§4.2 Critical Points

Introduction

Many of the applications that we will explore in this chapter require us to identify the **critical points** of a function. In this section, we will define what a critical point is, and practice finding the critical points of various functions, both algebraically and graphically.

To motivate the definition, let's recall some of the rate of change examples we have already done. In order to discover the intervals over which a function increases or decreases, we first needed to find the points where the derivative was zero. These are examples of critical points.

Definition 1: Critical numbers and critical points

The number $x = c$ is a **critical number** of the function $f(x)$ if $f(c)$ exists, and if either $f'(c) = 0$ or $f'(c)$ does not exist.

The ordered pair $(c, f(c))$ is then called a **critical point** of the function.

Note that we require that $f(c)$ exist in order for $x = c$ to be a critical number. This is saying that all critical numbers must be in the domain of the function. If a number is not in the domain of the function, then it cannot be a critical number.

A number c can only be a critical number if it is in the domain of our function.

Note as well that, at this point, we will only work with real numbers. Any complex numbers that arise in finding critical numbers (and they do arise on occasion) will be ignored. Calculus with complex numbers is beyond the scope of this course; it is usually taught in an upper level math course.

At this point, we will not allow imaginary critical numbers.

Finally, we should emphasize that a critical **number** is a number in the domain of the function and a critical **point** is a point on the graph of the function. This will become important to us in the next section. For now, we will focus on finding critical numbers.

A critical **number** is an x-value. A critical **point** is a point on the graph of the function (an (x, y) point).

Examples

Example 4.2.1

Determine all the critical numbers of the function

$$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$$

Solution:
We need to find the derivative of the function first. We will also factor the derivative as much as possible to make the next step easier.

$$\begin{aligned} f'(x) &= 30x^4 + 132x^3 - 90x^2 \\ &= 6x^2(5x^2 + 22x - 15) \\ &= 6x^2(5x - 3)(x + 5) \end{aligned}$$

Since the derivative is a polynomial, its domain is the entire set of real numbers. So there are no numbers for which $f'(x)$ does not exist. Therefore, the only critical numbers will be those values of x at which the derivative is zero. So we solve

$$6x^2(5x - 3)(x + 5) = 0$$

This yields three critical numbers,

$$x = -5, x = 0, x = \frac{3}{5}$$

Polynomials are usually fairly simple functions to find the critical points for, provided the degree isn't too large. Most of the more ”interesting” functions require a little more effort to find their critical numbers. Let's look at a few.

Example 4.2.2

Determine all the critical numbers of the function

$$g(t) = \sqrt[3]{t^2}(2t-1)$$

Solution:
While we could find the derivative using the product rule, it might be easier if we first distribute, then simplify as much as possible.

$$g(t) = t^{\frac{2}{3}}(2t-1) = 2t^{\frac{5}{3}} - t^{\frac{2}{3}}$$

Now we'll differentiate, using the power rule.

$$g'(t) = \frac{10}{3}t^{\frac{2}{3}} - \frac{2}{3}t^{-\frac{1}{3}} = \frac{10t^{\frac{2}{3}}}{3} - \frac{2}{3t^{\frac{1}{3}}}$$

Notice that we chose to rewrite the second term without the negative exponent. This isn't required, but it can make it easier to see what's happening.

One thing that we notice right away is that $t = 0$ is a critical number for this function, because of the denominator of the second term. If we had kept the negative exponent instead of moving the power of t to the denominator, we might have mistakenly thought that the derivative equaled zero when t equals zero.

So $t = 0$ is a critical number because the derivative does not exist there. Now we want to find any critical numbers of the other type: where $g'(t) = 0$. As a first step, let's combine the two terms into a single fraction to find

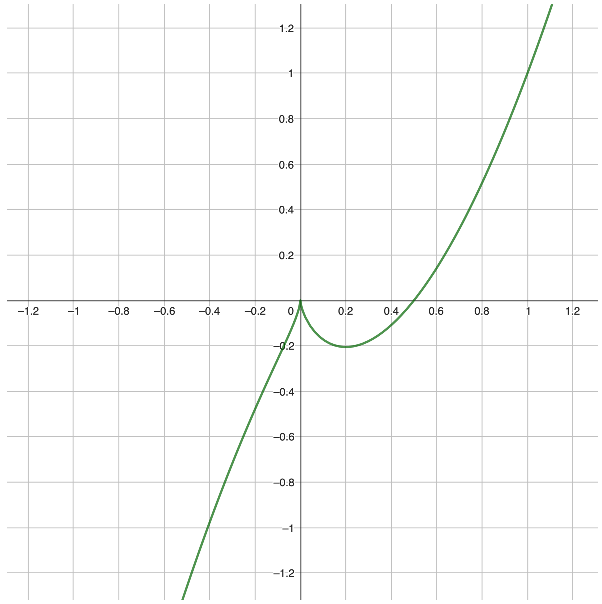
$$g'(t) = \frac{10t-2}{3t^{\frac{1}{3}}}$$

Notice that we still see the critical number at $t = 0$ in this form. Setting the numerator equal to zero yields the additional critical number $t = \frac{1}{5}$.

Therefore, there are two critical numbers for this function,

$$t = 0 \qquad \text{and} \qquad t = \frac{1}{5}$$

Before we move on to the next example, let's take a look at the graph of the function in Example 2. This function has one critical number (at $t = 0$) at which the derivative is undefined, and one critical number ($t = \frac{1}{5}$) at which the derivative equals zero.



On this graph, we see the cusp at the origin, which indicates that the function is not differentiable at $t = 0$. We then see that at the second critical point, the graph has a local minimum. We can also see that the derivative is zero at this point because the tangent line is horizontal there.

Example 4.2.3

Determine all the critical numbers of the function

$$R(w) = \frac{w^2 + 1}{w^2 - w - 6}$$

Solution:
Using the quotient rule and some simplification, we find that the derivative is

$$\frac{-w^2 - 14w + 1}{(w^2 - w - 6)^2} = -\frac{w^2 + 14w - 1}{(w^2 - w - 6)^2}$$

Notice that we factored negative one from the numerator. As a negative in front of the fraction, this will not affect the critical numbers, since it has no effect on whether the derivative is undefined or equal to zero.

To find critical numbers, we will set the denominator and the numerator equal to zero separately and solve each for w. The values of w in the first set are numbers at which the derivative does not exist, and those in the second set are the numbers at which the derivative equals zero.

Setting the denominator to zero yields

$$w^2 - w - 6 = (w - 3)(w + 2) = 0$$

Notice that since zero squared is still zero, we didn't need to set the squared expression equal to zero. We can therefore see that the denominator is zero at $w = 3$ and $w = -2$.

However, these are **not** critical numbers because the original function is also undefined at these values of w. Since they are not in the domain of the function, they cannot be critical numbers!

Now we'll set the numerator equal to zero. The quadratic doesn't factor, though, so we'll use the Quadratic Formula to find the solutions of this equation.

$$w = \frac{-14 \pm \sqrt{(14)^2 - 4(1)(-1)}}{2(1)} = \frac{-14 \pm \sqrt{200}}{2} = \frac{-14 \pm 10\sqrt{2}}{2} = -7 \pm 5\sqrt{2}$$

You certainly notice that these answers are not integers or simple fractions. But they are real numbers and therefore are perfectly reasonable answers. This type of answer often results from the Quadratic Formula.

Summarizing, we have two critical numbers, $w = -7 + 5\sqrt{2}$ and $w = -7 - 5\sqrt{2}$.

In the last example, we had to use the Quadratic Formula to determine some of the potential critical numbers. When we use the Quadratic Formula to solve a quadratic equation it also happens sometimes that the solutions are complex numbers. Since for us a critical number has to be a real number, we would disregard any complex solutions that we find.

The next few examples include trigonometric, exponential and logarithmic functions. Many of the applications we examine later in the chapter will involve these types of functions.

Example 4.2.4

Determine all the critical numbers of the function

$$y = 6x - 4\cos(3x)$$

Solution:

We first find the derivative, remembering to use the chain rule on the second term.

$$\frac{dy}{dx} = 6 + 12\sin(3x)$$

Both the function and its derivative are defined everywhere, so there are no critical numbers at which the derivative doesn't exist. The only critical numbers will be where the derivative is zero.

$$\begin{aligned} 6 + 12\sin(3x) &= 0 \\ \sin(3x) &= -\frac{1}{2} \end{aligned}$$

Applying the inverse sine to both sides of this last equation yields

$$
\begin{aligned}
3x &\approx 3.6652 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \\
3x &\approx 5.7596 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

Don't forget the $2\pi n$ on these, and make sure they are added at this stage (when applying the inverse trig function). There will be problems later in which we will miss solutions if we leave the $2\pi n$ off of these solutions. Now divide by 3 to find all the critical numbers of the function.

$$
\begin{aligned}
x &\approx 1.2217 + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots \\
x &\approx 1.9199 + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

In the last example we found an infinite number of critical numbers. This happens on occasion, especially when working with trig functions.

Example 4.2.5

Determine all the critical numbers of the function

$$h(t) = 10t\mathrm{e}^{3-t^2}$$

Solution: First we find the derivative of this function:

$$h'(t) = 10\mathrm{e}^{3-t^2} + 10t\mathrm{e}^{3-t^2}(-2t) = 10\mathrm{e}^{3-t^2} - 20t^2\mathrm{e}^{3-t^2}$$

This may look intimidating, but we can make it less so by factoring:

$$h'(t) = 10\mathrm{e}^{3-t^2}(1 - 2t^2)$$

This function is defined everywhere, so we just need to determine where it is zero. We know that exponential functions never equal zero, so we only need to set the second factor equal to zero.

$$
\begin{aligned}
1 - 2t^2 &= 0 \\
1 &= 2t^2 \\
\frac{1}{2} &= t^2
\end{aligned}
$$

We have two critical numbers for this function:

$$t = \pm\frac{1}{\sqrt{2}}$$

Example 4.2.6

Determine all the critical numbers of the function

$$f(x) = x^2 \ln(3x) + 6$$

Solution:
Before finding the derivative, we first observe that since the natural log is only defined for $x > 0$, we only need to look for critical numbers in this domain.

Now we find the derivative to be

$$\begin{aligned} f'(x) &= 2x\ln(3x) + x^2\left(\frac{3}{3x}\right) \\ &= 2x\ln(3x) + x \\ &= x(2\ln(3x) + 1) \end{aligned}$$

This derivative is undefined for $x \leq 0$, but these are also the values of x at which the original function is undefined. In other words, these potential critical numbers are not in the domain of the original function, so they are not critical numbers. So we just need to determine where the derivative is zero.

First note that since the derivative is not defined when $x = 0$, and we've already decided that this isn't a critical number, we only need to set the second factor equal to zero.

So the derivative is zero if

$$\begin{aligned} 2\ln(3x) + 1 &= 0 \\ \ln(3x) &= -\frac{1}{2} \end{aligned}$$

We solve this by exponentiating both sides:

$$
\begin{aligned}
\mathrm{e}^{\ln(3x)} &= \mathrm{e}^{-\frac{1}{2}} \\
3x &= \mathrm{e}^{-\frac{1}{2}} \\
x &= \frac{1}{3}\mathrm{e}^{-\frac{1}{2}} = \frac{1}{3\sqrt{\mathrm{e}}}
\end{aligned}
$$

This is the single critical number for this function.

We'll work one more example, just to make a point.

Example 4.2.7

Determine all the critical numbers of the function

$$f(x) = x\mathrm{e}^{x^2}$$

Solution:
Note that this function is not much different from the one we saw in Example 5. In this case, the derivative is

$$f'(x) = \mathrm{e}^{x^2} + x\mathrm{e}^{x^2}(2x) = \mathrm{e}^{x^2}(1 + 2x^2)$$

This function is defined for all x, so we only need to consider values of x that make $f'(x)$ equal zero. But on inspecting the two factors, we see that this function will never equal zero. The exponential is never zero, and the polynomial only has complex zeroes. Since for us a critical number must be a real number, there are no critical numbers for this function.

This last example illustrates the fact that not all functions have critical points. Most of the functions we will work with will have critical points, though, since those will make the more interesting examples.

§4.2.1 Practice Problems

Determine all critical numbers of each of the following functions.

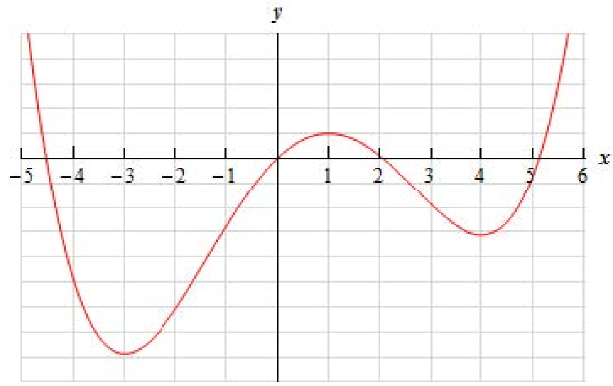
1. $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$
3. $g(w) = 2w^3 - 7w^2 - 3w - 2$
4. $g(x) = x^6 - 2x^5 + 8x^4$
5. $h(z) = 4z^3 - 3z^2 + 9z + 12$
6. $Q(x) = (2 - 8x)^4(x^2 - 9)^3$
7. $f(z) = \dfrac{z+4}{2z^2 + z + 8}$
8. $R(x) = \dfrac{1-x}{x^2 + 2x - 15}$
9. $r(y) = \sqrt[5]{y^2 - 6y}$
10. $h(t) = 15 - (3 - t)[t^2 - 8t + 7]^{\frac{1}{3}}$
11. $s(z) = 4\cos(z) - z$
12. $f(y) = \sin\left(\dfrac{y}{3}\right) + \dfrac{2y}{9}$
13. $V(t) = \sin^2(3t) + 1$
14. $f(x) = 5x\mathrm{e}^{9-2x}$
15. $g(w) = \mathrm{e}^{w^3 - 2w^2 - 7w}$
16. $R(x) = \ln(x^2 + 4x + 14)$
17. $A(t) = 3t - 7\ln(8t + 2)$

§4.2.2 Assignment Problems

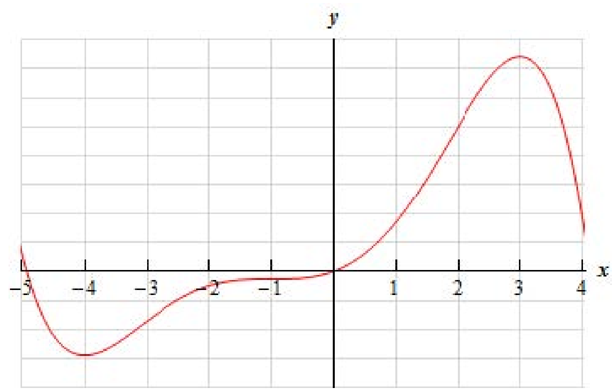
For problems 1 - 40, determine the critical numbers of the function.

1. $R(x) = 8x^3 - 18x^2 - 240x + 2$
2. $f(z) = 2z^4 - 16z^3 + 20z^2 - 7$
3. $g(z) = 8 - 12z^5 - 25z^6 + \frac{90}{7}z^7$
4. $P(w) = w^3 - 4w^2 - 7w - 1$
5. $A(t) = 7t^3 - 3t^2 + t - 15$
6. $a(t) = 4 - 2t^2 - 6t^3 - 3t^4$
7. $h(v) = v^5 + v^4 + 10v^3 - 15$
8. $g(z) = (z-3)^5(2z+1)^4$
9. $R(q) = (q+2)^4(q^2-8)^2$
10. $f(t) = (t-2)^3(t^2+1)^2$
11. $f(w) = \dfrac{w^2+2w+1}{3w-5}$
12. $h(t) = \dfrac{3-4t}{t^2+1}$
13. $R(y) = \dfrac{y^2-y}{y^2+3y+8}$
14. $Y(x) = \sqrt[3]{x-7}$
15. $f(t) = (t^3-25t)^{\frac{2}{3}}$
16. $h(x) = \sqrt[5]{x}(2x+8)^2$
17. $Q(w) = (6-w^2)\sqrt[3]{w^2-4}$
18. $Q(t) = 7\sin\left(\dfrac{t}{4}\right) - 2$
19. $g(x) = 3\cos(2x) - 5x$
20. $f(x) = 7\cos(x) + 2x$
21. $h(t) = 6\sin(2t) + 12t$
22. $w(z) = \cos^3\left(\dfrac{z}{5}\right)$
23. $U(z) = \tan(z) - 4z$
24. $h(x) = x\cos(x) - \sin(x)$
25. $h(x) = 2\cos(x) - \cos(2x)$
26. $f(w) = \cos^2(w) - \cos^4(w)$
27. $F(w) = \mathrm{e}^{14w+3}$
28. $g(z) = z^2\mathrm{e}^{1-z}$
29. $A(x) = (3-2x)\mathrm{e}^{x^2}$
30. $P(t) = (6t+1)\mathrm{e}^{8t-t^2}$
31. $f(x) = \mathrm{e}^{3+x^2} - \mathrm{e}^{2x^2-4}$
32. $f(z) = \mathrm{e}^{z^2-4z} + \mathrm{e}^{8z-2z^2}$
33. $h(y) = \mathrm{e}^{6y^3-8y^2}$
34. $g(t) = \mathrm{e}^{2t^3+4t^2-t}$
35. $Z(t) = \ln(t^2+t+3)$
36. $G(r) = r - \ln(r^2+1)$
37. $A(z) = 2 - 6z + \ln(8z+1)$
38. $f(x) = x - 4\ln(x^2+x+2)$
39. $g(x) = \ln(4x+2) - \ln(x+4)$
40. $h(t) = \ln(t^2-t+1) + \ln(4-t)$

41. The graph of a function $f(x)$ is shown below. Use the graph to estimate the critical numbers of the function.



42. The graph of a function $f(x)$ is shown below. Use the graph to estimate the critical numbers of the function.



43. The graph of a function $f(x)$ is shown below. Use the graph to estimate the critical numbers of the function.