then the partition of $\{1, 2, 3, 4, 5\}$ turns out to be the two sets

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The matrices that "worked" are ones where we fill in the last columns with terms of the form $x_i x_1^j$ in such a way that i is in the same subset as the row number of that entry.

The number theory underlying this partition in the general case where $\mathrm{g.c.d.}(d, n-1) = 1$ is elementary. But we are hoping to find some other 'context' in which this partition process arises.

Matrices over Leavitt algebras

Computations when $n = 5, d = 3$.

$\gcd(3, 5-1) = 1$. Now $5 = 1 \cdot 3 + 2$, so that $r = 2, r - 1 = 1$, and define $s = d - (r-1) = 3 - 1 = 2$.

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Now extend these two sets mod 3 to all integers up to 5.

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Matrices over Leavitt algebras

Does this look familiar?

Matrices over Leavitt algebras

Corollary. (Matrices over the Cuntz C*-algebras)

$$\mathcal{O}_n \cong \mathrm{M}_d(\mathcal{O}_n) \Leftrightarrow g.c.d.(d, n-1) = 1.$$

(And the isomorphisms are explicitly described.)

Matrices over Leavitt algebras

An important recent application:

For each pair of positive integers n, r, there exists an infinite, finitely presented simple group $G_{n,r}^{+}$. These were introduced in the mid-1970's. "Higman-Thompson groups".

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Theorem. (E. Pardo, 2011)

$$G^+_{n,r} \cong G^+_{m,s} \quad \Leftrightarrow \quad m = n \text{ and } \mathrm{g.c.d.}(r, n-1) = \mathrm{g.c.d.}(s, n-1).$$

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Proof. Show that $G^+_{n,r}$ can be realized as an appropriate subgroup of the invertible elements of $M_r(L_{\mathbb{C}}(1,n))$, and then use the explicit isomorphisms provided in the A -, Ánh, Pardo result.

What else is out there?

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(1) $L_K(E) \cong L_K(F) \Leftrightarrow \quad ?\ ?\ ?$

Ideas from symbolic dynamics come into play here. Using some results on flow equivalence, we have been able to show:

Theorem. (A -, Louly, Pardo, Smith, 2011) If $L_K(E)$ and $L_K(F)$ are purely infinite simple Leavitt path algebras such that

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then $L_K(E) \cong L_K(F)$.

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then $L_K(E) \cong L_K(F)$. Can we drop the determinant hypothesis?

What else is out there?

In particular, if

$$E_4 \ = \ \begin{tikzcd} \bullet^{v_1} \arrow[loop left] \arrow[r, bend left] & \bullet^{v_2} \arrow[loop above] \arrow[l, bend left] \arrow[r, bend left] & \bullet^{v_3} \arrow[loop above] \arrow[l, bend left] \arrow[r, bend left] & \bullet^{v_4} \arrow[loop right] \arrow[l, bend left] \end{tikzcd}$$

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is $L_{\mathbb{C}}(E_4) \cong L_{\mathbb{C}}(1,2)$?

The answer will be interesting, however it plays out.

What else is out there?

(2) For any graph E there is an intimate relationship between $L_{\mathbb{C}}(E)$ and $C^*(E)$. There are many theorems of the form:

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$$L_{\mathbb{C}}(E) \text{ has algebraic property } \mathcal{P} \Leftrightarrow C^*(E) \text{ has analytic property } \mathcal{P}$$

but the proofs are not direct! They all are based on showing that the two properties are both equivalent to

$$E \text{ has graph property } \mathcal{Q}.$$

Why this happens is still a mystery.

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(A-, Jessica Sklar), "The graph menagerie: Abstract algebra meets the Mad Veterinarian", *Mathematics Magazine* **83**(3), 2010, 168-179.

Questions?

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Thanks to: University of Victoria, and Simons Foundation

Some elementary number theory

The partition of $\{1, 2, ..., n\}$ induced by d when $\mathrm{g.c.d.}(d, n-1) = 1$

Suppose $\mathrm{g.c.d.}(d, n-1) = 1$. Write

$$n = dt + r \text{ with } 1 \leq r \leq d.$$

Let s denote $d - (r-1)$.

It is easy to show that $\mathrm{g.c.d.}(d, n-1) = 1$ implies $\mathrm{g.c.d.}(d, s) = 1$. We consider the sequence $\{h_i\}_{i=1}^{d}$ of integers, whose i^{th} entry is given by

$$h_i = 1 + (i-1)s \pmod{d}.$$

Some elementary number theory

The integers h_i are understood to be taken from the set $\{1, 2, ..., d\}$.

Because $\mathrm{g.c.d.}(d, s) = 1$, basic number theory yields that the set of entries $\{h_1, h_2, ..., h_d\}$ equals the set $\{1, 2, ..., d\}$ (in some order). Our interest lies in a decomposition of $\{1, 2, ..., d\}$ effected by the sequence $h_1, h_2, ..., h_d$, as follows.

Some elementary number theory

We let d_1 denote the integer for which

$$h_{d_1} = r - 1$$

in the previously defined sequence. We denote by $\hat{S}_1$ the following subset of $\{1, 2, ..., d\}$:

$$\hat{S}_1 = \{h_i | 1 \leq i \leq d_1\}.$$

We denote by $\hat{S}_2$ the complement of $\hat{S}_1$ in $\{1, 2, ..., d\}$. We now construct a partition $S_1 \cup S_2$ of $\{1, 2, ..., n\}$ by defining, for each $j \in \{1, 2, ..., n\}$ and for $i \in \{1, 2\}$,

$$j \in S_i \text{ precisely when } j \equiv j' \pmod{d} \text{ for } j' \in \{1, 2, ..., d\}, \text{ and } j' \in \hat{S}_i.$$

(In other words, we extend the partition $\hat{S}_1 \cup \hat{S}_2$ of $\{1, 2, ..., d\}$ to a partition $S_1 \cup S_2$ of $\{1, 2, ..., n\}$ by extending $\bmod\ d$.)

Some elementary number theory

Example. Suppose $n = 35, d = 13$. Then $\gcd(13, 35 - 1) = 1$, so we are in the desired situation. Now $35 = 2 \cdot 13 + 9$, so that $r = 9, r - 1 = 8$, and $s = d - (r - 1) = 13 - 8 = 5$. Then we consider the sequence starting at 1, and increasing by s each step, and interpret mod d. (This will give all integers between 1 and d.)

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So here we get the sequence $1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9.$

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Now break this set into two pieces: those integers up to and including $r - 1$, and those after. Since $r - 1 = 8$, here we get

$$\{1, 2, ..., 13\} = \{1, 3, 6, 8, 11\} \cup \{2, 4, 5, 7, 9, 10, 12, 13\}.$$

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Now extend these two sets mod 13 to all integers up to 35.

$$\{1, 3, 6, 8, 11, 14, 16, 19, 21, 24, 27, 29, 32, 34\} \cup$$

$$\{2, 4, 5, 7, 9, 10, 12, 13, 15, 17, 18, 20, 22, 23, 25, 26, 28, 30, 31, 33, 35\}$$