Leavitt path algebras: An Introduction

UW Algebra Seminar Pre-talk

May 20, 2014

General path algebras

Let E be a directed graph. $E = (E^0, E^1, r, s)$

$$\bullet^{s(e)} \xrightarrow{e} \bullet^{r(e)}$$

The *path algebra* KE is the K-algebra with basis $\{p_i\}$ consisting of the directed paths in E. (View vertices as paths of length 0.)

$$\text{In } KE, \ p \cdot q = pq \ \text{ if } \ r(p) = s(q), \quad 0 \text{ otherwise.}$$

$$\text{In particular, } s(e) \cdot e = e = e \cdot r(e).$$

Note: E^0 is finite $\Leftrightarrow$ KE is unital; in this case $1_{KE} = \sum_{v \in E^0} v$.

Building Leavitt path algebras

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$$E \;=\; \begin{tikzcd} & \bullet^{t} & \bullet^{u} \arrow[dl, "h"'] \\ \bullet^{v} \arrow[ur, "e"] \arrow[r, "f"'] \arrow[r, bend right, "g"'] & \bullet^{w} \arrow[loop below, "i"] \arrow[r, "j"'] & \bullet^{x} \end{tikzcd}$$

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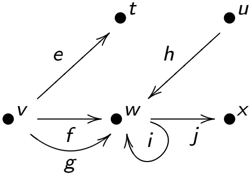

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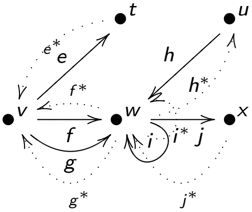

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(CK1) $e^*e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$.

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Definition

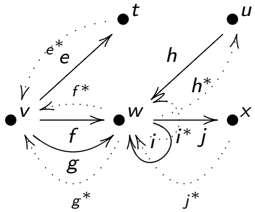
The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

$$\widehat{E} \;=$$



$$ee^* + ff^* + gg^* = v \qquad g^*g = w \quad g^*f = 0$$

$$h^*h = w \quad hh^* = u \quad ff^* = \ldots \;\; \text{(no simplification)}$$

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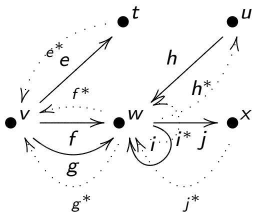

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But $(ff^*)^2 = f(f^*f)f^* = f \cdot r(f) \cdot f^* = ff^*.$

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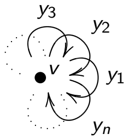
Then $L_K(E) \cong M_n(K)$.

$$E = \bullet^{v} \circlearrowright x$$

Then $L_K(E) \cong K[x, x^{-1}]$.

Leavitt path algebras: Examples

$E = R_n =$ 

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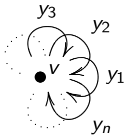

Then $L_K(E) \cong L_K(1, n)$, the classical "Leavitt algebra of order n".
$L_K(1, n)$ is generated by $y_1, \ldots, y_n, x_1, \ldots, x_n$, with relations

$$x_i y_j = \delta_{i,j} 1_K \quad \text{and} \quad \sum_{i=1}^{n} y_i x_i = 1_K.$$

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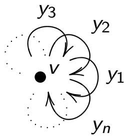

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Note: $A = L_K(1, n)$ has ${}_AA \cong {}_AA^n$ as left A-modules:

$$a \mapsto (ay_1, ay_2, \ldots, ay_n) \text{ and } (a_1, a_2, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i x_i.$$

Things we know about Leavitt path algebras

The main goal in the early years of the development: Establish results of the form

$$L_K(E) \text{ has algebraic property } \mathcal{P} \;\Leftrightarrow$$
$$E \text{ has graph-theoretic property } \mathcal{Q}.$$

(Only recently has the structure of K played a role.)

Things we know about Leavitt path algebras

We know precisely the graphs E for which $L_K(E)$ has these properties: (No role played by the structure of K in any of these.)

1. simplicity
2. purely infinite simplicity
3. (one-sided) artinian; (one-sided) noetherian
4. (two-sided) artinian; (two-sided) noetherian
5. exchange $(**)$
6. prime
7. primitive

The monoid $\mathcal{V}(R)$, and the Grothendieck group $K_0(R)$

Isomorphism classes of finitely generated projective (left) R-modules, with operation $\oplus$, denoted $\mathcal{V}(R)$.

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Theorem

(George Bergman, Trans. A.M.S. 1975) Given a field K and finitely generated conical monoid S with a distinguished element I, there exists a universal K-algebra R for which $\mathcal{V}(R) \cong S$, and for which, under this isomorhpism, $[R] \mapsto I$.

The construction is explicit, uses amalgamated products.

Bergman included the algebras $L_K(1, n)$ as examples of these universal algebras.

The monoid $\mathcal{V}(L_K(E))$

For any graph E construct the free abelian monoid M_E.

$$\text{generators } \{a_v \mid v \in E^0\}; \qquad \text{relations } a_v = \sum_{r(e)=w} a_w$$

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Using Bergman's construction,

Theorem

(Ara, Moreno, Pardo, Alg. Rep. Thy. 2007)

For any field K,

$$\mathcal{V}(L_K(E)) \cong M_E.$$

Under this isomorphism, $[L_K(E)] \mapsto \sum_{v \in E^0} a_v$.

The monoid $\mathcal{V}(L_K(E))$

Example.

$$\begin{tikzcd}
& 1 \arrow[dl, bend right] \arrow[dr, bend left=15] & \\
3 \arrow[ur, bend right=15] \arrow[rr, bend right=15] & & 2 \arrow[ul, bend right] \arrow[ll, bend right=15]
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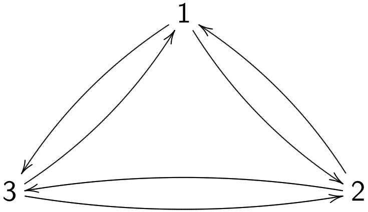

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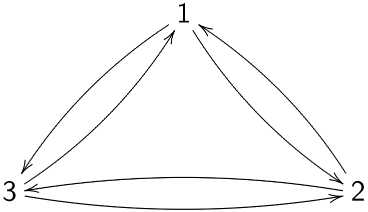

$$\text{Not hard to show:} \quad M_E = \{z, A_1, A_2, A_3, A_1 + A_2 + A_3\}$$

$$\text{Note: } M_E \setminus \{z\} \;\cong\; \mathbb{Z}_2 \times \mathbb{Z}_2.$$

$$\text{So } \mathcal{V}(L_K(E)) \setminus \{0\} \;\cong\; \mathbb{Z}_2 \times \mathbb{Z}_2.$$

$$\text{Here } L_K(E) \mapsto A_1 + A_2 + A_3 = (0,0) \in \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The monoid $\mathcal{V}(L_K(E))$

Example: For each $n \in \mathbb{N}$ let C_n denote the "directed cycle" graph with n vertices.

Then it's easy to show that $M_{C_n} = \mathbb{Z}^+$, and so $M_{C_n} \setminus \{0\} = \mathbb{N}$.

So in particular $\mathcal{V}(L_K(C_n)) \setminus \{0\}$ is not a group.

C*-algebras

Theorem

(Cuntz, Comm. Math. Physics, 1977) There exist simple C^*-algebras generated by partial isometries.

Denote by $\mathcal{O}_n$.

Subsequently, a similar construction was produced of the "graph C*-algebra" $C^*(E)$, for any graph E. In this context, $\mathcal{O}_n \cong C^*(R_n)$.

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Subsequently, a similar construction was produced of the "graph C*-algebra" $C^*(E)$, for any graph E. In this context, $\mathcal{O}_n \cong C^*(R_n)$.   For any graph E,

$$L_{\mathbb{C}}(E) \subseteq C^*(E)$$

as a dense $*$-subalgebra. In particular, $L_{\mathbb{C}}(1, n) \subseteq \mathcal{O}_n$.

(But $C^*(E)$ is usually "much bigger" than $L_{\mathbb{C}}(E)$.)

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1. simple
2. purely infinite simple
3. stable rank, prime, primitive, exchange, etc....

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But (2011, three different proofs) $L_K(1,2) \otimes L_K(1,2) \ncong L_K(1,2)$.