

§3.7 L'Hôpital's Rule and Indeterminate Forms

Indeterminate Forms

Back in the chapter on Limits, we saw methods for dealing with the following limits.

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} \qquad \lim_{x \rightarrow \infty} \frac{4x^2 - 5x}{1 - 3x^2}$$

In the first limit if we plugged in $x = 4$ we would get $0/0$ and in the second limit if we "plugged" in infinity we would get ∞/∞ . Both of these are called **indeterminate forms**. In both of these cases there are competing interests or rules and it's not clear which will win out.

In the case of $0/0$ we typically think of a fraction that has a numerator of zero as being zero. However, we also tend to think of fractions in which the denominator is going to zero, in the limit, as infinity or might not exist at all. Likewise, we tend to think of a fraction in which the numerator and denominator are the same as one. So, which will win out? Or will neither win out and they all "cancel out" and the limit will reach some other value?

If a limit is $0/0$, does the 0 in the numerator win and make the limit 0? Or does the 0 in the denominator win and make the limit blow up to infinity? Or does some secret other thing happen?

In the case of ∞/∞ we have a similar set of problems. If the numerator of a fraction is going to infinity we tend to think of the whole fraction going to infinity. Also, if the denominator is going to infinity, in the limit, we tend to think of the fraction as going to zero. We also have the case of a fraction in which the numerator and denominator are the same (ignoring the minus sign) and so we might get -1. Again, it's not clear which of these will win out, if any of them will win out. Also with this second limit, there is the further problem that infinity isn't really a number and so we really shouldn't even treat it like a number. Much of the time it simply won't behave as we would expect it to if it was a number.

If a limit is ∞/∞ , does the ∞ in the numerator win and make the limit blow up to infinity? Or does the ∞ in the denominator win and make the limit go to 0? Or does some secret other thing happen?

This is the problem with indeterminate forms. It's just not clear what is happening in the limit. There are other types of indeterminate forms as well. Some other types are,

$$\infty - \infty \quad (0)(\pm \infty) \quad 1^\infty \quad 0^0 \quad \infty^0.$$

The first we will call an **indeterminate difference**, the second (both the $+$ and $-$ versions) will be called **indeterminate products**, and the remaining three we will call **indeterminate exponentials**. These all have competing interests or rules that tell us what should happen and it's just not clear which, if any, of the interests or rules will win out. The topic of this section is how to deal with these kinds of limits.

As already pointed out we do know how to deal with some kinds of indeterminate forms already. For the two limits above we work them as follows.

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} = \lim_{x \rightarrow 4} (x + 4) = 8$$

$$\lim_{x \rightarrow \infty} \frac{4x^2 - 5x}{1 - 3x^2} = \lim_{x \rightarrow \infty} \frac{4 - \frac{5}{x}}{\frac{1}{x^2} - 3} = -\frac{4}{3}$$

In the first case we simply factored, canceled and took the limit and in the second case we factored out an x^2 from both the numerator and the denominator and took the limit. Notice as well that none of the competing interests or rules in these cases won out! That is often the case.

So, we can deal with some of these. However, what about the following two limits.

$$\lim_{x \rightarrow 1} \frac{\ln(x)}{x - 1} \qquad \lim_{x \rightarrow \infty} \frac{e^x}{x^2}$$

This first is of the form $0/0$, but we can't factor this one. The second is of the form ∞/∞ , but we can't just factor an x^2 out of the numerator. So, nothing that we've got in our bag of tricks will work with these two limits.

This is where the subject of this section comes into play.

L'Hôpital's Rule

Definition 1: L'Hôpital's Rule

Suppose $f(x)$ and $g(x)$ are differentiable near a real number a (but not necessarily differentiable at a) and that $g'(x)$ is nonzero near a (but not necessarily at a). If $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists and either

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{OR} \qquad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\pm \infty}{\pm \infty},$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Proof for a special case:

Suppose $f(a) = g(a) = 0$, $f(x)$ and $g(x)$ also have continuous derivatives at $x = a$, and $g'(a) \neq 0$. Then we get:

$$\begin{aligned}
 \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f(x) - 0}{g(x) - 0} \\
 &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} && \text{since } f(a) = g(a) = 0 \\
 &= \lim_{x \rightarrow a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} && \text{divide num. and denom. by } x - a \\
 &= \frac{\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}} \\
 &= \frac{f'(a)}{g'(a)} && \text{definition of the derivative} \\
 &= \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} && f \text{ and } g \text{ have continuous derivatives}
 \end{aligned}$$

Why we need $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ to exist

Let's look at an explicit example. Suppose we want to determine

$$\lim_{x \rightarrow \infty} \frac{x + \sin(x)}{x}.$$

Even though $\sin(x)$ in the numerator will oscillate, the numerator will still go to ∞ , as will the denominator. So we might think to try and use L'Hôpital's Rule: the derivative of the numerator is $1 + \cos(x)$ and the derivative of the denominator is 1.

So our new limit would look like

$$\lim_{x \rightarrow \infty} \frac{1 + \cos(x)}{1} = \lim_{x \rightarrow \infty} (1 + \cos(x))$$

which does not exist since the values of $1 + \cos(x)$ will oscillate between 0 and 2 indefinitely as $x \rightarrow \infty$.

However, we can compute the limit of the original expression:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x + \sin(x)}{x} &= \lim_{x \rightarrow \infty} \left(\frac{x}{x} + \frac{\sin(x)}{x} \right) \\ &= \lim_{x \rightarrow \infty} \left(1 + \frac{\sin(x)}{x} \right) \\ &= 1 + \lim_{x \rightarrow \infty} \frac{\sin(x)}{x}.\end{aligned}$$

The limit $\lim_{x \rightarrow \infty} \frac{\sin(x)}{x}$ can be shown to equal 0 by an application of the Squeeze Theorem. Thus

$$\lim_{x \rightarrow \infty} \frac{x + \sin(x)}{x} = 1 + 0 = 1,$$

which does exist, even though $\lim_{x \rightarrow \infty} \frac{1 + \cos(x)}{1}$ does not exist.

While L'Hôpital's Rule here is written for two sided limits at a finite number a , it is still valid for one-sided limits and limits at $\pm\infty$.

So, L'Hôpital's Rule tells us that if we have an indeterminate form $0/0$ or $\pm\infty/\infty$ all we need to do is differentiate the numerator and differentiate the denominator and then take the limit. There are some important things to note about L'Hôpital's Rule.

Note:

You can only use in the case of the indeterminate forms $0/0$ or $\pm\infty/\infty$. So you **must** always check, with direct substitution usually, that your limit is of one of these forms.

If your limit is not of one of these forms, you can **not** use L'Hôpital's Rule.

Why we need $0/0$ or $\pm\infty/\infty$

Let's suppose we are looking at the following limit.

$$\lim_{x \rightarrow 2} \frac{3 - 2x}{x + 1}$$

If we were to "apply L'Hôpital's Rule", we would get the following limit.

$$\lim_{x \rightarrow 2} \frac{-2}{1} = -2$$

And so, we would be tempted to say that the original limit is also -2 . However, when we use direct substitution on the original limit, we get,

$$\lim_{x \rightarrow 2} \frac{3 - 2x}{x + 1} = \frac{-1}{3} \neq -2 = \lim_{x \rightarrow 2} \frac{-2}{1}.$$

And so we see that applying L'Hôpital's Rule when we are not allowed to will often yield incorrect results.

Note

Do not confuse L'Hôpital's Rule with the Quotient Rule for derivatives. L'Hôpital's Rule **uses** derivatives (of just the numerator and denominator **separately**) to help determine certain limits. The Quotient Rule tells you **how** to take derivatives of a fraction of functions.

It is also possible to use L'Hôpital's Rule more than once in the same problem. If your original limit gives, say, $0/0$, you can use L'Hôpital's Rule. Now let's say this new limit also gives $0/0$. You can use L'Hôpital's Rule again to obtain a second new limit that should still equal the original.

In certain instances, as we'll see, L'Hôpital's Rule can be used (meaning we get $0/0$ or $\pm\infty/\infty$) but does not end up being helpful to determine the limit. In these instances, we can usually revert back to methods discussed in the Limits Chapter.

There is a lot to say and more things to be careful about with L'Hôpital's Rule, but we will bring those up in some examples.

Example 3.7.1: Using L'Hôpital's Rule

Evaluate each of the following limits.

- $\lim_{x \rightarrow 0} \frac{\sin x}{x}$
- $\lim_{t \rightarrow 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3}$
- $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$
- $\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 4}}{2 - 5x}$

Solutions:

- We can quickly see that direct substitution will lead to a $0/0$ indeterminate form. So let's apply L'Hôpital's Rule. Remember, we only need the derivative of the numerator, which is $\cos(x)$, and the derivative of the denominator, which is 1.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\sin(x))}{\frac{d}{dx}(x)} \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{1} \\ &= \frac{1}{1} = 1 \end{aligned}$$

- b. In this case we also have a $0/0$ indeterminate form and if we were really good at factoring we could factor the numerator and denominator, simplify and take the limit. However, that might be more work than just using L'Hôpital's Rule. So let's try L'Hôpital's Rule.

$$\begin{aligned}\lim_{t \rightarrow 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3} &= \lim_{t \rightarrow 1} \frac{\frac{d}{dt}(5t^4 - 4t^2 - 1)}{\frac{d}{dt}(10 - t - 9t^3)} \\ &= \lim_{t \rightarrow 1} \frac{20t^3 - 8t}{-1 - 27t^2} \\ &= \frac{20 - 8}{-1 - 27} \\ &= -\frac{3}{7}\end{aligned}$$

- c. Note that both $e^x \rightarrow \infty$ and $x^2 \rightarrow \infty$ as $x \rightarrow \infty$. So this limit gives a ∞/∞ indeterminate form. Let's apply L'Hôpital's Rule.

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

Now we have a small problem. This new limit is also a ∞/∞ indeterminate form. However, it's not really a problem. We know how to deal with these kinds of limits. Just apply L'Hôpital's Rule again.

$$\lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(2x)} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

Altogether, we can view this as

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty.$$

Here was an example where we needed to apply L'Hôpital's Rule more than once. And as long as we satisfy the conditions for L'Hôpital's Rule ($0/0$ or ∞/∞ indeterminate form), we may use it. Also remember that this particular limit does not exist. We just use " ∞ " to explain **how** it doesn't exist.

- d. First note that direct substitution would give a $-\infty/\infty$ indeterminate form. We know we're allowed to use L'Hôpital's Rule in this instance, so let's do so. We'll be sure to simplify our expression as well.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 4}}{2 - 5x} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(\sqrt{3x^2 + 4})}{\frac{d}{dx}(2 - 5x)} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{2}(3x^2 + 4)^{-1/2}(6x)}{-5} \\ &= \lim_{x \rightarrow \infty} \frac{3x}{-5\sqrt{3x^2 + 4}}\end{aligned}$$

We see we've run into a "problem" again in that this new limit is also a $-\infty/\infty$ indeterminate form. Well, we saw in the last example that we could use L'Hôpital's Rule again, so let's do that here as well.

$$\begin{aligned}
\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 4}}{2 - 5x} &= \lim_{x \rightarrow \infty} \frac{3x}{-5\sqrt{3x^2 + 4}} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(3x)}{\frac{d}{dx}(-5\sqrt{3x^2 + 4})} \\
&= \lim_{x \rightarrow \infty} \frac{3}{-5 \cdot \frac{1}{2}(3x^2 + 4)^{-1/2}(6x)} \\
&= \lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 4}}{-5x}
\end{aligned}$$

Again we see we end up with a $-\infty/\infty$ indeterminate form. While we are allowed to use L'Hôpital's Rule here, we should take a look at where we ended up in comparison to where we started. We start by looking at

$$\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 4}}{2 - 5x}$$

and ended up looking at

$$\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 4}}{-5x}.$$

While the expressions are not the same, they only differ by a constant in the denominator. We still have a quadratic expression under a square root in the numerator and a linear expression in the denominator. L'Hôpital's Rule didn't give us a "simpler" expression.

Unfortunately, this can happen sometimes; repeated uses of L'Hôpital's Rule (even if allowed to apply it) can lead to expressions "cycling" between similar forms.

In cases like these, we need to stop using L'Hôpital's Rule as it is clearly not going to help us here. Instead, we'll have to revert back to our methods from the Limits Chapter.

Let's go back to the original problem:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 4}}{2 - 5x}.$$

Remember that we had done a type of factoring to solve limits like this. Here, we'll factor out an x^2 on the inside of the square root, recall that $\sqrt{x^2} = x$ for this problem since $x \rightarrow \infty$, and also factor out x from the denominator.

$$\begin{aligned}
\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2 + 4}}{2 - 5x} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 \left(3 + \frac{4}{x^2}\right)}}{x \left(\frac{2}{x} - 5\right)} \\
&= \lim_{x \rightarrow \infty} \frac{x \sqrt{3 + \frac{4}{x^2}}}{x \left(\frac{2}{x} - 5\right)} \\
&= \lim_{x \rightarrow \infty} \frac{\sqrt{3 + \frac{4}{x^2}}}{\frac{2}{x} - 5}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{\sqrt{3+0}}{0-5} \\
 &= -\frac{\sqrt{3}}{5}
 \end{aligned}$$

Now try one on your own!

Check Point 3.7.1: Practice L'Hôpital's Rule

Evaluate the limit using L'Hôpital's rule.

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{4x^2}$$

Indeterminate Products

Suppose we'd like to determine the following limit:

$$\lim_{x \rightarrow 0^+} x \ln(x).$$

With direct substitution, we see that x will go to 0 while $\ln(x)$ will go to $-\infty$. So we have two terms being multiplied, and they are at odds with one another. On one hand, the x piece is trying to make the whole expression get closer to 0 (since 0 times any number is 0). On the other hand, the $\ln(x)$ piece is trying to make the whole expression a larger and larger (negative) number. So which one wins?

If our limit is $0 \cdot \infty$, does the 0 win and make the limit very small, or does the ∞ win and make the limit very large? Or does some secret third thing happen?

This $0(\pm\infty)$ is another indeterminate form, and so we'll need to do something else to figure out this limit. Unfortunately, it is not in the form of $0/0$ or $\pm\infty/\infty$, so we can NOT use L'Hôpital's Rule. However, we can rewrite the expression as a quotient of 2 things instead of a product; we just need to make sure we're using algebra appropriately.

We can **not** immediately use L'Hôpital's Rule on a limit in $0 \cdot \infty$ form. We must rewrite it first!

Generally speaking, this is how we can rewrite a product as a quotient:

$$f(x)g(x) = \frac{g(x)}{1/f(x)} \quad \text{OR} \quad f(x)g(x) = \frac{f(x)}{1/g(x)}.$$

Rewriting $f(x)g(x)$:

$$f(x)g(x) = \frac{g(x)}{1/f(x)}$$

$$f(x)g(x) = \frac{f(x)}{1/g(x)}$$

Using this algebra technique will allow us to turn any limit in the form $0(\pm\infty)$ into a limit of the form $0/0$ or $\pm\infty/\infty$. Which one of these two we get after doing the rewrite will depend upon which fact we used to

do the rewrite. One of the rewrites will give $0/0$ and the other will give $\pm\infty/\infty$. It all depends on which function stays in the numerator and which gets moved down to the denominator. Either way, once we make this change, then we'll usually be able to apply L'Hôpital's Rule as it will be a $0/0$ or $\pm\infty/\infty$ form.

Knowing which piece to move to the denominator depends on the actual expressions you are working with. One tip for thinking about this is to look at the fraction you end up with in preparation for L'Hôpital's Rule. One version might have slightly easier derivatives of the numerator and denominator than the other version. In general, more practice will get you used to the idea. And if one version doesn't work or isn't helpful, just go back and try the other version.

Let's use this idea to go back and determine that first limit introduced in this part.

Example 3.7.2: Determine the limit.

$$\lim_{x \rightarrow 0^+} x \ln(x)$$

Solution: We already saw that direct substitution will yield an indeterminate form of $0(-\infty)$. Now note that

$$x \ln(x) = \frac{\ln(x)}{\frac{1}{x}}$$

and so

$$\lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x}.$$

So we have only rewritten the expression. However, a little investigation (and remembering some information about [infinite limits](#)) shows that this new limit is of the form $-\infty/\infty$; $\ln(x) \rightarrow -\infty$ and $\frac{1}{x} \rightarrow \infty$ as $x \rightarrow 0^+$. Thus we can apply L'Hôpital's Rule!

$\lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{x}}$	now in $-\infty/\infty$ form
$= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} [\ln(x)]}{\frac{d}{dx} [\frac{1}{x}]}$	L'Hôpital's Rule
$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$	derivative rules
$= \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot \left(-\frac{x^2}{1}\right)$	dividing fractions
$= \lim_{x \rightarrow 0^+} -x$	simplify expression
$= 0$	evaluate limit

So we see we were able to take a limit in the form of an indeterminate product ($0(-\infty)$) and rewrite it so that it became a limit of an indeterminate form of $0/0$ or $\pm\infty/\infty$. In doing so, we were allowed to then apply L'Hôpital's Rule.

We will see that much of what we do in the rest of this section is to somehow rewrite a limit so that it is in the

form of $0/0$ or $\pm\infty/\infty$ and then apply L'Hôpital's Rule. If you keep this idea in mind, there will not be much you need to memorize; only knowing our end goal is to get a limit of either $0/0$ or $\pm\infty/\infty$ form.

Let's look at another example.

Example 3.7.3: Determine the limit.

$$\lim_{x \rightarrow -\infty} xe^x$$

Solution: We can see this limit is of the form $(\infty)(0)$. As with these indeterminate forms, we'll need to write it as a quotient. Moving the x to the denominator worked in the previous example so let's try that with this problem as well.

$$\lim_{x \rightarrow -\infty} xe^x = \lim_{x \rightarrow -\infty} \frac{e^x}{1/x}$$

Writing the product in this way gives us a product that has the form $0/0$ in the limit. So, let's use L'Hôpital's Rule.

$$\lim_{x \rightarrow -\infty} xe^x = \lim_{x \rightarrow -\infty} \frac{e^x}{1/x} = \lim_{x \rightarrow -\infty} \frac{e^x}{-1/x^2}$$

Well, we still have a $0/0$ form, so let's apply L'Hôpital's Rule as we get those forms.

$$\lim_{x \rightarrow -\infty} xe^x = \lim_{x \rightarrow -\infty} \frac{e^x}{1/x} = \lim_{x \rightarrow -\infty} \frac{e^x}{-1/x^2} = \lim_{x \rightarrow -\infty} \frac{e^x}{2/x^3} = \lim_{x \rightarrow -\infty} \frac{e^x}{-6/x^4} = \dots$$

This doesn't seem to be getting us anywhere. With each application of L'Hôpital's Rule we just end up with another $0/0$ indeterminate form and in fact the derivatives seem to be getting worse and worse. Also note that if we simplified the quotient back into a product we would just end up with either $(\infty)(0)$ or $(-\infty)(0)$ and so that won't do us any good.

This does not mean however that the limit can't be done. It just means that we may have moved the wrong function to the denominator. Let's move the exponential function instead.

$$\lim_{x \rightarrow -\infty} xe^x = \lim_{x \rightarrow -\infty} \frac{x}{1/e^x} = \lim_{x \rightarrow -\infty} \frac{x}{e^{-x}}$$

Note that we used the fact that $\frac{1}{e^{-x}} = e^x$, which will help us when it comes time to take some derivatives. The quotient is now an indeterminate form of $-\infty/\infty$ and L'Hôpital's Rule gives

$$\lim_{x \rightarrow -\infty} xe^x = \lim_{x \rightarrow -\infty} \frac{x}{e^{-x}} = \lim_{x \rightarrow -\infty} \frac{1}{-e^{-x}} = 0.$$

When faced with a product $(0)(\pm\infty)$ we can turn it into a quotient that will allow us to use L'Hôpital's Rule. However, as we saw in the last example we need to be careful with how we do that on occasion. Sometimes we can use either quotient and in other cases only one will work.

Check Point 3.7.2: Practice with indeterminate products

Use L'Hôpital's Rule to find the following limit.

$$\lim_{x \rightarrow 0^+} x^2 \ln(x)$$

Indeterminate Differences

We have seen indeterminates of the form $\infty - \infty$ before (or equivalently $-\infty + \infty$). In certain instances, we found that factoring out the term which goes to ∞ (or $-\infty$) fastest was a good method for determining limits.

But there are other situations we might find ourselves in that still yield the $\infty - \infty$ form. Let's see some examples and how we might be able to apply our algebra knowledge (and perhaps L'Hôpital's Rule if we meet the conditions).

Example 3.7.4: Determine the limits.

- a. $\lim_{\theta \rightarrow \pi^+} (\cot(\theta) + \csc(\theta))$
 b. $\lim_{x \rightarrow \infty} (\ln(x^2 + 3x + 1) - \ln(5 + 4x^2))$

Solutions:

- a. We may need to remember a bit about trigonometry, but this limit is of the form $\infty - \infty$. In particular, $\cot(\theta) \rightarrow \infty$ and $\csc(\theta) \rightarrow -\infty$ as $\theta \rightarrow \pi^+$. Here, there is not necessarily anything to factor, but we can remember the relationship between cotangent and secant with sine and cosine.

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \quad \text{and} \quad \csc(\theta) = \frac{1}{\sin(\theta)}$$

Let's use this to rewrite our limit.

$$\lim_{\theta \rightarrow \pi^+} (\cot(\theta) + \csc(\theta)) = \lim_{\theta \rightarrow \pi^+} \left(\frac{\cos(\theta)}{\sin(\theta)} + \frac{1}{\sin(\theta)} \right) = \lim_{\theta \rightarrow \pi^+} \frac{\cos(\theta) + 1}{\sin(\theta)}$$

If we were to do a direct substitution at this point, we would see that the numerator goes to 0 ($\cos(\pi) = -1$) and the denominator goes to 0. So now we have a limit of the form $0/0$, allowing us to use L'Hôpital's Rule.

$$\begin{aligned} \lim_{\theta \rightarrow \pi^+} (\cot(\theta) + \csc(\theta)) &= \lim_{\theta \rightarrow \pi^+} \frac{\cos(\theta) + 1}{\sin(\theta)} \\ &= \lim_{\theta \rightarrow \pi^+} \frac{-\sin(\theta) + 0}{\cos(\theta)} && \text{apply L'Hôpital's Rule} \\ &= \frac{-\sin(\pi)}{\cos(\pi)} \\ &= \frac{-(0)}{-1} \\ &= 0 \end{aligned}$$

- b. Notice that $x^2 + 3x + 1$ and $5 + 4x^2$ both go to ∞ as x does. Additionally, the natural logarithm function goes to ∞ as its input does. So this limit is in the $\infty - \infty$ form. Here, there is nothing to cancel and there is no common denominator between the two components to be able to write the entire expression as a fraction.

However, we can at least combine this into just one term using the properties of logarithms. In particular,

$$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right).$$

So we can rewrite our limit as

$$\lim_{x \rightarrow \infty} (\ln(x^2 + 3x + 1) - \ln(5 + 4x^2)) = \lim_{x \rightarrow \infty} \ln\left(\frac{x^2 + 3x + 1}{5 + 4x^2}\right).$$

Now note that the natural logarithm function is continuous everywhere on its domain (inputs are positive) and we see that as $x \rightarrow \infty$, the entire input of our logarithm, $\frac{x^2 + 3x + 1}{5 + 4x^2}$, will be positive. What this allows us to do is bring the limit on the inside of the logarithm; check out the section on [continuity](#) for a reminder of this fact.

Now we have the following.

$$\begin{aligned} \lim_{x \rightarrow \infty} (\ln(x^2 + 3x + 1) - \ln(5 + 4x^2)) &= \lim_{x \rightarrow \infty} \ln\left(\frac{x^2 + 3x + 1}{5 + 4x^2}\right) \\ &= \ln\left(\lim_{x \rightarrow \infty} \frac{x^2 + 3x + 1}{5 + 4x^2}\right) \end{aligned}$$

If we look at this limit on the inside of the logarithm, we see that it is of the form ∞/∞ , which means we can use L'Hôpital's Rule. But be careful! We will only be doing the derivatives of the numerator and denominator on the inside of the logarithm. We will leave the natural logarithm alone (but it still needs to stay there).

$$\begin{aligned} \lim_{x \rightarrow \infty} (\ln(x^2 + 3x + 1) - \ln(5 + 4x^2)) &= \ln\left(\lim_{x \rightarrow \infty} \frac{x^2 + 3x + 1}{5 + 4x^2}\right) \\ &= \ln\left(\lim_{x \rightarrow \infty} \frac{\frac{d}{dx}[x^2 + 3x + 1]}{\frac{d}{dx}[5 + 4x^2]}\right) && \text{L'Hôpital's Rule on inside} \\ &= \ln\left(\lim_{x \rightarrow \infty} \frac{2x + 3}{8x}\right) && \text{derivative rules; inside still } \infty/\infty \\ &= \ln\left(\lim_{x \rightarrow \infty} \frac{\frac{d}{dx}[2x + 3]}{\frac{d}{dx}[8x]}\right) && \text{L'Hôpital's Rule on inside again} \\ &= \ln\left(\lim_{x \rightarrow \infty} \frac{2}{8}\right) && \text{derivative rules} \\ &= \ln\left(\frac{2}{8}\right) && \text{evaluate limit} \\ &= \ln\left(\frac{1}{4}\right) && \text{simplify fraction} \\ &= -\ln(4) && \text{equivalent answer: uses properties of logs.} \end{aligned}$$

As we saw, limits of the form $\infty - \infty$ can come from a variety of different function types. What is important is to recognize that $\infty - \infty$ is an indeterminate form and that more work needs to be done in order to determine the limit.

Check Point 3.7.3: Practice with indeterminate differences

Use L'Hôpital's Rule to find the following limit.

$$\lim_{x \rightarrow 0^+} (\ln(x) - \ln(\sin(x))) =$$

Indeterminate Exponentials

Now we come to the last group of indeterminate forms:

$$1^\infty \quad 0^0 \quad \infty^0.$$

We'll refer to these collectively as indeterminate exponentials. In all of these, the base and the exponent have competing interests and it is not clear which one, if either, will win out for the limit. Thankfully, these can all be dealt with in the same way, thankfully. Remember, one of the goals for dealing with these new indeterminate forms was to somehow rewrite the limit so it is a fraction, perhaps of the form $0/0$ or $\pm\infty/\infty$ in order to use L'Hôpital's Rule.

We **cannot** immediately apply L'Hôpital's Rule to indeterminate exponentials. We'll need to rewrite first!

Unlike the other problems, we can't easily turn an exponential expression into a fraction; the exponent is fairly stuck. However, we can use logarithms (even if they're not there) and their properties to our advantage. Recall:

$$y = a^b \quad \text{means} \quad \ln(y) = \ln(a^b) = b \ln(a).$$

Notice that the exponent b is no longer an exponent. In fact, $b \ln(a)$ is a product, and we know how to deal with limits of products (even ones which give indeterminate forms). The downside here is that $b \ln(a)$ is equal to $\ln(y)$, not y itself. So if we want to get back to y , we need to apply an exponential:

$$\ln(y) = b \ln(a) \quad \Rightarrow \quad y = e^{\ln(y)} = e^{b \ln(a)}.$$

In particular, this means that

$$\lim_{x \rightarrow a} f(x)^{g(x)} = \lim_{x \rightarrow a} e^{g(x) \ln(f(x))}.$$

It can be fairly messy to do all the work here in the exponent. What we will sometimes do is look and see exactly what is happening to just the exponent in terms of the limit. However, it is incredibly important to **remember to go back to the original limit**. If we find that $L = \lim_{x \rightarrow a} g(x) \ln(f(x))$, then $\lim_{x \rightarrow a} f(x)^{g(x)} = e^L$.

Hopefully this will be easier to see with an example.

Example 3.7.5: Determine the limit.

$$\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$$

Solution: This limit is of the form ∞^0 , and indeterminate. As seen in the previous discussion,

$$x^{\frac{1}{x}} = e^{\frac{1}{x} \ln(x)} = e^{\frac{\ln(x)}{x}}.$$

In order to look at the limit $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$, let's first look at a **different limit**, $\lim_{x \rightarrow \infty} \frac{\ln(x)}{x}$, the limit of just the exponent on e from above. Note that if $\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} = L$, then $\lim_{x \rightarrow \infty} x^{\frac{1}{x}} = e^L$.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{\ln(x)}{x} && \frac{\infty}{\infty} \text{ form; use L'Hôpital's Rule} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} && \text{derivative rules} \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} && \text{simplify expression} \\ &= 0 && \text{evaluate limit} \end{aligned}$$

So we have found that $L = 0$, but this is not our final answer! Remember, this is all happening in the exponent of e . We need to go back to the original limit.

$$\begin{aligned} \lim_{x \rightarrow \infty} x^{\frac{1}{x}} &= e^L \\ &= e^0 \\ &= 1 \end{aligned}$$

Now we are done and can conclude that $\lim_{x \rightarrow \infty} x^{\frac{1}{x}} = 1$.

See if you can work out a problem yourself.

Check Point 3.7.4: Practice with indeterminate exponentials

Find

$$\lim_{x \rightarrow \infty} \left(1 + \frac{5}{x}\right)^{8x}$$

§3.7.1 Practice Problems

Determine the following limits.

- $\lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$
- $\lim_{w \rightarrow -4} \frac{\sin(\pi w)}{w^2 - 16}$
- $\lim_{t \rightarrow \infty} \frac{\ln(3t)}{t^2}$
- $\lim_{z \rightarrow 0} \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2}$
- $\lim_{x \rightarrow -\infty} \frac{x^2}{e^{1-x}}$
- $\lim_{z \rightarrow \infty} \frac{z^2 + e^{4z}}{2z - e^z}$
- $\lim_{t \rightarrow \infty} \left[t \ln \left(1 + \frac{3}{t} \right) \right]$
- $\lim_{w \rightarrow 0^+} [w^2 \ln(4w^2)]$
- $\lim_{x \rightarrow 1^+} \left[(x-1) \tan \left(\frac{\pi}{2} x \right) \right]$
- $\lim_{y \rightarrow 0^+} [\cos(2y)]^{1/y^2}$
- $\lim_{x \rightarrow \infty} [e^x + x]^{1/x}$
- $\lim_{x \rightarrow \infty} \frac{2x + \sqrt{x^2 - 7}}{4 - 5x}$
- $\lim_{x \rightarrow \infty} (\ln(x) - x^2)$
- $\lim_{x \rightarrow \infty} (e^x - x^3)$

Assignment Problems

For problems 1 – 20 determine the limit.

- $\lim_{x \rightarrow -4} \frac{x^3 + 6x^2 - 32}{x^3 + 5x^2 + 4x}$
- $\lim_{w \rightarrow -\infty} \frac{e^{-6w}}{4 + e^{-3w}}$
- $\lim_{t \rightarrow 0} \frac{\sin(6t)}{\sin(11t)}$
- $\lim_{x \rightarrow 1} \frac{x^2 + 8x - 9}{x^3 - 2x^2 - 5x + 6}$
- $\lim_{t \rightarrow 2} \frac{t^3 - 7t^2 + 16t - 12}{t^4 - 4t^3 + 4t^2}$
- $\lim_{w \rightarrow -\infty} \frac{w^2 - 4w + 1}{3w^2 + 7w - 4}$
- $\lim_{y \rightarrow \infty} \frac{y^2 - e^{6y}}{4y^2 + e^{7y}}$
- $\lim_{x \rightarrow 0} \frac{2 \cos(4x) - 4x^2 - 2}{\sin(2x) - x^2 - 2x}$
- $\lim_{x \rightarrow -3} \frac{3e^{2x+6} + x^2 - 12}{x^3 + 6x^2 + 9x}$
- $\lim_{z \rightarrow 6} \frac{\sin(\pi z)}{\ln(z-5)}$
- $\lim_{w \rightarrow \infty} \left[w \ln \left(1 - \frac{2}{3w} \right) \right]$
- $\lim_{t \rightarrow 0^+} [\ln(t) \sin(t)]$
- $\lim_{z \rightarrow -\infty} z^2 e^z$
- $\lim_{x \rightarrow -\infty} \frac{3x + 5}{\sqrt{9 + x^2}}$

15. $\lim_{x \rightarrow \infty} \left[x \sin \left(\frac{7}{x} \right) \right]$

18. $\lim_{x \rightarrow \infty} (\ln(x) - \sqrt{x})$

16. $\lim_{z \rightarrow 0^+} [z^2 (\ln z)^2]$

19. $\lim_{t \rightarrow 0^+} [e^t + t]^{1/t}$

17. $\lim_{x \rightarrow 0^+} x^{1/x}$

20. $\lim_{x \rightarrow \infty} [e^{-2x} + 3x]^{1/dx}$

21. Suppose that we know that $f'(x)$ is a continuous function. Use L'Hôpital's Rule to show that,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x).$$

22. Suppose that we know that $f''(x)$ is a continuous function. Use L'Hôpital's Rule to show that,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

23. Suppose you that $f(10) = -9$, $f'(10) = -8$, and $f'(x)$ is continuous for all x . Find the following.

a. $\lim_{x \rightarrow 10} \frac{f(x)+9}{x^2-100}$

b. $\lim_{x \rightarrow 10} \frac{\ln(f(x)+10)}{x-10}$

24. Suppose you that $f(-6) = 7$, $f'(-6) = -8$, and $f'(x)$ is continuous for all x . Find the following.

a. $\lim_{x \rightarrow -6} \frac{\ln(f(x)-6)}{[f(x)]^2-49}$

Note: The following content is adapted from OpenStax and is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

Access for free at <https://openstax.org/books/calculus-volume-1/pages/1-introduction>.

Evaluate the following limits.

25. $\lim_{x \rightarrow 0} \frac{\sin(x) - \tan(x)}{x^3}$

28. $\lim_{x \rightarrow 0^+} x \ln(x^4)$

26. $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$

29. $\lim_{x \rightarrow 0} \frac{3^x - 2^x}{x}$

27. $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

30. $\lim_{x \rightarrow \infty} x e^{1/x}$

31. $\lim_{x \rightarrow 0^+} x^{1/\cos(x)}$

Section 3.8 Solutions

1. It may be tempting to jump right in and use L'Hôpital's Rule, but we need to verify that L'Hôpital's Rule **can** be used; that is, the limit is either in the form of $0/0$ or $\pm\infty/\infty$. A quick check with direct substitution shows us that,

$$\text{as } x \rightarrow 2 \quad \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} \rightarrow \frac{0}{0}$$

and so this is a form that allows the use of L'Hôpital's Rule. So let's apply L'Hôpital's Rule.

$$\lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x \rightarrow 2} \frac{3x^2 - 14x + 10}{2x + 1}$$

At this point all we need to do is try the new limit (with direct substitution) and see if it can be done.

$$\lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x \rightarrow 2} \frac{3x^2 - 14x + 10}{2x + 1} = \frac{-6}{5}$$

So, the limit can be done, and we done with the problem! The limit is then,

$$\lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = -\frac{6}{5}.$$

Note that we could have just used our techniques from the limits chapter to determine this particular limit. The numerator and denominator of the original expression can be factored and a common factor from the numerator and denominator can cancel. That process looks like:

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} &= \lim_{x \rightarrow 2} \frac{x(x-5)(x-2)}{(x+3)(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{x(x-5)}{x+3} \\ &= \frac{2(2-5)}{2+3} \\ &= -\frac{6}{5}. \end{aligned}$$

We will not redo every problem here; this is only a remind that we don't always **have** to use L'Hôpital's Rule even when it applies and is helpful. L'Hôpital's Rule is simply another tool we can use for determining certain limits.

2. Let's begin by direct substitution. A quick check shows us that,

$$\text{as } w \rightarrow -4 \quad \frac{\sin(\pi w)}{w^2 - 16} \rightarrow \frac{0}{0}$$

and so this is a form that allows the use of L'Hôpital's Rule. Let's apply L'Hôpital's Rule.

$$\lim_{w \rightarrow -4} \frac{\sin(\pi w)}{w^2 - 16} = \lim_{w \rightarrow -4} \frac{\pi \cos(\pi w)}{2w}$$

Now let's try direct substitution on this new limit.

$$\lim_{w \rightarrow -4} \frac{\sin(\pi w)}{w^2 - 16} = \lim_{w \rightarrow -4} \frac{\pi \cos(\pi w)}{2w} = \frac{\pi \cos(-4\pi)}{-8} = \frac{\pi}{-8}$$

So, the limit can be done, and we done with the problem! The limit is then,

$$\lim_{w \rightarrow -4} \frac{\sin(\pi w)}{w^2 - 16} = -\frac{\pi}{8}.$$

3. Let's begin by directly seeing what happens as $t \rightarrow \infty$. A quick check shows us that,

$$\text{as } t \rightarrow \infty \quad \frac{\ln(3t)}{t^2} \rightarrow \frac{\infty}{\infty}$$

and so this is a form that allows the use of L'Hôpital's Rule. Let's apply L'Hôpital's Rule.

$$\lim_{t \rightarrow \infty} \frac{\ln(3t)}{t^2} = \lim_{t \rightarrow \infty} \frac{\frac{1}{t}}{2t} = \lim_{t \rightarrow \infty} \frac{1}{2t^2}$$

Don't forget to simplify after taking the derivatives. This can often be the difference between being able to do the problem or not. Now let's investigation what happens with this new limit as $t \rightarrow \infty$.

$$\lim_{t \rightarrow \infty} \frac{\ln(3t)}{t^2} = \lim_{t \rightarrow \infty} \frac{1}{2t^2} = 0$$

So, the limit can be done, and we done with the problem! The limit is then,

$$\lim_{t \rightarrow \infty} \frac{\ln(3t)}{t^2} = 0.$$

4. Let's begin by direct substitution. A quick check shows us that,

$$\text{as } z \rightarrow 0 \quad \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2} \rightarrow \frac{0}{0}$$

and so this is a form that allows the use of L'Hôpital's Rule. Let's apply L'Hôpital's Rule. Before actually using L'Hôpital's Rule it might be better if we multiply out the denominator to make the derivative (and later steps a little easier). Doing this gives,

$$\lim_{z \rightarrow 0} \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2} = \lim_{z \rightarrow 0} \frac{\sin(2z) + 7z^2 - 2z}{z^4 + 2z^3 + z^2}.$$

Now let's apply L'Hôpital's Rule.

$$\lim_{z \rightarrow 0} \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2} = \lim_{z \rightarrow 0} \frac{2\cos(2z) + 14z - 2}{4z^3 + 6z^2 + 2z}$$

Now let's try direct substitution on this new limit. Here, we can see that

$$\text{as } z \rightarrow 0 \quad \frac{2\cos(2z) + 14z - 2}{4z^3 + 6z^2 + 2z} \rightarrow \frac{0}{0}.$$

So, using L'Hôpital's Rule doesn't give us a limit that we can do. However, the new limit is one that can use L'Hôpital's Rule on so let's do that.

$$\lim_{z \rightarrow 0} \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2} = \lim_{z \rightarrow 0} \frac{-4\sin(2z) + 14}{12z^2 + 12z + 2} = \frac{14}{2}$$

Okay, the second application of L'Hôpital's Rule gives us a limit we can do and so the value of this limit is,

$$\lim_{z \rightarrow 0} \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2} = 7.$$

5. Let's begin by directly seeing what happens as $x \rightarrow -\infty$. A quick check shows us that,

$$\text{as } x \rightarrow -\infty \quad \frac{x^2}{e^{1-x}} \rightarrow \frac{\infty}{\infty}$$

and so this is a form that allows the use of L'Hôpital's Rule. Let's apply L'Hôpital's Rule.

$$\lim_{x \rightarrow -\infty} \frac{x^2}{e^{1-x}} = \lim_{x \rightarrow -\infty} \frac{2x}{-e^{1-x}}$$

At this point let's try the limit and see if it can be done. However, in this case, we can see that,

$$\text{as } x \rightarrow -\infty \quad \frac{2x}{-e^{1-x}} \rightarrow \frac{-\infty}{-\infty}.$$

So, using L'Hôpital's Rule doesn't give us a limit that we can do. However, the new limit is one that can use L'Hôpital's Rule on so let's do that.

$$\lim_{x \rightarrow -\infty} \frac{x^2}{e^{1-x}} = \lim_{x \rightarrow -\infty} \frac{2x}{-e^{1-x}} = \lim_{x \rightarrow -\infty} \frac{2}{e^{1-x}} = 0$$

The second application of L'Hôpital's Rule gives us a limit we can do and so the value of the limit is,

$$\lim_{x \rightarrow -\infty} \frac{x^2}{e^{1-x}} = 0.$$

6. Let's begin by directly seeing what happens as $z \rightarrow \infty$. A quick check shows us that,

$$\text{as } z \rightarrow \infty \quad \frac{z^2 + e^{4z}}{2z - e^z} \rightarrow \frac{\infty}{-\infty}$$

and so this is a form that allows the use of L'Hôpital's Rule. Let's apply L'Hôpital's Rule.

$$\lim_{z \rightarrow \infty} \frac{z^2 + e^{4z}}{2z - e^z} = \lim_{z \rightarrow \infty} \frac{2z + 4e^{4z}}{2 - e^z}$$

At this point let's try the limit and see if it can be done. However, in this case, we can see that,

$$\text{as } z \rightarrow \infty \quad \frac{2z + 4e^{4z}}{2 - e^z} \rightarrow \frac{\infty}{-\infty}.$$

So, using L'Hôpital's Rule doesn't give us a limit that we can do. However, the new limit is one that can use L'Hôpital's Rule on so let's do that.

$$\lim_{z \rightarrow \infty} \frac{z^2 + e^{4z}}{2z - e^z} = \lim_{z \rightarrow \infty} \frac{2z + 4e^{4z}}{2 - e^z} = \lim_{z \rightarrow \infty} \frac{2 + 16e^{4z}}{-e^z}$$

Now let's try the limit and see if it can be done. However, again we can see that,

$$\text{as } z \rightarrow \infty \quad \frac{2z + 4e^{4z}}{2 - e^z} \rightarrow \frac{\infty}{-\infty}.$$

We might be thinking that we'll be stuck in a loop forever, but notice that these limits we get after applying L'Hôpital's Rule are getting progressively simpler as expressions. Let's apply L'Hôpital's Rule again, since it is of a correct form to do so, and we'll be sure to do any simplifying of the expression if possible.

$$\lim_{z \rightarrow \infty} \frac{z^2 + e^{4z}}{2z - e^z} = \lim_{z \rightarrow \infty} \frac{2z + 4e^{4z}}{2 - e^z} = \lim_{z \rightarrow \infty} \frac{2 + 16e^{4z}}{-e^z} = \lim_{z \rightarrow \infty} \frac{64e^{4z}}{-e^z} = \lim_{z \rightarrow \infty} -64e^{3z}$$

Now investigate what happens as $z \rightarrow \infty$ with this new limit.

$$\lim_{z \rightarrow \infty} \frac{z^2 + e^{4z}}{2z - e^z} = \lim_{z \rightarrow \infty} -64e^{3z} = -\infty$$

While $-\infty$ is not a number, and so this limit does not exist, we still will say,

$$\lim_{z \rightarrow \infty} \frac{z^2 + e^{4z}}{2z - e^z} = -\infty.$$

7. Let's begin by directly seeing what happens as $t \rightarrow \infty$. A quick check shows us that,

$$\text{as } t \rightarrow \infty \quad t \ln \left(1 + \frac{3}{t} \right) \rightarrow (\infty)(0)$$

which is an indeterminate form, but not one that allows us to use L'Hôpital's Rule. As discussed in the notes for this section we can always turn this kind of indeterminate form into a rational expression that will allow L'Hôpital's Rule to be applied.

The real question is do we move the first term or the second term to the denominator. From the looks of things, it appears that it would be best to move the first term to the denominator.

$$\lim_{t \rightarrow \infty} \left[t \ln \left(1 + \frac{3}{t} \right) \right] = \lim_{t \rightarrow \infty} \frac{\ln \left(1 + \frac{3}{t} \right)}{1/t}$$

Notice as well that,

$$\text{as } t \rightarrow \infty \quad \frac{\ln \left(1 + \frac{3}{t} \right)}{1/t} \rightarrow \frac{0}{0}$$

and we can use L'Hôpital's Rule on this. Applying L'Hôpital's Rule gives,

$$\lim_{t \rightarrow \infty} \left[t \ln \left(1 + \frac{3}{t} \right) \right] = \lim_{t \rightarrow \infty} \frac{\ln \left(1 + \frac{3}{t} \right)}{1/t} = \lim_{t \rightarrow \infty} \frac{\frac{-3/t^2}{1+3/t}}{-1/t^2}.$$

Can you see why we chose to move the t to the denominator? Moving the logarithm would have left us with a very messy derivative to take! It might have ended up working okay for us, but the work would be greatly increased.

We can't forget to simplify the expression now, as it will greatly help us determine the limit.

$$\frac{\frac{-3/t^2}{1+3/t}}{-1/t^2} = \frac{3}{1+3/t}$$

Trying the limit now, we have

$$\lim_{t \rightarrow \infty} \left[t \ln \left(1 + \frac{3}{t} \right) \right] = \lim_{t \rightarrow \infty} \frac{\ln \left(1 + \frac{3}{t} \right)}{1/t} = \lim_{t \rightarrow \infty} \frac{3}{1+3/t} = 3$$

and so we are done!

8. Let's begin by direct substitution. A quick check shows us that,

$$\text{as } w \rightarrow 0^+ \quad w^2 \ln(4w^2) \rightarrow (0)(-\infty)$$

which is an indeterminate form, but not one that allows us to use L'Hôpital's Rule. As discussed in the notes for this section we can always turn this kind of indeterminate form into a rational expression that will allow L'Hôpital's Rule to be applied.

The real question is do we move the first term or the second term to the denominator. From the looks of things, it appears that it would be best to move the first term to the denominator.

$$\lim_{w \rightarrow 0^+} [w^2 \ln(4w^2)] = \lim_{w \rightarrow 0^+} \frac{\ln(4w^2)}{1/w^2}$$

Notice as well that,

$$\text{as } w \rightarrow 0^+ \quad \frac{\ln(4w^2)}{1/w^2} \rightarrow \frac{-\infty}{\infty}$$

and we can use L'Hôpital's Rule on this. Applying L'Hôpital's Rule gives,

$$\lim_{w \rightarrow 0^+} [w^2 \ln(4w^2)] = \lim_{w \rightarrow 0^+} \frac{\ln(4w^2)}{1/w^2} = \lim_{w \rightarrow 0^+} \frac{2/w}{-2/w^3}.$$

Can you see why we chose to move the t to the denominator? Moving the logarithm would have left us with a very messy derivative to take! It might have ended up working okay for us, but the work would be greatly increased.

We can't forget to simplify the expression now, as it will greatly help us determine the limit.

$$\lim_{w \rightarrow 0^+} [w^2 \ln(4w^2)] = \lim_{w \rightarrow 0^+} \frac{\ln(4w^2)}{1/w^2} = \lim_{w \rightarrow 0^+} \frac{2/w}{-2/w^3} = \lim_{w \rightarrow 0^+} (-w^2) = 0$$

9. Let's begin by direct substitution. A quick check shows us that,

$$\text{as } x \rightarrow 1^+ \quad (x-1) \tan\left(\frac{\pi}{2}x\right) \rightarrow (0)(-\infty)$$

which is an indeterminate form, but not one that allows us to use L'Hôpital's Rule. As discussed in the notes for this section we can always turn this kind of indeterminate form into a rational expression that will allow L'Hôpital's Rule to be applied.

The real question is do we move the first term or the second term to the denominator. At first glance it might appear that neither term will be particularly useful in the denominator. In particular, if we move the tangent to the denominator we would end up needing to differentiate a term in the form $1/\tan(\theta)$. That doesn't look to be all that fun to differentiate and we're liable to end up with a mess when we are done.

However, that is exactly the term we are going to move to the denominator for reasons that will quickly become apparent.

$$\lim_{x \rightarrow 1^+} \left[(x-1) \tan\left(\frac{\pi}{2}x\right) \right] = \lim_{x \rightarrow 1^+} \frac{x-1}{1/\tan(\frac{\pi}{2}x)} = \lim_{x \rightarrow 1^+} \frac{x-1}{\cot(\frac{\pi}{2}x)}$$

With a little simplification after moving the tangent to the denominator we ended up with something that doesn't look all that bad. We'll also see that the remainder of this problem is going to be quite simple.

Before we proceed however we should notice as well that,

$$\text{as } x \rightarrow 1^+ \quad \frac{x-1}{\cot(\frac{\pi}{2}x)} \rightarrow \frac{0}{0}$$

and we can use L'Hôpital's Rule on this. Applying L'Hôpital's Rule gives,

$$\lim_{x \rightarrow 1^+} \left[(x-1) \tan \left(\frac{\pi}{2} x \right) \right] = \lim_{x \rightarrow 1^+} \frac{x-1}{\cot \left(\frac{\pi}{2} x \right)} = \lim_{x \rightarrow 1^+} \frac{1}{-\frac{\pi}{2} \csc^2 \left(\frac{\pi}{2} x \right)} = -\frac{2}{\pi}$$

and we are done!

10. Let's begin by direct substitution. A quick check shows us that,

$$\text{as } y \rightarrow 0^+ \quad [\cos(2y)]^{1/y^2} \rightarrow 1^\infty$$

which is an indeterminate form, but not one that allows us to use L'Hôpital's Rule. As discussed in the notes for this section we can do some manipulation on this to turn it into a problem that can be done with L'Hôpital's Rule.

First, let's define,

$$z = [\cos(2y)]^{1/y^2}$$

and take the natural logarithm of both sides. We'll also do a little simplification.

$$\ln z = \ln \left([\cos(2y)]^{1/y^2} \right) = \frac{1}{y^2} \ln [\cos(2y)] = \frac{\ln [\cos(2y)]}{y^2}$$

We can now take the limit as $y \rightarrow 0^+$ of this.

$$\lim_{y \rightarrow 0^+} [\ln z] = \lim_{y \rightarrow 0^+} \left[\frac{\ln [\cos(2y)]}{y^2} \right]$$

Before we proceed let's notice that we have the following,

$$\text{as } y \rightarrow 0^+ \quad \frac{\ln [\cos(2y)]}{y^2} \rightarrow \frac{\ln(1)}{0} = \frac{0}{0}$$

and we can use L'Hôpital's Rule on this. Applying L'Hôpital's Rule gives,

$$\lim_{y \rightarrow 0^+} [\ln z] = \lim_{y \rightarrow 0^+} \left[\frac{\ln [\cos(2y)]}{y^2} \right] = \lim_{y \rightarrow 0^+} \frac{-2 \sin(2y) / \cos(2y)}{2y} = \lim_{y \rightarrow 0^+} \frac{-\tan(2y)}{y}.$$

We now have a limit that behaves like,

$$\text{as } y \rightarrow 0^+ \quad \frac{-\tan(2y)}{y} \rightarrow \frac{0}{0}$$

and so we can use L'Hôpital's Rule on this as well. Doing this gives,

$$\lim_{y \rightarrow 0^+} [\ln z] = \lim_{y \rightarrow 0^+} \frac{-\tan(2y)}{y} = \lim_{y \rightarrow 0^+} \frac{-2\sec^2(2y)}{1} = -2.$$

Now all we need to do is recall that,

$$z = e^{\ln(z)}.$$

This in turn means that we can do the original limit as follows,

$$\lim_{y \rightarrow 0^+} [\cos(2y)]^{1/y^2} = \lim_{y \rightarrow 0^+} z = \lim_{y \rightarrow 0^+} e^{\ln z} = e^{\lim_{y \rightarrow 0^+} [\ln z]} = e^{-2}$$

and we are done!

11. Let's begin by seeing what happens as $x \rightarrow \infty$. A quick check shows us that,

$$\text{as } x \rightarrow \infty \quad [e^x + x]^{1/x} \rightarrow \infty^0$$

which is an indeterminate form, but not one that allows us to use L'Hôpital's Rule. As discussed in the notes for this section we can do some manipulation on this to turn it into a problem that can be done with L'Hôpital's Rule.

First, let's define,

$$z = [e^x + x]^{1/x}$$

and take the natural logarithm of both sides. We'll also do a little simplification.

$$\ln z = \ln \left([e^x + x]^{1/x} \right) = \frac{1}{x} \ln [e^x + x] = \frac{\ln [e^x + x]}{x}$$

We can now take the limit as $x \rightarrow \infty$ of this.

$$\lim_{x \rightarrow \infty} [\ln z] = \lim_{x \rightarrow \infty} \left[\frac{\ln [e^x + x]}{x} \right]$$

Before we proceed let's notice that we have the following,

$$\text{as } x \rightarrow \infty \quad \frac{\ln [e^x + x]}{x} \rightarrow \frac{\infty}{\infty}$$

and we can use L'Hôpital's Rule on this. Applying L'Hôpital's Rule gives,

$$\lim_{x \rightarrow \infty} [\ln z] = \lim_{x \rightarrow \infty} \left[\frac{\ln [e^x + x]}{x} \right] = \lim_{x \rightarrow \infty} \frac{e^x + 1 / e^x + x}{1} = \lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x + x}.$$

We now have a limit that behaves like,

$$\text{as } x \rightarrow \infty \quad \frac{e^x + 1}{e^x + x} \rightarrow \frac{\infty}{\infty}$$

and so we can use L'Hôpital's Rule on this as well. Doing this gives,

$$\lim_{x \rightarrow \infty} [\ln z] = \lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x + x} = \lim_{x \rightarrow \infty} \frac{e^x}{e^x + 1} = \lim_{x \rightarrow \infty} \frac{e^x}{e^x} = \lim_{x \rightarrow \infty} (1) = 1$$

Notice that we did have to use L'Hôpital's Rule twice here and we also made sure to do some simplification so we could actually take the limit.

Now all we need to do is recall that,

$$z = e^{\ln z}$$

This in turn means that we can do the original limit as follows,

$$\lim_{x \rightarrow \infty} [e^x + x]^{1/x} = \lim_{x \rightarrow \infty} z = \lim_{x \rightarrow \infty} e^{\ln z} = e^{\lim_{x \rightarrow \infty} [\ln z]} = e$$

and we are done!

12. Let's begin by seeing what happens as $x \rightarrow \infty$. A quick check shows us that,

$$\lim_{x \rightarrow \infty} \frac{2x + \sqrt{x^2 - 7}}{4 - 5x} \rightarrow \frac{\infty}{-\infty}$$

which is an indeterminate form that allows us to use L'Hôpital's Rule. So let's do that. Keep in mind we'll have to use the chain rule for the square root in the numerator. We'll also be sure to simplify our expression after applying L'Hôpital's Rule.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x + \sqrt{x^2 - 7}}{4 - 5x} &= \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{2}(x^2 - 7)^{-1/2}(2x)}{-5} && \text{apply L'Hôpital's Rule} \\ &= \lim_{x \rightarrow \infty} \frac{2\sqrt{x^2 - 7} + x}{-5\sqrt{x^2 - 7}} && \text{simplify}\end{aligned}$$

A quick check shows that this new limit also goes to $\frac{\infty}{-\infty}$ as $x \rightarrow \infty$. Ok, no problem. We've seen before that we may need to use L'Hôpital's Rule more than once (as long as we still meet the conditions to do so). So let's apply L'Hôpital's Rule again, as well as simplifying the expression.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x + \sqrt{x^2 - 7}}{4 - 5x} &= \lim_{x \rightarrow \infty} \frac{2\sqrt{x^2 - 7} + x}{-5\sqrt{x^2 - 7}} && \text{from before} \\ &= \lim_{x \rightarrow \infty} \frac{2 \cdot \frac{1}{2}(x^2 - 7)^{-1/2}(2x) + 1}{-5 \cdot \frac{1}{2}(x^2 - 7)^{-1/2}(2x)} && \text{apply L'Hôpital's Rule} \\ &= \lim_{x \rightarrow \infty} \frac{2x + \sqrt{x^2 - 7}}{-5x} && \text{simplify}\end{aligned}$$

While this new limit would also yield $\frac{\infty}{-\infty}$, the bigger concern is that this is practically the same limit we started out with. What this tells us is that applying L'Hôpital's Rule (even though we're "allowed" to), is not going to be helpful. We'll just be stuck in a cycle if we tried to repeatedly apply L'Hôpital's Rule.

In cases like these, we need to go back to our algebraic methods from before. Here, we can factor out the largest power of x and see if the resulting limit is easier to compute.

We'll want to start with factoring on the inside of the root, and then continue with factoring from the numerator and denominator.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x + \sqrt{x^2 - 7}}{4 - 5x} &= \lim_{x \rightarrow \infty} \frac{2x + \sqrt{x^2 \left(1 - \frac{7}{x^2}\right)}}{4 - 5x} && \text{factor } x^2 \text{ on inside of root} \\ &= \lim_{x \rightarrow \infty} \frac{2x + x\sqrt{1 - \frac{7}{x^2}}}{4 - 5x} && \sqrt{x^2} = x \text{ since } x \rightarrow \infty \\ &= \lim_{x \rightarrow \infty} \frac{x \left(2 + \sqrt{1 - \frac{7}{x^2}}\right)}{x \left(\frac{4}{x} - 5\right)} && \text{factor } x \text{ from numerator and denominator} \\ &= \lim_{x \rightarrow \infty} \frac{2 + \sqrt{1 - \frac{7}{x^2}}}{\frac{4}{x} - 5} && \text{cancel common factor}\end{aligned}$$

$$= \frac{2 + \sqrt{1-0}}{0-5}$$

$$= -\frac{2}{5}$$

apply limit: $\frac{7}{x^2} \rightarrow 0$ and $\frac{4}{x} \rightarrow 0$ as $x \rightarrow \infty$

simplify

And so the value of our limit is $-\frac{2}{5}$. This problem is a reminder that we do not **have** to use L'Hôpital's Rule when our algebraic methods will work. And in this situation, L'Hôpital's Rule didn't help anyways.

13. Let's begin by seeing what happens as $x \rightarrow \infty$. A quick check shows us that,

$$\lim_{x \rightarrow \infty} (\ln(x) - x^2) \rightarrow \infty - \infty$$

which is an indeterminate form, but not one that allows us to use L'Hôpital's Rule. So we'll have to do some algebraic manipulation to rewrite it. As we had done with polynomials, let's factor out x^2 from the entire expression.

$$\lim_{x \rightarrow \infty} (\ln(x) - x^2) = \lim_{x \rightarrow \infty} x^2 \left(\frac{\ln(x)}{x^2} - 1 \right)$$

Let's just look at the expression $\frac{\ln(x)}{x^2}$ for a moment. As $x \rightarrow \infty$, $\frac{\ln(x)}{x^2} \rightarrow \frac{\infty}{\infty}$. So we can apply L'Hôpital's Rule to this particular limit. Let's do this.

$$\lim_{x \rightarrow \infty} \frac{\ln(x)}{x^2} = \lim_{x \rightarrow \infty} \frac{1/x}{2x}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{2x^2}$$

$$= 0$$

apply L'Hôpital's Rule

simplify

apply limit

Keep in mind that this was not the limit we were asked about, but only part of an overall limit. But we can use what we found here to help determine the overall limit.

$$\lim_{x \rightarrow \infty} x^2 \left(\frac{\ln(x)}{x^2} - 1 \right) = \left(\lim_{x \rightarrow \infty} x^2 \right) \left(\lim_{x \rightarrow \infty} \frac{\ln(x)}{x^2} - \lim_{x \rightarrow \infty} 1 \right) \quad \text{properties of limits}$$

$$= \infty(0 - 1) \quad \text{apply limit; use what we found above}$$

$$= -\infty \quad \text{simplify}$$

So we see the overall limit tends to $-\infty$ (which does not exist) as $x \rightarrow \infty$.

14. Let's begin by seeing what happens as $x \rightarrow \infty$. A quick check shows us that,

$$\lim_{x \rightarrow \infty} (e^x - x^3) \rightarrow \infty - \infty$$

which is an indeterminate form, but not one that allows us to use L'Hôpital's Rule. So we'll have to do some algebraic manipulation to rewrite it. As we had done with polynomials, let's factor out x^3 from the entire expression.

$$\lim_{x \rightarrow \infty} (e^x - x^3) = \lim_{x \rightarrow \infty} x^3 \left(\frac{e^x}{x^3} - 1 \right)$$

Let's just look at the expression $\frac{e^x}{x^3}$ for a moment. As $x \rightarrow \infty$, $\frac{e^x}{x^3} \rightarrow \frac{\infty}{\infty}$. So we can apply L'Hôpital's Rule to this particular limit. Let's do this.

$\lim_{x \rightarrow \infty} \frac{e^x}{x^3} = \lim_{x \rightarrow \infty} \frac{e^x}{3x^2}$	apply L'Hôpital's Rule; still goes to $\frac{\infty}{\infty}$
$= \lim_{x \rightarrow \infty} \frac{e^x}{6x}$	apply L'Hôpital's Rule again; still goes to $\frac{\infty}{\infty}$
$= \lim_{x \rightarrow \infty} \frac{e^x}{6}$	apply L'Hôpital's Rule again
$= \frac{\infty}{6}$	apply limit
$= \infty$	simplify

Keep in mind that this was not the limit we were asked about, but only part of an overall limit. But we can use what we found here to help determine the overall limit.

$\lim_{x \rightarrow \infty} x^3 \left(\frac{e^x}{x^3} - 1 \right) = \left(\lim_{x \rightarrow \infty} x^3 \right) \left(\lim_{x \rightarrow \infty} \frac{e^x}{x^3} - \lim_{x \rightarrow \infty} 1 \right)$	properties of limits
$= \infty(\infty - 1)$	apply limit; use what we found above
$= \infty$	simplify

So we see the overall limit tends to ∞ (which does not exist) as $x \rightarrow \infty$.