Symbolic dynamics and Leavitt path algebras: The Algebraic KP Question

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May 20, 2014

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A unital algebra A which is not a division ring, and which has this property, is called *purely infinite simple.*

There is a module-theoretic description of these algebras:

An idempotent $e \in A$ is called *infinite* if there exist NONZERO idempotents $f, g \in A$ for which $Ae \cong Af \oplus Ag$, and for which $Ae \cong Af$.

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Proposition: A is purely infinite simple if and only if every nonzero left ideal of A contains an infinite idempotent.

Simple, and purely infinite simple, Leavitt path algebras

Theorem (A-, Aranda Pino, 2005): $L_K(E)$ is simple if and only if E has:

1. every vertex in E connects to every cycle and every sink in E, and
2. every cycle in E has an *exit*.

(Note: no dependence on K.)

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1. every vertex in E connects to every cycle in E,
2. every cycle in E has an *exit*, and
3. E contains at least one cycle.

So this generalizes Leavitt's result.

The Isomorphism Question for Leavitt path algebras

There is a nice connection between $\mathcal{V}(L_K(E))$ and $K_0(L_K(E))$ in the context of purely infinite simplicity.

Theorem. $L_K(E)$ is purely infinite simple if and only if $\mathcal{V}(L_K(E)) \setminus \{0\}$ is a group.   (Necessarily $K_0(L_K(E))$.)

The Isomorphism Question for Leavitt path algebras

$$L_K(E) \cong L_K(F) \quad \Leftrightarrow \quad ???$$

It's fair to say that this question is the Holy Grail for many (most?) people working in Leavitt path algebras.

$L_K(E) \cong L_K(F) \quad \Leftrightarrow \quad ???$

There are easy examples to show that different graphs E and F can produce isomorphic Leavitt path algebras.

Proposition: Suppose E is a finite graph which contains no (directed) closed paths. Let $v_1, v_2, \ldots, v_t$ denote the sinks of E. (At least one must exist.) For each $1 \leq i \leq t$, let n_i denote the number of paths in E which end in v_i. Then

$$L_K(E) \cong \oplus_{i=1}^{t} \mathrm{M}_{n_i}(K).$$

$L_K(E) \cong L_K(F) \quad \Leftrightarrow \quad ???$

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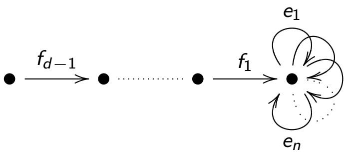
For instance: If

$$E = \bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{and} \quad F = \bullet \longrightarrow \bullet \longleftarrow \bullet$$

then E and F are not isomorphic as graphs, but $L_K(E) \cong L_K(F) \cong \mathrm{M}_3(K)$.

$L_K(E) \cong L_K(F) \quad \Leftrightarrow \quad ???$

Let $R_n(d)$ denote this graph:



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(so there are $d - 1$ edges added)

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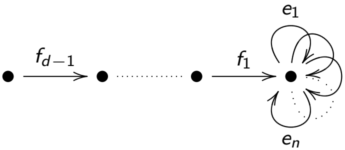

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Proposition:

$$L_K(R_n(d)) \cong \mathrm{M}_d(L_K(1,n)).$$

$L_K(E) \cong L_K(F) \Leftrightarrow \quad ? ? ?$

Recall that $A = L_K(1,n)$ has ${}_AA \cong {}_AA^n$ as left A-modules. So, in particular, $A \cong \mathrm{M}_n(A)$. But then

$$L_K(R_n) \cong L_K(1,n) \cong \mathrm{M}_n(L_K(1,n)) \cong L_K(R_n(n)),$$

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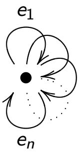
$$L_K(R_n) \cong L_K(1,n) \cong \mathrm{M}_n(L_K(1,n)) \cong L_K(R_n(n)),$$

so that the Leavitt path algebras of these two graphs are isomorphic:



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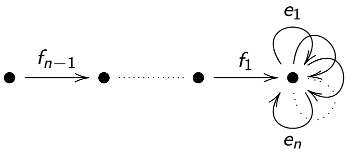

$L_K(E) \cong L_K(F) \Leftrightarrow$? ? ?

More generally: for what values of n, n', d, d' do we have

$$L_K(R_n(d)) \cong L_K(R_{n'}(d'))?$$

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More generally: for what values of n, n', d, d' do we have

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Theorem

(A-, Ánh, Pardo; Crelle's J. 2008) For any field K,

$$\mathrm{M}_d(L_K(1,n)) \cong \mathrm{M}_{d'}(L_K(1,n')) \;\Leftrightarrow$$

$$n = n' \;\text{ and }\; g.c.d.(d, n-1) = g.c.d.(d', n-1).$$

(Moreover, we can write down the isomorphisms explicitly.)

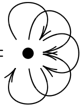
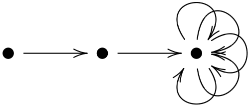
Matrices over Leavitt algebras

Breakthrough came from an analysis of isomorphisms between more general Leavitt path algebras.

There are a few "graph moves" which preserve the isomorphism classes of certain types of Leavitt path algebras.

"Shift" and "outsplitting".

Matrices over Leavitt algebras

$E = R_5 =$  and $R_5(3) =$ 

There exists a sequence of graphs

$$R_5 = E_1, E_2, ..., E_7 = R_5(3)$$

for which E_{i+1} is gotten from E_i by one of these two "graph moves".

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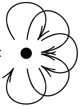
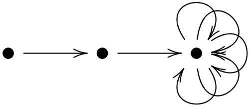
$E = R_5 =$  and $R_5(3) =$ 

There exists a sequence of graphs

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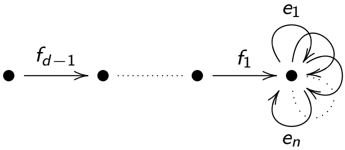
$$\text{So } \; L_K(R_5) \cong L_K(E_2) \cong \cdots \cong L_K(R_5(3)) \cong \mathrm{M}_3(R_5).$$

Note: For $2 \leq i \leq 6$ it is not immediately obvious how to view $L_K(E_i)$ in terms of a matrix ring over a Leavitt algebra.

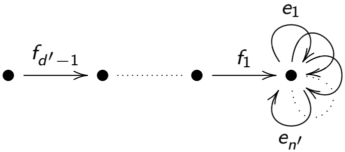
Once we parsed out what was happening with this particular set of moves, we were able to see how to do things in general.

Matrices over Leavitt algebras

So in particular we have if

$R_n(d) =$ 

and

$R_{n'}(d') =$ 

Then $L_K(R_n(d)) \cong L_K(R_{n'}(d'))$ if and only if $n = n'$ and $\gcd(d, n-1) = \gcd(d', n-1)$.

Application to the theory of simple groups

Brief digression:

Here is an important recent application of the A-, Ánh, Pardo isomorphism theorem.

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For each pair of positive integers n, r, there exists an infinite, finitely presented simple group $G_{n,r}^{+}$. "Higman Thompson groups."

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Here is an important recent application of the A-, Ánh, Pardo isomorphism theorem.

For each pair of positive integers n, r, there exists an infinite, finitely presented simple group $G^+_{n,r}$. "Higman Thompson groups."

Higman knew *some* conditions regarding isomorphisms between these groups, but did not have a complete classification.

Application to the theory of simple groups

Theorem. (E. Pardo, 2011)

$$G^+_{n,r} \cong G^+_{m,s} \quad \Leftrightarrow \quad m = n \text{ and } \mathrm{g.c.d.}(r, n-1) = \mathrm{g.c.d.}(s, n-1).$$

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Idea of Proof. Show that $G^+_{n,r} \cong U_r(n)$ (an explicitly described subgroup of the units of $\mathrm{M}_r(L_K(1,n))$), and that the explicit isomorphisms provided in the A -, Ánh, Pardo result take $U_r(n)$ onto $U_s(n)$.

The Kirchberg Phillips Theorem for C*-algebras

Kirchberg and Phillips (2000) each proved this deep result:

KP Theorem for C*-algebras: Suppose A and B are C*-algebras which are:

1. unital
2. simple
3. purely infinite
4. separable
5. nuclear
6. in the "bootstrap class"

Suppose there is an isomorphism $\varphi : K_0(A) \to K_0(B)$ for which $\varphi([A]) = [B]$, and suppose $K_1(A) \cong K_1(B)$.

Then $A \cong B$ (homeomorphically).

The Kirchberg Phillips Theorem for C*-algebras

In the particular case of graph C*-algebras, necessarily some of these hypotheses are automatically satisfied. The KP Theorem becomes:

KP Theorem for graph C*-algebras: Suppose E and F are finite graphs for which $C^*(E)$ and $C^*(F)$ are purely infinite simple. Suppose there is an isomorphism $\varphi : K_0(C^*(E)) \to K_0(C^*(F))$ for which $\varphi([C^*(E)]) = [C^*(F)]$.

Then $C^*(E) \cong C^*(F)$ (homeomorphically).

The Algebraic Kirchberg Phillips Question

It turns out that:

1) $K_0(L_K(E)) \cong K_0(C^*(E))$ for any finite graph E.
(Ara / Moreno / Pardo, 2007)

2) The K_1 data for $L_K(E)$ and $C^*(E)$ does not necessarily match up. But: if $L_K(E)$ and $L_K(F)$ are unital purely infinite simple, then

$$K_0(L_K(E)) \cong K_0(L_K(F)) \;\Rightarrow\; K_1(L_K(E)) \cong K_1(L_K(F)).$$

The Algebraic Kirchberg Phillips Question

3) When $L_K(E)$ is unital purely infinite simple, the K_0 groups are easily described in terms of the adjacency matrix A_E of E. Let $n = |E^0|$. View $I_n - A_E^t$ as a linear transformation $\mathbb{Z}^n \to \mathbb{Z}^n$. Then

$$K_0(L_K(E)) \cong K_0(C^*(E)) \cong \mathrm{Coker}(I_n - A_E^t).$$

Moreover, $\mathrm{Coker}(I_n - A_E^t)$ can be computed* by finding the Smith normal form of $I_n - A_E^t$.

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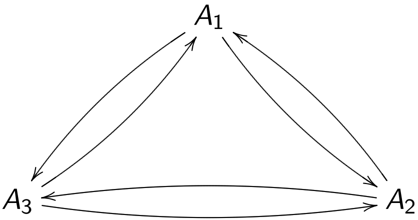
* But this might take awhile in general ...

The Algebraic Kirchberg Phillips Question

Example:



$$I_3 - A_E^t = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \text{ whose Smith normal form is: } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Conclude that $K_0(L_K(E)) \cong \mathrm{Coker}(I_3 - A_E^t) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The Algebraic Kirchberg Phillips Question

The question becomes: Can information about K_0 be used to establish isomorphisms between Leavitt path algebras as well?

The Algebraic KP Question: Suppose E and F are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple. Suppose also that there exists an isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$.

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$$\text{Is } L_K(E) \cong L_K(F)?$$

Connections to symbolic dynamics

VERY informally:

Some mathematicians and computer scientists have interest in, roughly, how information "flows" through a directed graph.

Makes sense to ask: When is it the case that information flows through two different graphs in essentially the same way?
"Flow equivalent graphs".

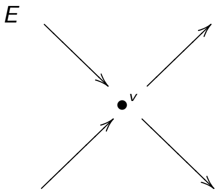
(Often cast in the language of matrices.)

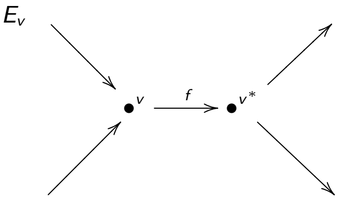
Standard reference for these ideas:

D. Lind and B. Marcus, "An Introduction to Symbolic Dynamics and Coding", Cambridge U. Press, 1995.

Connections to symbolic dynamics

Example: "Expansion at v"





Proposition: If E_v is the expansion graph of E at v, then E and E_v are flow equivalent. Rephrased, "expansion" (and its inverse "contraction") preserve flow equivalence.

Connections to symbolic dynamics

There are four other 'graph moves' which preserve flow equivalence:

out-split (and its inverse out-amalgamation), and

in-split (and its inverse in-amalgamation).

Theorem PS (Parry / Sullivan): Two graphs E, F are flow equivalent if and only if one can be gotten from the other by a sequence of transformations involving these six graph operations.

Connections to symbolic dynamics

Graph transformations may be reformulated in terms of adjacency matrices.

For an $n \times n$ matrix M with integer entries, think of M as a linear transformation $M : \mathbb{Z}^n \to \mathbb{Z}^n$. In particular, when $M = I_n - A_E^t$.

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Proposition (Parry / Sullivan): If E is flow equivalent to F, then $\det(I - A_E^t) = \det(I - A_F^t)$.

Proposition (Bowen / Franks): If E is flow equivalent to F, then $\mathrm{Coker}(I - A_E^t) \cong \mathrm{Coker}(I - A_F^t)$.

Connections to symbolic dynamics

Theorem F (Franks): Suppose E and F have some additional properties (*irreducible, nontrivial*). If

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So by Theorem PS, if E and F are "nice", and if the Cokernels and determinants of appropriate transformations match up correctly, then there is a sequence of "well-understood" graph transformations which starts with E and ends with F.

Connections to symbolic dynamics

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Proof: Show that an isomorphic copy of $L_K(E)$ can be viewed as a corner of $L_K(F)$ (or vice-versa); the corner is necessarily full by simplicity.

Connections to symbolic dynamics

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Theorem: (A- / Louly / Pardo / C. Smith 2011): Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F)) \quad \text{and} \quad \det(I - A_E^t) = \det(I - A_F^t),$$

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Remark:

If $K_0(L_K(E))$ is finite, then $|K_0(L_K(E))| = |\det(I - A_E^t)|$.

If $K_0(L_K(E))$ is infinite, then $|\det(I - A_E^t)| = 0$.

So we need only assume that the signs of $\det(I - A_E^t)$ and $\det(I - A_F^t)$ are the same.

Connections to symbolic dynamics

Using some intricate computations provided by Huang (2001), one can show the following:

Suppose $L_K(E)$ is purely infinite simple.

Suppose there is *some* Morita equivalence $\Psi : L_K(E)-\mathrm{Mod} \to L_K(F)-\mathrm{Mod}$.

Further, suppose there is *some* isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$.

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Then there is a Morita equivalence
$\Phi : L_K(E)-\mathrm{Mod} \to L_K(F)-\mathrm{Mod}$ for which $\Phi|_{K_0(L_K(E))} = \varphi$.

"Restricted" Algebraic KP Theorem

Consequently:

Theorem: (A- / Louly / Pardo / Smith 2011): Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F))$$

via an isomorphism φ for which $\varphi([L_K(E)]) = [L_K(F)]$,

and $\quad \det(I - A_E^t) = \det(I - A_F^t)$,

then $L_K(E) \cong L_K(F)$.

"Restricted" Algebraic KP Theorem

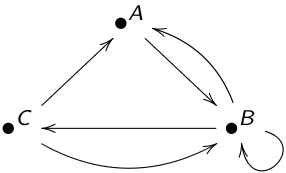
Examples

1. $E = R_n$.
   1. $K_0(L_K(R_n)) \cong \mathbb{Z}_{n-1}$
   2. under this isomorphism, $[L_K(R_n)] \mapsto 1$
   3. $\det(I - A^t_{R_n}) = 1 - n < 0$.

2. $E = R_n(d)$.
   1. $K_0(L_K(R_n(d))) \cong \mathbb{Z}_{n-1}$
   2. under this isomorphism, $[L_K(R_n(d))] \mapsto d$
   3. $\det(I - A^t_{R_n(d)}) = 1 - n < 0$.

Examples

3.

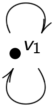
$E =$ 

1. $K_0(L_K(E)) \cong \mathbb{Z}_3$
2. under this isomorphism, $[L_K(E)] \mapsto 1$
3. $\det(I - A^t_{R_n}) = -3 < 0.$

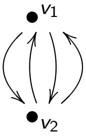
Conclude: $L_K(E) \cong L_K(R_4) = L_K(1,4)$.

The graphs C_n^{-1}:

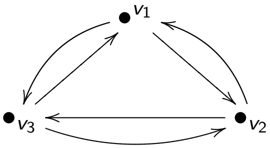
$C_1^{-1} =$



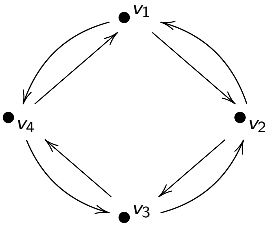
$C_2^{-1} =$



$C_3^{-1} =$



$C_4^{-1} =$



$K_0(L_K(C_n^{-1}))$

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$$E = C_n^{-1}$$

1

$n \bmod 6$	1	2	3	4	5	6
$K_0(C_n^{-1}) \cong$	$\{0\}$	$\mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_3$	$\{0\}$	$\mathbb{Z} \times \mathbb{Z}$

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This gives information about isomorphisms to various matrix rings over Leavitt algebras.

Can we drop the determinant hypothesis?

Algebraic KP Question: Can we drop the hypothesis on the determinants in the Restricted Algebraic KP Theorem?

Can we drop the determinant hypothesis?

Here's the "smallest" example of a situation of interest. Consider the Leavitt path algebras $L(R_2)$ and $L(E_4)$, where

$$R_2 = \begin{tikzcd} \bullet^{v} \arrow[loop above] \arrow[loop right] \end{tikzcd} \quad \text{and} \quad E_4 = \begin{tikzcd} \bullet^{v_1} \arrow[loop left] \arrow[r, bend left] & \bullet^{v_2} \arrow[loop above] \arrow[l, bend left] \arrow[r, bend left] & \bullet^{v_3} \arrow[loop above] \arrow[l, bend left] \arrow[r, bend left] & \bullet^{v_4} \arrow[loop right] \arrow[l, bend left] \end{tikzcd}$$

It is not hard to establish that

$$(K_0(L(R_2)), [1_{L(R_2)}]) = (\{0\}, 0) = (K_0(L(E_4)), [1_{L(E_4)}]);$$

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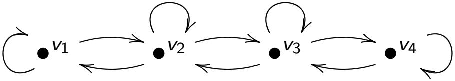
and $E_4 =$ 

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Question: Is $L_K(R_2) \cong L_K(E_4)$?

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Some remarks:

1. $C^*(R_2) \cong C^*(E_4)$; this follows from the KP Theorem for C*-algebras, and can also be done more "directly" using "KK Theory". But the isomorphism is NOT given explicitly.

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2. Start with E for which $L_K(E)$ is purely infinite simple. There is a systematic (easy) way to produce a graph F for which $L_K(F)$ is purely infinite simple, $K_0(L_K(E)) \cong K_0(L_K(F))$, but $\det(I - A_E^t) = -\det(I - A_F^t)$. "Cuntz Splice".

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4. There are three possible outcomes to the Algebraic KP Question: NEVER, SOMETIMES, or ALWAYS. The answer will be interesting, no matter how things play out.

Since it's true for C*-algebras ...

There's a strong (uncanny / not-well-understood) connection between results for Leavitt path algebras and results for graph C*-algebras.

But the results are not identical.

For example: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

Question (open for about five years):

$$\text{Is } L_K(1,2) \otimes_K L_K(1,2) \cong L_K(1,2)?$$

Since it's true for C*-algebras ...

This has been answered in the negative.

THREE different proofs given, independently, in Spring 2011:

1. J. Bell + G. Bergman
2. W. Dicks
3. P. Ara + G. Cortiñas

Ara / Cortiñas showed more: if the tensor product of n nontrivial Leavitt path algebras is isomorphic to the tensor product of m nontrivial Leavitt path algebras, then $m = n$.

Dicks' approach: Show an isomorphism invariant doesn't match up (global dimension).

Conjecture?

Is there an Algebraic KP **Conjecture**?

Not really.

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More open questions about Leavitt path algebras were generated at a meeting at BIRS in April 2013.

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Recent ideas in "continuous orbit equivalence" imply that things work out for the graph C*-algebra case; the Leavitt path algebra case is currently being worked out as well.

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For any graph E, $L_K(E)$ is $\mathbb{Z}$-graded:

$$\deg(v) = 0 \ \forall v \in E^0; \quad \deg(e) = 1, \deg(e^*) = -1 \ \forall e \in E^1$$

So if α is a path of length m and β is a path of length n in E, then $\deg(\alpha\beta^*) = m - n$.

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Revisit these two graphs:

$$E = \bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{and} \quad F = \bullet \longrightarrow \bullet \longleftarrow \bullet$$

We saw $L_K(E) \cong L_K(F)$. But $L_K(E) \ncong L_K(F)$ as *graded* K-algebras.

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Graded projective modules; $K_0^{gr}(L_K(E))$.

The shift operation yields that $K_0^{gr}(L_K(E))$ is a module over $K[x, x^{-1}]$.

A *graded* Algebraic KP Theorem?

Question: Suppose E and F are any finite graphs. Suppose there exists a $K[x, x^{-1}]$-module isomorphism

$$\varphi : K_0^{gr}(L_K(E)) \to K_0^{gr}(L_K(F))$$

for which $\varphi([1_{L_K(E)}]) = [1_{L_K(F)}]$.

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Affirmative results have been achieved:

R. Hazrat (2011): "polycephalic" graphs (includes acyclic, R_n, ...)

P. Ara & E. Pardo (in preparation): Many more ... (but not yet all)

Questions?

Thanks to the Simons Foundation