

§2.3 Differentiation Formulas

Introduction

In the first section of this chapter, we saw the definition of the derivative, and we used it to compute several derivatives. As we saw in those examples, there was a fair amount of work involved in computing the necessary limits, even though the functions that we worked with were not terribly complicated.

For more intricate functions, using the definition of the derivative would be a very difficult task. Luckily for us, we won't have to use the definition very often. Although we will still need to use it on occasion, there is a large collection of formulas and properties that we can use to find derivatives more easily, and that will allow us to avoid using the definition whenever possible.

We will introduce most of these formulas over the course of the next several sections. We will start in this section with some of the basic properties and rules. These rules will allow us to find derivatives of polynomials and some other algebraic functions quickly, and to find the derivatives of sine and cosine.

Properties of Derivatives

The two properties we give below allow us to break functions into smaller pieces before tackling the derivative.

Definition 1: Sum/Difference Rule

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

OR

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$$

In other words, to differentiate a sum or difference, all we need to do is differentiate the individual terms and then put them back together with the appropriate signs. Note that this property is not limited to just two functions.

Proof:

The proof uses the definition of the derivative and some algebra. We'll start with the sum of two functions. First, put the sum of the functions into the definition of the derivative and rewrite the numerator a little:

$$\begin{aligned}(f(x) + g(x))' &= \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}\end{aligned}$$

Now break the expression into two fractions and recall that the limit of the sum is the sum of the limits. Using this fact, we see that we end up with the definition of the derivative for each of the two functions.

$$\begin{aligned}
 (f(x) + g(x))' &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= f'(x) + g'(x)
 \end{aligned}$$

The argument for the difference of two functions is nearly identical, so we will give it here without any explanation.

$$\begin{aligned}
 (f(x) - g(x))' &= \lim_{h \rightarrow 0} \frac{f(x+h) - g(x+h) - (f(x) - g(x))}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - (g(x+h) - g(x))}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= f'(x) - g'(x)
 \end{aligned}$$

Therefore, $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.

Definition 2: Constant Multiple Rule

$$(cf(x))' = cf'(x)$$

OR

$$\frac{d}{dx}(cf(x)) = c \frac{df}{dx},$$

where c is any number.

In other words, we can "factor" a constant multiplier out of a derivative if we need to.

Proof:

The proof uses the definition of the derivative and the fact that we can factor a constant multiple out of a limit.

$$(cf(x))' = \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} = c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = cf'(x)$$

Note that we have not included formulas for the derivative of compositions, products or quotients of functions here. We will take a look at these in the next two sections.

Basic Derivative Rules

The two rules below, when used with the properties we just saw, will allow us to compute quite a few derivatives.

Definition 3: Derivative of a Constant

$$\begin{aligned} \text{If } f(x) = c, \text{ then } f'(x) &= 0 \\ \text{OR} \\ \frac{d}{dx}(c) &= 0. \end{aligned}$$

This property says that the derivative of any constant is zero.

Proof: Let c be any number, define $f(x) = c$, and use the definition of the derivative.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0$$

Definition 4: Power Rule

$$\begin{aligned} \text{If } f(x) = x^n, \text{ then } f'(x) &= nx^{n-1} \\ \text{OR} \\ \frac{d}{dx}(x^n) &= nx^{n-1}, \end{aligned}$$

where n is any number.

All we are doing here is bringing the original exponent down to the front, multiplying by any constant that is already there, and then subtracting one from the original exponent on x .

Note that in order to use this rule, n must be a number, it cannot be a variable. Also note that the base, x , must be a variable, it cannot be a number. It will be tempting later to misuse the Power Rule when we run into some functions where the exponent isn't a number and/or the base isn't a variable. By then, though, we will have learned new rules for handling these kinds of functions.

Proof:

In the proof we give here, we need to assume that n is a positive integer. The full proof of the Power Rule requires us to have read more of the Derivatives chapter, and so we will see that later in the chapter.

In this proof, we use the definition of the derivative and the Binomial Theorem. The Binomial Theorem tells us that

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

are called the binomial coefficients and $n! = n(n-1)(n-2)\dots(2)(1)$ is the factorial.

Expanding the sum and using the definition of $\binom{n}{k}$, we can write

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots + nab^{n-1} + b^n\end{aligned}$$

Now we can give the details of the proof. First, substitute $f(x) = x^n$ into the definition of the derivative and use the Binomial Theorem to expand the first term (here "a" is x and "b" is h):

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n\right) - x^n}{h}\end{aligned}$$

Notice that we can subtract the two x^n terms, and that leaves every term in the numerator with an h in it. The h can then be factored out and divided with the h in the denominator. At this point, we can evaluate the limit.

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h \rightarrow 0} nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \dots + nxh^{n-2} + h^{n-1} \\ &= nx^{n-1}\end{aligned}$$

Therefore, for any positive integer n , if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Examples using the basic derivative rules

The properties we just learned allow us to find the derivative of a lot of different functions. Let's take a look at some of them.

Example 2.3.1

Differentiate each of the following functions, using the properties of derivatives.

a. $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

b. $g(t) = 2t^6 + 7t^{-6}$

c. $y = 8z^3 - \frac{1}{3z^5} + z - 23$

d. $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \frac{2}{\sqrt[5]{x^2}}$

e. $h(x) = x^\pi - x^{\sqrt{2}}$

Solutions:

- a. In this example, we have the sum or difference of four terms. Using the first property above, we will differentiate each of the terms and put them back together again with the proper sign. Also, for each term with a multiplicative constant, remember that all we need to do is "factor" out the constant using the second property, then take the derivative using the power rule. For the last term, since it is a constant, its derivative is zero.

$$\begin{aligned} f'(x) &= 15(100)x^{99} - 3(12)x^{11} + 5(1)x^0 - 0 \\ &= 1500x^{99} - 36x^{11} + 5 \end{aligned}$$

Notice that in the third term, the exponent was one, so upon subtracting one from the original exponent, we are left with a new exponent of zero. Recall that $x^0 = 1$ for any $x \neq 0$.

- b. The point of this problem is to make sure that you deal with negative exponents correctly. Here is the derivative:

$$\begin{aligned} g'(t) &= 2(6)t^5 + 7(-6)t^{-7} \\ &= 12t^5 - 42t^{-7} \end{aligned}$$

Make sure that you deal with the exponents in these cases correctly, especially the negative exponents. It is an easy mistake to "go the other way" when subtracting one from a negative exponent and say the derivative is $-6t^{-5}$ instead of $-6t^{-7}$.

- c. In this function, the second term is not set up correctly for us to use the power rule. The power rule requires that the term be a variable raised to a power only, it cannot be in the denominator of a fraction. So, prior to differentiating, we need to rewrite the second term in a form that we can deal with:

$$y = 8z^3 - \frac{1}{3}z^{-5} + z - 23$$

Note that we left the 3 in the denominator and only moved the variable to the numerator. Remember that the only thing an exponent applies to is what is immediately to its left. If we had wanted the 3 to move up as well, we'd have written

$$\frac{1}{(3z)^5}.$$

So be careful with this! It's a very common mistake to move the 3 into the numerator as well at this stage.

Now that we've gotten the function rewritten into a form that allows us to use the power rule, we can differentiate the function. Here is the derivative:

$$\frac{dy}{dz} = 24z^2 + \frac{5}{3}z^{-6} + 1$$

- d. All of the terms in this function have roots in them. In order to use the power rule, we need to convert all the roots to fractional exponents. Again, remember that the power rule requires us to have a variable raised to an exponent that's a number, and that it cannot be in the denominator of a fraction. Here is the function written in a "proper" form for the power rule:

$$\begin{aligned} T(x) &= x^{\frac{1}{2}} + 9(x^7)^{\frac{1}{3}} - \frac{2}{(x^2)^{\frac{1}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - \frac{2}{x^{\frac{2}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - 2x^{-\frac{2}{5}} \end{aligned}$$

Notice that in the last two terms we combined the exponents. You should always do this with this kind of term. Also, don't forget to move the expression in the denominator of the third term to the numerator. We can now take the derivative:

$$\begin{aligned} T'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + 9\left(\frac{7}{3}\right)x^{\frac{4}{3}} - 2\left(-\frac{2}{5}\right)x^{-\frac{7}{5}} \\ &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{63}{3}x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}} \\ &= \frac{1}{2}x^{-\frac{1}{2}} + 21x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}} \end{aligned}$$

Make sure that you can deal with fractional exponents. You will see a lot of them!

- e. In all of the previous examples, the exponents have been integers or fractions. That is usually what we'll see in this class. However, the exponent only needs to be a number, so don't worry about problems like this one. They work exactly the same way.

$$h'(x) = \pi x^{\pi-1} - \sqrt{2}x^{\sqrt{2}-1}$$

The answer looks a little messy, but this is the correct answer; we don't convert the exponents to decimals. Problems like this are not terribly difficult, they just look that way initially.

There is a general rule about derivatives that you will need to get into the habit of using. When you see a radical expression (like $\sqrt[3]{x^7}$ in part (d) above), you should always convert the radical to a fractional exponent (like $x^{7/3}$) and then simplify exponents as much as possible before taking the derivative. Following this rule will save you a lot of grief in the future.

Pro-tip: Use the exponent property $\sqrt[n]{x^m} = x^{m/n}$ to rewrite radical expressions as exponents before differentiating.

Check Point 2.3.1

Find the derivative of $f(x) = 19x - 4x^{-1} + 3x^{-6} + 6$.

Check Point 2.3.2

Find the derivative of $f(x) = 18\sqrt[6]{x^8} + 17\sqrt[4]{x^{10}} - 2\sqrt{x^5}$.

When we wrote down the properties above we noted that we hadn't included a rule for products or quotients, and that we would see rules for them later. That doesn't mean that we can't differentiate any product or quotient at this point. There are some that we can do without a special rule. We just need to use some algebra first.

Example 2.3.2

Differentiate each of the following functions.

- $y = \sqrt[3]{x^2}(2x - x^2)$
- $h(t) = \frac{2t^5 + t^2 - 5}{t^2}$

Solution:

- In this function we can't just differentiate the $\sqrt[3]{x^2}$, differentiate the $2x - x^2$ and then multiply the two back together. That just won't work. We'll discuss this in detail in a future section, so if you're not sure you believe that it won't work, hold on for a bit. We'll be looking at this soon, and will see an example that shows why it won't work.

It is still possible to take the derivative, though. All we need to do is convert the radical to a fractional exponent (as we should anyway) and then multiply this through the expression in parentheses.

$$y = x^{\frac{2}{3}}(2x - x^2) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

Now we can differentiate the function:

$$\frac{dy}{dx} = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

- Similar to the first part, we cannot just differentiate the numerator and denominator separately then put them back together as a fraction. Again, if you're not sure you believe this, hold on a bit. We'll take a more detailed look at this in a [future section](#).

What we can do is simplify this expression as follows:

$$h(t) = \frac{2t^5}{t^2} + \frac{t^2}{t^2} - \frac{5}{t^2} = 2t^3 + 1 - 5t^{-2}$$

This is now a function that we can differentiate using the power rule.

$$h'(t) = 6t^2 + 10t^{-3}$$

So, as we saw in this example, there are a few products and quotients that we can differentiate. If we can first do some simplification, the functions will sometimes simplify into a form that can be differentiated using the properties and rules in this section.

Check Point 2.3.3

Find the derivative of $f(x) = \sqrt[6]{x^3}(9x^6 + 5x^5 - 4)$

Check Point 2.3.4

Find the derivative of

$$f(x) = \frac{-3x^6 - x^4 - 4}{\sqrt[3]{x^2}}$$

Many of the functions we have differentiated in the last couple of examples are not polynomials; they have exponents that are negative or are fractions. We were able to use the power rule to find their derivatives, and could also keep taking higher order derivatives, although the coefficients and exponents could become very messy.

When the function is a true polynomial, though, there is an interesting pattern in the higher order derivatives.

Example 2.3.3

Find the fifth derivative of the polynomial

$$p(x) = 2x^4 - 5x^3 + x^2 + 7x - 13$$

Solution:

Since the function is a polynomial, we will use the power rule. Notice that each of the derivatives is again a polynomial, so we may use the power rule each time we take a derivative.

The derivatives of $p(x)$ are as follows:

$$p'(x) = 8x^3 - 15x^2 + 2x + 7$$

$$p''(x) = 24x^2 - 30x + 2$$

$$p'''(x) = 48x - 30$$

$$p^{(4)}(x) = 48$$

$$p^{(5)}(x) = 0$$

What we saw in this last example illustrates a general fact about polynomials.

Definition 5

If $p(x)$ is a polynomial of degree n (i.e. the largest exponent in the polynomial is n) then

$$p^{(k)}(x) = 0 \quad \text{for } k \geq n + 1$$

Before moving on to the next section, let's work a few examples to remind us once again of some of the interpretations of the derivative.

Example 2.3.4

Is $f(x) = 2x^3 + \frac{300}{x^3} + 4$ increasing, decreasing or not changing at $x = -2$?

Solution:

We know that the rate of change of a function is given by the derivative, so all we need to do is rewrite the function (to deal with the second term) and then take the derivative.

$$f(x) = 2x^3 + 300x^{-3} + 4 \Rightarrow f'(x) = 6x^2 - 900x^{-4} = 6x^2 - \frac{900}{x^4}$$

Note that we rewrote the last term in the derivative back as a fraction. This is not something we've done up to this point, and is only being done here to help with the evaluation in the next step. It is often easier to evaluate a function with positive exponents.

So, upon evaluating the derivative, we find

$$f'(-2) = 6(4) - \frac{900}{16} = -\frac{129}{4}$$

Since the derivative at $x = -2$ is negative, the function is decreasing at $x = -2$.

Example 2.3.5

Write the equation of the tangent line to the graph of $f(x) = 4x - 8\sqrt{x}$ at $x = 16$.

Solution:

Recall that the equation of a tangent line has the form

$$y = f(a) + f'(a)(x - a)$$

So, we will need the derivative of the function (don't forget to rewrite the radical!):

$$f(x) = 4x - 8x^{\frac{1}{2}} \quad \Rightarrow \quad f'(x) = 4 - 4x^{-\frac{1}{2}} = 4 - \frac{4}{x^{\frac{1}{2}}}$$

Again, we eliminated the negative exponent in the derivative solely for the sake of evaluation. Now we need to evaluate the function and the derivative at the point in question, where $x = 16$.

$$f(16) = 64 - 8(4) = 32 \quad f'(16) = 4 - \frac{4}{4} = 3$$

The tangent line then has the equation

$$y = 32 + 3(x - 16) = 3x - 16$$

Example 2.3.6

The position (in meters) of an object at any time t (in minutes) is given by

$$s(t) = 2t^3 - 21t^2 + 60t - 10$$

- Determine when the object is moving to the right and when the object is moving to the left.
- Find the acceleration of the object at any time t .
- Find the acceleration of the object when $t = 3$ minutes.

Solutions:

- In order to know for certain which direction the object is moving, we need to have the velocity of the object. Recall that if the velocity is positive, then the object is moving to the right, and if the velocity is negative, then the object is moving to the left.

We need the derivative in order to find the velocities of the object at different times. The derivative, and hence the velocity function, is

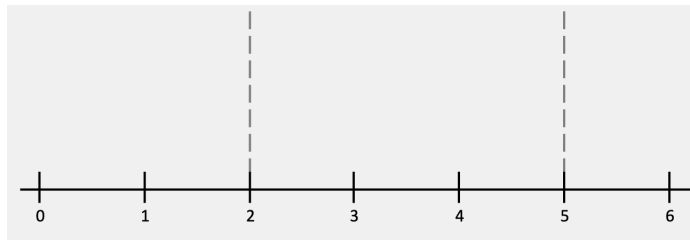
$$s'(t) = 6t^2 - 42t + 60 = 6(t^2 - 7t + 10) = 6(t - 2)(t - 5)$$

The reason for factoring the derivative will be apparent shortly.

We need to determine where the derivative is positive and where it is negative. There are several ways to do this. The method that we tend to prefer is the following.

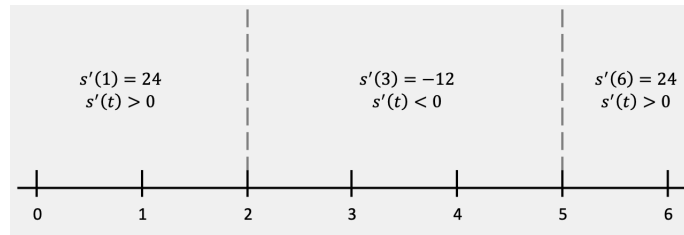
Since polynomials are continuous functions, we know from the Intermediate Value Theorem that if the derivative changes sign, then it must pass through zero. So, if we know where the derivative is zero, then we know the only points at which the derivative *might* change sign.

We can see from the factored form of the derivative that it will be zero at $t = 2$ and $t = 5$. Let's graph these points on a number line:



We can see that these two points divide the number line into three distinct pieces. For x values in each of these pieces, we **know** that the derivative will be the same sign because the derivative can only change sign at the two points that are used to divide the number line into intervals.

Therefore, all we need to do is to check the derivative at a test point in each interval. The derivative will have the same sign at every point in that interval as at the test point. Here is the number line with the test points and results shown:



We now know the intervals in which the derivative is positive and negative:

positive: $-\infty < t < 2$ & $5 < t < \infty$

negative: $2 < t < 5$

We included the negative t values here because we have the information, even though they may not make much sense in the context of this problem. We can now answer the question. The object is moving to the right and to the left in the following intervals:

moving to the right: $-\infty < t < 2$ & $5 < t < \infty$

moving to the left: $2 < t < 5$

- b. In the last section we learned that one interpretation of the second derivative is acceleration, when the function represents the position of an object. So we need to find the second derivative of $s(t)$. We also learned in the last section that the second derivative of a function $s(t)$ is the derivative of $s'(t)$.

$$s''(t) = \frac{d}{dt}(s'(t)) = 12t - 42$$

Thus the acceleration of the object at any time t is given by $12t - 42$.

- c. To find the acceleration when $t = 3$, we just need to substitute $t = 3$ into the acceleration function.

$$s''(3) = 36 - 42 = -6$$

The acceleration of the object at $t = 3$ is -6 meters per minute per minute.

Make sure that you can do the kind of work we just did in the first part of this example. You will be asked numerous times over the course of the next two chapters to determine where functions are positive and/or negative. If you need some review or want to practice these kinds of problems, you should check out the Solving Inequalities section of the Algebra/Trig Review.

Derivatives of Sine and Cosine

Two other fundamental rules of derivatives are those for finding the derivatives of sine and cosine. We will prove those and work some examples here, then you will learn the derivatives of the other four trig functions in a later section.

Before we can prove the rules for the derivatives of sine and cosine, we need to learn a couple of limits involving these functions.

Definition 6: Two Trigonometric Limits

$$1. \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$2. \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$$

A quick note here. Students often ask why we use radians instead of degrees in Calculus. The proof we just did shows the reason: if the angles are in degrees, then the limit involving sine is not 1 and this changes the derivative formulas for sine and cosine. The formulas would pick up an extra constant that would just get in the way of our work, so we use radians to avoid that.

Why radians and not degrees? Basically, convenience.

Before we look at the derivatives of sine and cosine, let's work some limit problems that these two new formulas allow us to do.

Example 2.3.7

Evaluate each of the following limits.

$$a. \lim_{\theta \rightarrow 0} \frac{\sin \theta}{6\theta}$$

$$b. \lim_{x \rightarrow 0} \frac{\sin(6x)}{x}$$

$$c. \lim_{x \rightarrow 0} \frac{\sin(7x)}{\sin(7x)}$$

$$d. \lim_{t \rightarrow 0} \frac{\sin(3t)}{\sin(8t)}$$

$$e. \lim_{x \rightarrow 4} \frac{\sin(x-4)}{x-4}$$

$$f. \lim_{z \rightarrow 0} \frac{\cos(2z) - 1}{z}$$

Solutions:

- a. There really isn't a whole lot to this limit. In fact, it's only here to contrast with the next example so that you can see the difference in how these work. In this case, since there is only a 6 in the denominator, we'll just factor it out and then use the limit fact that we proved above.

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{6\theta} = \frac{1}{6} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{1}{6}(1) = \frac{1}{6}$$

- b. Now, in this case we cannot just factor the 6 out of the argument of the sine function. We'll need to figure out another way to deal with it. To do this, we need to notice that in the limit rule we proved above, $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$, the argument of the sine is identical to the denominator (**i.e.** they are both θ). So we need to make the denominator match the argument of the sine function. We can do this by multiplying the numerator and the denominator by 6 as follows:

$$\lim_{x \rightarrow 0} \frac{\sin(6x)}{x} = \lim_{x \rightarrow 0} \frac{6 \sin(6x)}{6x} = 6 \lim_{x \rightarrow 0} \frac{\sin(6x)}{6x}$$

Note that we factored the 6 in the numerator out of the limit. At this point, while it may not look like it, we can use the limit rule to finish the limit.

To see how, let's do a **change of variables**. A change of variables is really just a renaming of portions of the problem to make them look more like something we know how to deal with. They

can't always be done, but sometimes, such as in this case, they really simplify the problem.

The change of variables here is to let $\theta = 6x$ and then notice that as $x \rightarrow 0$, we also have $\theta \rightarrow 6(0) = 0$. When doing a change of variables in a limit, we need to change all the x 's into θ 's (and that includes the one in the limit).

Doing the change of variables on this limit gives

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(6x)}{x} &= 6 \lim_{x \rightarrow 0} \frac{\sin(6x)}{6x} && \text{let } \theta = 6x \\ &= 6 \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \\ &= 6(1) \\ &= 6\end{aligned}$$

And there we are. Note that we didn't really need to do a change of variables here. All we really need to notice is that the argument of the sine is the same as the denominator, and then we can use the rule. A change of variables, in this case, is really only needed to make it clear that the rule really does work on a problem like this.

- c. We appear to have a problem here, in that the function we're taking the limit of is "upside down" compared to the limit rule. To put things right, we need to see that we can rewrite the fraction as follows:

$$\frac{x}{\sin(7x)} = \frac{1}{\frac{\sin(7x)}{x}}$$

This may look like we've made things worse until we remember a nice property of limits that allows us to distribute the limit to numerator and denominator.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x}{\sin(7x)} &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin(7x)}{x}} \\ &= \frac{\lim_{x \rightarrow 0} (1)}{\lim_{x \rightarrow 0} \frac{\sin(7x)}{x}} \\ &= \frac{1}{\lim_{x \rightarrow 0} \frac{\sin(7x)}{x}}\end{aligned}$$

So we see that we do end up needing to do a limit that uses the rule. This limit is just like the one in the previous problem, but this time we'll do it without writing out the change of variables. All we need to do is multiply the numerator and denominator by 7. Here is the work.

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x}{\sin(7x)} &= \frac{1}{\lim_{x \rightarrow 0} \frac{7 \sin(7x)}{7x}} \\
&= \frac{1}{7 \lim_{x \rightarrow 0} \frac{\sin(7x)}{7x}} \\
&= \frac{1}{(7)(1)} \\
&= \frac{1}{7}
\end{aligned}$$

- d. At first glance, this limit looks nothing like the limit in the rule. However, by doing a little rewriting, it can be thought of as a combination of the limits in the previous two problems. First, we'll split up the fraction as follows:

$$\lim_{t \rightarrow 0} \frac{\sin(3t)}{\sin(8t)} = \lim_{t \rightarrow 0} \frac{\sin(3t)}{1} \frac{1}{\sin(8t)}$$

Now the limit rule requires a t in the denominator of the first fraction and in the numerator of the second. This can be achieved by multiplying the whole thing by $\frac{t}{t}$ (which is just 1 after all, and so won't change the problem) and then rearranging as follows:

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{\sin(3t)}{\sin(8t)} &= \lim_{t \rightarrow 0} \frac{\sin(3t)}{1} \frac{1}{\sin(8t)} \frac{t}{t} \\
&= \lim_{t \rightarrow 0} \frac{\sin(3t)}{t} \frac{t}{\sin(8t)} \\
&= \left(\lim_{t \rightarrow 0} \frac{\sin(3t)}{t} \right) \left(\lim_{t \rightarrow 0} \frac{t}{\sin(8t)} \right)
\end{aligned}$$

At this point we can see that this really is two limits that we've seen before. On the second limit, we observe that it doesn't really matter whether the sine is in the numerator or the denominator. As long as the argument of the sine is the same as what's in the other part of the fraction, the limit is one. So we multiply the first limit by $\frac{3}{3}$ and the second by $\frac{8}{8}$ and proceed as we did previously.

Here is the work for this limit.

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{\sin(3t)}{\sin(8t)} &= \left(\lim_{t \rightarrow 0} \frac{3 \sin(3t)}{3t} \right) \left(\lim_{t \rightarrow 0} \frac{8t}{8 \sin(8t)} \right) \\
&= \left(3 \lim_{t \rightarrow 0} \frac{\sin(3t)}{3t} \right) \left(\frac{1}{8} \lim_{t \rightarrow 0} \frac{8t}{\sin(8t)} \right)
\end{aligned}$$

$$\begin{aligned}
 &= (3)(1) \left(\frac{1}{8} \right) (1) \\
 &= \frac{3}{8}
 \end{aligned}$$

- e. This limit resembles the one in the limit rule in that the argument of the sine is the same as the denominator. However, notice that in the limit, x is approaching 4 rather than 0 as the rule requires. However, with a change of variables, we can rewrite this limit so that it matches the rule.

We let $\theta = x - 4$ and notice that as $x \rightarrow 4$, we have $\theta \rightarrow 0$. Therefore, after carrying out the change of variables, the limit becomes

$$\lim_{x \rightarrow 4} \frac{\sin(x - 4)}{x - 4} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

- f. The previous parts of this example all used the limit rule involving sine. In this problem, though, we want to use the limit rule involving cosine. The procedure is similar to our work in the previous problems, so we will not add much explanation here.

$$\begin{aligned}
 \lim_{z \rightarrow 0} \frac{\cos(2z) - 1}{z} &= \lim_{z \rightarrow 0} \frac{2(\cos(2z) - 1)}{2z} \\
 &= 2 \lim_{z \rightarrow 0} \frac{\cos(2z) - 1}{2z} \\
 &= 2(0) \\
 &= 0
 \end{aligned}$$

All that is required to apply the limit rule is to have the argument of the cosine be identical to what is in the denominator.

Check Point 2.3.5

Evaluate the following limits:

- $\lim_{x \rightarrow 0} \frac{\sin(-5x)}{\sin(-3x)}$
- $\lim_{x \rightarrow 0} \left(\frac{\sin(-4x)}{x} \right)$

Now we can return to our goal of differentiating sine and cosine. The proofs of these rules use the definition of the derivative, a trig identity and the new limit rules from above. We will include the proof of the first rule and leave the second for you to do on your own.

Definition 7

- The derivative of $y = \sin(x)$ is $y' = \cos(x)$ OR $\frac{d}{dx}(\sin(x)) = \cos(x)$.
- The derivative of $y = \cos(x)$ is $y' = -\sin(x)$ OR $\frac{d}{dx}(\cos(x)) = -\sin(x)$.

Proof: Our first step is to substitute the sine function into the definition of the derivative.

$$\frac{d}{dx}(\sin(x)) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Since we cannot just set h to zero to evaluate the limit (why not?), we need to use the sum identity for sines to expand $\sin(x+h)$.

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

Doing this in the difference quotient above gives us

$$\begin{aligned} \frac{d}{dx}(\sin(x)) &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\ &= \lim_{h \rightarrow 0} \sin(x) \frac{\cos(h) - 1}{h} + \lim_{h \rightarrow 0} \cos(x) \frac{\sin(h)}{h} \end{aligned}$$

As you can see, after substituting in the trig identity, we grouped the first and third terms together, then factored out a sine from both. We then broke the overall limit into two limits, each of which can be dealt with separately.

Both of the limits here are limits as h approaches zero. In the first limit, we have $\sin(x)$ and in the second we have $\cos(x)$. As h moves towards zero, there is no effect on x . So since both $\sin(x)$ and $\cos(x)$ are functions of only x , they are unaffected by the limit. That is, as far as the limits are concerned, these two functions are constants and can be factored out of their respective limits. Doing this yields

$$\frac{d}{dx}(\sin(x)) = \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

At this point all we need to do is use the new limit rules, which give us

$$\frac{d}{dx}(\sin(x)) = \sin(x)(0) + \cos(x)(1) = \cos(x)$$

Therefore, $\frac{d}{dx}(\sin(x)) = \cos(x)$.

At this point, we cannot do very many derivatives involving sine and cosine. We will be able to do a lot more once we have derivative rules for composite functions, products, and quotients. The examples we can do combine the new derivative rules with the constant multiple rule and the power rule.

Example 2.3.8

Find the derivatives of the following functions.

a. $g(x) = 10 \sin(x) - 3x^4 + x^{-3}$

b. $r(t) = 5 \cos(t) + 7 \sin(t)$

Solutions:

- a. Here we have the sum or difference of three terms, and one of them is a multiple of sine. We may use the sum/difference rule here to handle the terms separately. In the first term, we combine the constant multiple rule with the derivative of sine. The derivative is then

$$g'(x) = 10 \cos(x) - 12x^3 - 3x^{-4}$$

- b. To do this derivative, we just need to use the rules for the derivatives of sine and cosine, combined with the constant multiple rule. The sum/difference rule tells us that we can handle the two trig functions separately. The derivative is then

$$r'(t) = -5 \sin(t) + 7 \cos(t)$$

Practice Problems

For problems 1 - 3, evaluate the limit.

1. $\lim_{z \rightarrow 0} \frac{\sin(10z)}{z}$

3. $\lim_{x \rightarrow 0} \frac{\cos(4x) - 1}{x}$

2. $\lim_{\alpha \rightarrow 0} \frac{\sin(12\alpha)}{\sin(5\alpha)}$

For problems 4 - 17, find the first derivative of the function.

4. $f(x) = 6x^3 - 9x + 4$

11. $R(z) = \frac{6}{\sqrt{z^3}} + \frac{1}{8z^4} - \frac{1}{3z^{10}}$

5. $y = 2t^4 - 10t^2 + 13t$

12. $z = x(3x^2 - 9)$

6. $g(z) = 4z^7 - 3z^{-7} + 9z$

13. $g(y) = (y - 4)(2y + y^2)$

7. $h(y) = y^{-4} - 9y^{-3} + 8y^{-2} + 12$

14. $h(x) = \frac{4x^3 - 7x + 8}{x}$

8. $y = \sqrt{x} + 8\sqrt[3]{x} - 2\sqrt[4]{x}$

15. $f(y) = \frac{y^5 - 5y^3 + 2y}{y^3}$

9. $f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3$

10. $f(t) = \frac{4}{t} - \frac{1}{6t^3} + \frac{8}{t^5}$

16. $y = \frac{13 \sin(x)}{7} + x(2x - 4)$

17. $Q(t) = 8\sqrt{t} + \sqrt{3} \cos(t)$

18. Find the second derivative of each of the functions given below.

a. $y = 2t^4 - 10t^2 + 13t$

c. $f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3$

b. $h(y) = y^{-4} - 9y^{-3} + 8y^{-2} + 12$

d. $f(t) = \frac{4}{t} - \frac{1}{6t^3} + \frac{8}{t^5}$

19. a. Find the second, third, and fourth derivatives of the sine function.

b. Find the 105^{th} derivative of the sine function.20. Determine where, if anywhere, the function $f(x) = x^3 + 9x^2 - 48x + 2$ is not changing.21. Determine where, if anywhere, the function $y = 2z^4 - z^3 - 3z^2$ is not changing.22. The position of an object is given by $s(t) = 2 + 7 \cos(t)$. Determine all the times when the object is not moving.23. Write the equation of the tangent line to the graph of $g(x) = \frac{16}{x} - 4\sqrt{x}$ at $x = 4$.24. Write the equation of the tangent line to the graph of $f(x) = 7x^4 + 8x^{-6} + 2x$ at $x = -1$.25. Write the equation of the tangent line to the graph of $h(x) = \sin(x)$ at $x = \frac{\pi}{3}$.26. The position (in feet) of an object at any time t (in seconds) is given by $s(t) = 3t^4 - 40t^3 + 126t^2 - 9$.a. Determine the velocity of the object at any time t .

b. Are there any times at which the object is at rest?

c. When is the object moving to the right and when is the object moving to the left?

d. What is the acceleration of the object at $t = 1$?27. Determine where the function $h(z) = 6 + 40z^3 - 5z^4 - 4z^5$ is increasing and where it is decreasing.28. Determine where the function $R(x) = (x + 1)(x - 2)^2$ is increasing and where it is decreasing.29. Determine where, if anywhere, the tangent line to $f(x) = x^3 - 5x^2 + x$ is parallel to the line $y = 4x + 23$.30. Determine where the function $f(x) = 4 \cos(x) - x$ is increasing and where it is decreasing in the interval $[-2, 7]$

Assignment Problems

For problems 1 - 20, find the derivative of the given function, using the rules learned in this section.

1. $g(x) = 8 - 4x^3 + 2x^8$
2. $f(z) = z^{10} - 7z^5 + 2z^3 - z^2$
3. $y = 8x^4 - 10x^3 - 9x + 4$
4. $f(x) = 3x^{-4} + x^4 - 3x$
5. $R(t) = 9t^{10} + 8t^{-10} + 12$
6. $h(y) = 3y^{-6} - 8y^{-3} + 9y^{-1}$
7. $g(t) = t^{-7} + 2t^{-3} - 6t^{-2} + 8t^4 - 1$
8. $z = \sqrt[6]{x} - 7\sqrt[4]{x} + 3\sqrt{x}$
9. $f(x) = 7\sqrt[9]{x^4} - 2\sqrt{x^7} + \sqrt[3]{x^4}$
10. $h(y) = 6\sqrt{y} + \sqrt[6]{y^5} + \frac{7}{\sqrt[9]{y^2}}$
11. $g(z) = \frac{4}{z^2} + \frac{1}{7z^5} - \frac{1}{2z}$
12. $y = \frac{2}{3t^9} + \frac{1}{7t^3} - 9t^2 - \sqrt{t^3}$
13. $W(x) = x^3 - \frac{1}{x^6} + \frac{1}{\sqrt[5]{x^2}}$
14. $g(w) = (w - 5)(w^2 + 1)$
15. $h(x) = \sqrt{x}(1 - 9x^3)$
16. $f(t) = (3 - 2t^3)^2$
17. $g(x) = (1 + 2x)(2 - x + x^2)$
18. $y = \frac{4 - 8x + 2x^2}{x}$
19. $Y(t) = \frac{t^4 - 2t^2 + 7t}{t^3}$
20. $S(x) = \frac{w^2(2 - w) + w^5}{3w}$

For problems 21 - 26, determine where, if anywhere, the function is not changing.

21. $f(x) = 2x^3 - 9x^2 - 108x + 14$
22. $u(t) = 45 + 300t^2 + 20t^3 - 3t^4$
23. $Q(t) = t^3 - 9t^2 + t - 10$
24. $h(w) = 2w^3 + 3w^2 + 4w + 5$
25. $g(x) = 9 + 8x^2 + 3x^3 - x^4$
26. $G(z) = z^2(z - 1)^2$
27. Write the equation of the tangent line to the graph of $f(x) = 3x^5 - 4x^2 + 9x - 12$ at $x = -1$.
28. Write the equation of the tangent line to the graph of $g(x) = \frac{x^2 + 1}{x}$ at $x = 2$.
29. Write the equation of the tangent line to the graph of $h(x) = 2\sqrt{x} - 8\sqrt[4]{x}$ at $x = 16$.
30. The position of an object at any time t is given by $s(t) = 3t^4 - 44t^3 + 108t^2 + 20$.
 - a. Determine the velocity of the object at any time t .

- b. Is the object ever at rest? If so, at what time(s) is it at rest?
- c. When is the object moving to the right and when is the object moving to the left?
31. The position of an object at any time t is given by $s(t) = 1 - 150t^3 + 45t^4 - 2t^5$.
- a. Determine the velocity of the object at any time t .
- b. Is the object ever at rest? If so, at what time(s) is it at rest?
- c. When is the object moving to the right and when is the object moving to the left?
32. Determine where the function $f(x) = 4x^3 - 18x^2 - 336x + 27$ is increasing and where it is decreasing.
33. Determine where the function $g(w) = w^4 + 2w^3 - 15w^2 - 9$ is increasing and where it is decreasing.
34. Determine where the function $V(t) = t^3 - 24t^2 + 192t - 50$ is increasing and where it is decreasing.
35. Determine the percentage of the interval $[-6, 4]$ on which $f(x) = 7 + 10x^3 - 5x^4 - 2x^5$ is increasing.
36. Determine the percentage of the interval $[-5, 2]$ on which $f(x) = 3x^4 - 8x^3 - 144x^2$ is decreasing.
37. Is $h(x) = 3 - x + x^2 + 2x^3$ increasing or decreasing more on the interval $[-1, 1]$?
38. Determine where, if anywhere, the tangent line to the graph of $f(x) = 12x^2 - 9x + 3$ is parallel to the line $y = 1 - 7x$.
39. Determine where, if anywhere, the tangent line to the graph of $f(x) = 8 + 4x + x^2 - 2x^3$ is perpendicular to the line $y = -\frac{1}{4}x + \frac{8}{3}$.
40. Determine where, if anywhere, the tangent line to the graph of $f(x) = \sqrt[3]{x} - 8x$ is perpendicular to the line $y = 2x - 11$.
41. Determine where, if anywhere, the tangent line to $f(x) = \frac{13x}{9} + \frac{1}{x}$ is parallel to the line $y = x$.

Determine whether each statement is true or false. Justify your answer.

42. If $f(x) = \frac{1}{3x^2}$, then $f'(x) = -6x^{-3}$.
43. $\lim_{\theta \rightarrow 0} \frac{\cos(\theta)}{\theta} = 1$
44. The function $f(x) = \sin(x)$ is increasing at $x = 0$ (use derivatives to justify your answer).

Section 2.3 solutions

1. We would like to use the first trigonometric limit from this section. In order to do so, we will multiply by $\frac{10}{10}$ to put the function into a form to which the rule applies.

$$\lim_{z \rightarrow 0} \frac{\sin(10z)}{z} = \lim_{z \rightarrow 0} \frac{10 \sin(10z)}{10z} = 10 \lim_{z \rightarrow 0} \frac{\sin(10z)}{10z} = 10(1) = 10$$

2. In order to use the limit rule from this section, we first need to split the fraction, then we multiply each fraction by one in a carefully chosen form:

$$\lim_{\alpha \rightarrow 0} \frac{\sin(12\alpha)}{\sin(5\alpha)} = \lim_{\alpha \rightarrow 0} \left[\frac{12\alpha \sin(12\alpha)}{12\alpha} \frac{5\alpha}{5\alpha \sin(5\alpha)} \right]$$

Then we rearrange one more time so that it is clear that we are dealing with three fractions:

$$\lim_{\alpha \rightarrow 0} \frac{\sin(12\alpha)}{\sin(5\alpha)} = \lim_{\alpha \rightarrow 0} \left[\frac{12\alpha \sin(12\alpha)}{5\alpha} \frac{5\alpha}{12\alpha} \frac{1}{\sin(5\alpha)} \right]$$

Now we can apply the limit rule:

$$\begin{aligned} &= \lim_{\alpha \rightarrow 0} \left[\frac{12 \sin(12\alpha)}{5} \frac{5\alpha}{12\alpha} \frac{1}{\sin(5\alpha)} \right] \\ &= \frac{12}{5} \left[\lim_{\alpha \rightarrow 0} \frac{\sin(12\alpha)}{12\alpha} \right] \left[\lim_{\alpha \rightarrow 0} \frac{5\alpha}{\sin(5\alpha)} \right] = \frac{12}{5} (1)(1) = \frac{12}{5} \end{aligned}$$

3. This limit is close to the second limit rule we proved in this section. We just need to multiply by one in a useful form before we can apply the limit rule.

$$\lim_{x \rightarrow 0} \frac{\cos(4x) - 1}{x} = \lim_{x \rightarrow 0} \frac{4(\cos(4x) - 1)}{4x} = 4 \lim_{x \rightarrow 0} \frac{\cos(4x) - 1}{4x} = 4(0) = 0$$

4. Since this function is a polynomial, we will use the constant multiple rule and the power rule on each term, then combine them using the sum/difference rule. Also notice that since the last term is a constant, its derivative is 0. For this problem, we have

$$f'(x) = (3)(6)x^{3-1} - (1)(9)x^{1-1} + 0 = 18x^2 - 9$$

5. Since this function is a polynomial, we will use the constant multiple rule and the power rule on each term, then combine them using the sum/difference rule. Doing this, we find

$$\frac{dy}{dt} = 8t^3 - 20t + 13$$

6. Remember that we can use the power rule whenever we take the derivative of a variable raised to a power that is a number (even a negative number). So we will again use the constant multiple rule and the power rule on each term here, then combine them with the sum/difference rule. Doing this, we find that

$$g'(z) = (7)(4)z^{7-1} + (-7)(-3)z^{-7-1} + (1)(9)z^{1-1} = 28z^6 + 21z^{-8} + 9$$

7. As with the last problem, we will apply the power rule and the constant multiple rule to each term, then combine them according to the sum/difference rule. The derivative of the last term is 0 because it's a constant. We find that

$$h'(y) = -4y^{-5} + 27y^{-4} - 16y^{-3}$$

8. We will again use the rules we learned in this section, but remember that we first need to rewrite the roots as fractional exponents before we can apply the power rule. The rewritten function is

$$y = x^{1/2} + 8x^{1/3} - 2x^{1/4}$$

The derivative is then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{8}{3}x^{-2/3} - \frac{1}{2}x^{-3/4} = \frac{1}{2\sqrt{x}} + \frac{8}{3\sqrt[3]{x^2}} - \frac{1}{2\sqrt[4]{x^3}}$$

9. First we will rewrite the function as

$$f(x) = 10x^{3/5} - x^{7/2} + 6x^{8/3} - 3$$

The derivative is then

$$f'(x) = 6x^{-2/5} - \frac{7}{2}x^{5/2} + 16x^{5/3} = \frac{6}{\sqrt[5]{x^2}} - \frac{7}{2}\sqrt{x^5} + 16\sqrt[3]{x^5}$$

10. Here we first need to rewrite the function using negative exponents, so that we can then use the power rule. The rewritten function is

$$f(t) = 4t^{-1} - \frac{1}{6}t^{-3} + 8t^{-5}$$

And the derivative is

$$f'(t) = -4t^{-2} + \frac{1}{2}t^{-4} - 40t^{-6} = \frac{-4}{t^2} + \frac{1}{2t^4} - \frac{40}{t^6}$$

11. We first rewrite the function so that there are no roots and all the variables are in the numerator with negative exponents. The rewritten function is

$$R(z) = 6z^{-1/3} + \frac{1}{8}z^{-4} - \frac{1}{3}z^{-10}$$

The derivative is then

$$R'(z) = -2z^{-4/3} - \frac{1}{2}z^{-5} + \frac{10}{3}z^{-11},$$

which can be rewritten as

$$R'(z) = -\frac{2}{z^{4/3}} - \frac{1}{2z^5} + \frac{10}{3z^{11}} = -\frac{2}{\sqrt[3]{z^4}} - \frac{1}{2z^5} + \frac{10}{3z^{11}}$$

12. For this one, we first need to distribute the x , and write the function in the correct form for the power rule. The rewritten function is

$$z = 3x^3 - 9x$$

The derivative is then

$$\frac{dz}{dt} = 9x^2 - 9$$

13. Here we will distribute in order to write the function as a polynomial, so that we can use the power rule. The rewritten function is

$$g(y) = 2y^2 + y^3 - 8y - 4y^2 = y^3 - 2y^2 - 8y$$

The derivative is then

$$g'(y) = 3y^2 - 4y - 8$$

14. For this one, we will rewrite the function by dividing each term in the numerator by x . The rewritten function is

$$h(x) = 4x^2 - 7 + 8x^{-1}$$

Then the derivative is

$$h'(x) = 8x - 8x^{-2} = 8x - \frac{8}{x^2}$$

15. Here again we divide each term in the numerator by the denominator, y^3 . The rewritten function is

$$f(y) = y^2 - 5 + 2y^{-2}$$

The derivative is then

$$f'(y) = 2y - 4y^{-3}$$

16. We first need to rewrite the function a little:

$$y = \frac{13}{7} \sin(x) + 2x^2 - 4x$$

The derivative is then

$$y' = \frac{13}{7} \cos(x) + 4x - 4$$

17. We need to rewrite the first term to eliminate the radical. The rewritten function is

$$Q(t) = 8t^{1/2} + \sqrt{3} \cos(t)$$

The derivative is then

$$Q'(t) = 4t^{-1/2} - \sqrt{3} \sin(t)$$

18. a. This function is the function from problem 2 above. In that problem, we found that the first derivative is

$$\frac{dy}{dt} = 8t^3 - 20t + 13,$$

so the second derivative is the derivative of this function. Using the power rule, we find that

$$\frac{d^2y}{dt^2} = 24t^2 - 20$$

- b. This is the function from problem 4 above. There, we found that the first derivative is

$$h'(y) = -4y^{-5} + 27y^{-4} - 16y^{-3}$$

The second derivative is the derivative of this function. Using the power rule, we find that

$$h''(y) = 20y^{-6} - 108y^{-5} + 48y^{-4}$$

- c. This is the function from problem 6 above. There, we found that the first derivative (after rewriting the function) is

$$f'(x) = 6x^{-2/5} - \frac{7}{2}x^{5/2} + 16x^{5/3}$$

The second derivative is the derivative of this function. Using the power rule, we find that

$$f''(x) = -\frac{12}{5}x^{-7/5} - \frac{35}{4}x^{3/2} + \frac{80}{3}x^{2/3} = \frac{-12}{5\sqrt[5]{x^7}} - \frac{35}{4}\sqrt{x^3} + \frac{80}{3}\sqrt[3]{x^2}$$

- d. This is the function from problem 7 above. There, we found that the first derivative (after rewriting the function) is

$$f'(t) = -4t^{-2} + \frac{1}{2}t^{-4} - 40t^{-6}$$

The second derivative is the derivative of this function. Using the power rule, we find that

$$f''(t) = 8t^{-3} - 2t^{-5} + 240t^{-7} = \frac{8}{t^3} - \frac{2}{t^5} + \frac{240}{t^7}$$

19. a. Since the first derivative of $\sin x$ is $\cos x$, and the derivative of $\cos x$ is $-\sin x$, we just need to keep track of minus signs as we take repeated derivatives of sine. If $y = \sin x$, then the first four derivatives of sine are

$$y' = \cos x \quad y'' = -\sin x \quad y''' = -\cos x \quad y^{(4)} = \sin x$$

- b. Notice that $105 = 4(26) + 1$. We saw in part (a) that the fourth derivative of sine is $\sin x$, which means that if we keep taking derivatives, every fourth one will give us the sine function back. In between, the derivatives cycle through $\cos x$, $-\sin x$, and $-\cos x$. So since 105 is one more than a multiple of 4, the 105th derivative of sine is the same as the first derivative of sine, $y^{(105)} = \cos x$.
20. Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. The function won't be changing if its rate of change is zero, so all we need to do is find out where the derivative is zero, if anywhere.

Using the power rule, we find that the derivative is

$$f'(x) = 3x^2 + 18x - 48 = 3(x^2 + 6x - 16) = 3(x - 2)(x + 8)$$

Notice that we have factored the derivative in preparation for finding where it equals zero.

$$f'(x) = 0 \iff 3(x - 2)(x + 8) = 0$$

We can see from this that the derivative is zero at $x = 2$ and $x = -8$. Therefore, the function is not changing when $x = -8$ and again when $x = 2$.

21. Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. The function won't be changing if its rate of change is zero, so all we need to do is find out where the derivative is zero, if anywhere.

Using the power rule, we find that the derivative is

$$\frac{dy}{dz} = 8z^3 - 3z^2 - 6z = z(8z^2 - 3z - 6)$$

Notice that we have factored the derivative in preparation for finding where it equals zero.

$$\frac{dy}{dz} = 0 \iff z = 0, 8z^2 - 3z - 6 = 0$$

This tells us immediately that the function is standing still when $z = 0$. We still need to solve $8z^2 - 3z - 6 = 0$, though. For this, we'll use the quadratic formula.

$$z = \frac{3 \pm \sqrt{(-3)^2 - 4(8)(-6)}}{2(8)} = \frac{3 \pm \sqrt{201}}{16}$$

Therefore, the function is not changing when $z = 0$, $z = \frac{3 + \sqrt{201}}{16} \approx 1.07359$ and $z = \frac{3 - \sqrt{201}}{16} \approx -0.69859$.

22. The object is not moving at any time at which its velocity is zero. Since the velocity is given by the derivative of the position function, we need to take the derivative, set it equal to zero, and solve for t .

$$s'(t) = -7 \sin(t)$$

The derivative, and therefore the velocity, is zero when $\sin(t) = 0$. The unit circle tells us that this will be whenever

$$t = 0 + 2\pi n \quad \text{and} \quad t = \pi + 2\pi n$$

for n any integer.

23. Recall that one of the interpretations of the derivative is that it gives the slope of the tangent line to the graph of the function at any point. So, we will need to find the derivative. However, before we can do that, we need to rewrite the function. Here is the rewritten function and the derivative:

$$g(x) = 16x^{-1} - 4x^{1/2} \implies g'(x) = -16x^{-2} - 2x^{-1/2} = -\frac{16}{x^2} - \frac{2}{\sqrt{x}}$$

Since we now need to evaluate the derivative at $x = 4$, we rewrote the derivative as rational expressions with roots. Now, evaluating at $x = 4$, we find that

$$g(4) = \frac{16}{4} - 4\sqrt{4} = -4 \quad g'(4) = -\frac{16}{4^2} - \frac{2}{\sqrt{4}} = -2$$

Now all we need to do is substitute $g(4)$ and $g'(4)$ into the equation of the tangent line.

$$y = g(4) + g'(4)(x - 4) = -4 - 2(x - 4)$$

Thus the equation of the tangent line is $y = -2x + 4$.

24. Recall that one of the interpretations of the derivative is that it gives the slope of the tangent line to the graph of a function at any point. So, we need to find the derivative of the function.

Using the power rule, we find that

$$f'(x) = 28x^3 - 48x^{-7} + 2 = 28x^3 - \frac{48}{x^7} + 2$$

Next we need to evaluate the function and the derivative at $x = -1$.

$$f(-1) = 7 + 8 - 2 = 13 \quad f'(-1) = -28 + 48 + 2 = 22$$

Now all we need to do is write down the equation of the tangent line.

$$y = f(-1) + f'(-1)(x + 1) = 13 + 22(x + 1)$$

or

$$y = 22x + 35$$

25. To find the slope of the tangent line, we'll need the derivative of $h(x) = \sin(x)$, $h'(x) = \cos(x)$. So, $m = h'(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$. We also need the y-coordinate of the point on the line, $h(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$. So, an equation of the tangent line is $y = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3})$.

26. a. Recall that one of the interpretations of the derivative is that it gives the velocity function of an object. We've been given the position function of an object, so all we need to do is find its derivative, and we'll have the velocity of the object at any time t .

The velocity function of the object is then

$$v(t) = s'(t) = 12t^3 - 120t^2 + 252t = 12t(t - 3)(t - 7)$$

Note that we factored the velocity function here for use in later parts of this problem.

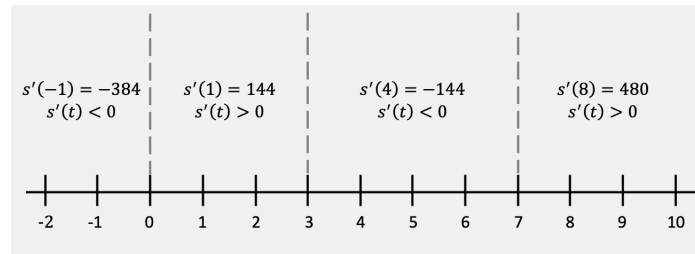
- b. The position of the object will not change at any time that its velocity is zero. So we need to set the derivative equal to zero and solve for t .

$$s'(t) = 0 \Leftrightarrow 12t(t - 3)(t - 7) = 0$$

From this, we see that the object is not moving when $t = 0, t = 3$ or $t = 7$.

- c. To answer this part, we need to know where the derivative is positive (and hence the object is moving to the right) or negative (and hence the object is moving to the left). Because the derivative is continuous, we know that the only places that it can change sign are where it is zero. So, as we did in Example 5 above, we will use the zeros of the derivative to split the number line into intervals, then find the sign of the derivative on each interval.

Here is the number line for this problem:



From this, we deduce the following information:

Moving right when $0 < t < 3$ or $7 < t < \infty$

Moving left when $-\infty < t < 0$ or $3 < t < 7$

Note that depending upon your interpretation of t as time, you may or may not have included the interval $-\infty < t < 0$ in the "moving left" answer.

- d. The acceleration of an object is the rate of change of its velocity. So we need to find the derivative of the velocity function. Recall that this is also the second derivative of the position function.

So the acceleration function is then

$$s''(t) = v'(t) = 36t^2 - 240t + 252$$

and the acceleration at $t = 1$ is

$$s''(1) = v'(1) = 36 - 240 + 252 = 48 \text{ feet per second per second}$$

27. Recall that one of the interpretations of the derivative is that it gives the rate of change of the function at any point. Since we are asked to find where the function is increasing and decreasing, we are talking about the rate of change of the function. So, we'll need the derivative.

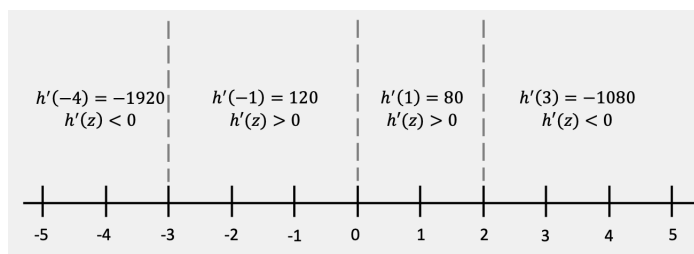
$$h'(z) = 120z^2 - 20z^3 - 20z^4 = -20z^2(z + 3)(z - 2)$$

Note that the derivative was factored here for later steps. This doesn't really need to be done in general.

To find the answer to this problem, we need to know where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous, we know that the only places where it can change sign are where it is zero. So let's first find those points.

$$h'(z) = 0 \Leftrightarrow -20z^2(z + 3)(z - 2) = 0 \Leftrightarrow z = 0, z = -3, z = 2$$

Now we will use the zeros of the derivative to split the number line into intervals, then find the sign of the derivative on each interval. Here is the number line for this problem:



From this, we deduce the increasing/decreasing information.

Increasing when $-3 < z < 0$ or $0 < z < 2$

Decreasing when $-\infty < z < -3$ or $2 < z < \infty$

28. Recall that one of the interpretations of the derivative is that it gives the rate of change of the function at any point. Since we are asked to find where the function is increasing or decreasing, we are talking about the rate of change of the function. So, we need the derivative. We will first need to multiply out the factors so that we can use the power rule. Here are the rewritten function and the derivative:

$$R(x) = x^3 - 3x^2 + 4$$

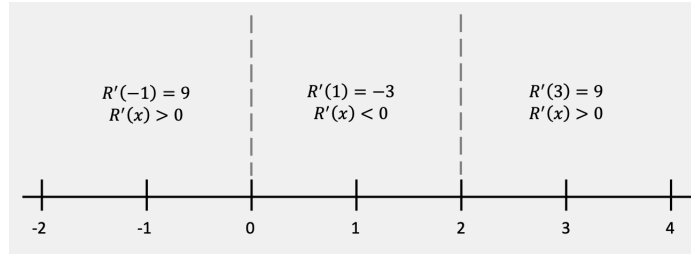
$$R'(x) = 3x^2 - 6x = 3x(x - 2)$$

Note that the derivative was factored for use in later steps. This doesn't really need to be done in general.

To answer the question, we need to know where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous, we know that the only places it can change sign are where it is zero. So let's first find those points.

$$R'(x) = 0 \Leftrightarrow 3x(x - 2) = 0 \Leftrightarrow x = 0, x = 2$$

Now we will use the zeros of the derivative to split the number line into intervals, then find the sign of the derivative on each interval. Here is the number line for this problem:



From this, we deduce the increasing/decreasing information.

Increasing when $-\infty < x < 0$ or $2 < x < \infty$

Decreasing when $0 < x < 2$

29. Since two lines are parallel if they have the same slope, we need the slopes of the tangent lines to the graph of f . Recall that this is one of the interpretations of the derivative, so we need to take the derivative of the function.

$$f'(x) = 3x^2 - 10x + 1$$

The slope of the line $y = 4x + 23$ is 4, so we need to find any points where the slope of the tangent line to f is 4. In other words, we need to solve $f'(x) = 4$ for x .

$$f'(x) = 4 \Leftrightarrow 3x^2 - 10x + 1 = 4 \Leftrightarrow 3x^2 - 10x - 3 = 0$$

This quadratic doesn't factor, so we use the quadratic formula to find

$$x = \frac{10 \pm \sqrt{136}}{6} = \frac{10 \pm 2\sqrt{34}}{6} = \frac{5 \pm \sqrt{34}}{3}$$

So the tangent line is parallel to the line $y = 4x + 23$ at the points where

$$x = \frac{5 - \sqrt{34}}{3} \approx -0.276984$$

$$x = \frac{5 + \sqrt{34}}{3} \approx 3.61032$$

30. The function will be increasing wherever its rate of change is positive, and will be decreasing wherever its rate of change is negative. Since the derivative gives us the rate of change of a function, we need the derivative.

$$f'(x) = -4\sin(x) - 1$$

Next we need to know where the derivative is zero:

$$-4\sin(x) - 1 = 0 \quad \Leftrightarrow \quad \sin(x) = -\frac{1}{4}$$

Using a calculator to solve for x , we find

$$x = \arcsin\left(-\frac{1}{4}\right) \approx -0.2527$$

Recall that we can find an equivalent positive angle by adding 2π . So we also have $x \approx 2\pi - 0.2527 = 6.0305$.

By looking at the unit circle, we see that $x \approx \pi + 0.2527 = 3.3943$ is another solution.

Putting these together, we see that the derivative will be zero at

$$x \approx 6.0305 + 2\pi n \quad \text{and} \quad x \approx 3.3943 + 2\pi n$$

for n any integer.

By testing a few values of n , we see that the first set of solutions only falls into the interval $[-2, 7]$ for $n = -1$ and $n = 0$, and the second set of solutions is only in the desired interval when $n = 0$.

In the interval $[-2, 7]$, the derivative is zero when

$$x \approx -0.2527, 3.3943, 6.0305$$

Now to find where the function is increasing and decreasing, we need to find the intervals over which the derivative is positive, and those over which it is negative. Because the derivative is continuous, the only places it can change sign are where the derivative is zero. We will test points in each of the intervals formed by the values we just found.

From this, we deduce the increasing/decreasing information.

Increasing when $-2 \leq x < -0.2527, 3.3943 < x < 6.0305$

Decreasing when $-0.2527 < x < 3.3943, 6.0305 < x \leq 7$

Note that because we only looked at what is happening in the interval $[-2, 7]$, we can't say anything about the increasing/decreasing behavior of the function outside of this interval.