

§2.5 Derivatives Of Exponential And Logarithm Functions

Exponential Functions

The next set of functions that we want to take a look at are exponential and logarithm functions. The most common exponential and logarithm functions in a calculus course are the natural exponential function, e^x , and the natural logarithm function, $\ln(x)$.

We will take a more general approach however and look at the general exponential and logarithm function, where the base is not necessarily e . We'll start off by looking at the exponential function,

$$f(x) = a^x$$

where a is some real number greater than 0 and not 1. We want to differentiate this. The power rule that we looked at a couple of sections ago won't work as that required the exponent to be a fixed number and the base to be a variable. That is exactly the opposite from what we've got with this function. So, we're going to have to start with the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x (a^h - 1)}{h} \end{aligned}$$

Now, the a^x is not affected by the limit since it doesn't have any h 's in it and so is a constant as far as the limit is concerned. We can therefore factor this out of the limit. This gives,

$$f'(x) = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}.$$

Now let's notice that the limit we've got above is exactly the definition of the derivative of $f(x) = a^x$ at $x = 0$, i.e. $f'(0)$. Therefore, the derivative becomes,

$$f'(x) = f'(0) a^x.$$

So, we are kind of stuck. We need to know the derivative in order to get the derivative!

There is one value of a that we can deal with at this point. Recall how the number e is defined. There are in fact a variety of ways to define e . Here are three of them, the first being how we have defined e in the past.

Definition 1: Some Definitions of e

1. $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$
2. e is the unique positive number for which $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$
3. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

The second one is the important one for us because that limit is exactly the limit that we're working with above. So, this version of the definition leads to the following fact:

$$\text{for } f(x) = e^x, \text{ we have } f'(0) = \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

So, provided we are using the natural exponential function we get the following.

$$f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x$$

We still need to know what to do with derivatives of exponential functions whose base is not (necessarily) e . For this, we are able to employ the chain rule to help us, along with the help of logarithmic functions.

Remember that exponential and logarithmic functions undo one another, so if we raise a base number to a power containing a log with the same base, we're left with the input of the logarithm. That is, $b^{\log_b(x)} = x$. For a positive real number (not 1), a , notice that using this property of logarithms we can write a as,

$$a = e^{\ln a}.$$

Using this we can write the function $f(x) = a^x$ as,

$$\begin{aligned} f(x) &= a^x \\ &= (a)^x \\ &= \left(e^{\ln a}\right)^x \\ &= e^{(\ln a)x} \\ &= e^{x \ln a}. \end{aligned}$$

Now that we've gotten that taken care of all we need to remember is that a is a constant and so $\ln(a)$ is also a constant. Now, differentiating $f(x)$ is a fairly simple Chain Rule problem.

$$f'(x) = e^{x \ln a} (\ln a)$$

All we need to do is rewrite the first term back as a^x to get,

$$f'(x) = a^x \ln(a).$$

And so we can now take the derivative of any exponential function. Before working some examples, let's recap these derivatives.

Definition 2: Derivatives of Exponential functions

Assuming a is positive real number which is not 1,

$$\frac{d}{dx} [e^x] = e^x \quad \text{and} \quad \frac{d}{dx} [a^x] = a^x \ln(a).$$

Note that the derivative of e^x still follows the derivative for a^x if you were to replace a with e . Simply remember that $\ln(e) = 1$ and the rules align.

Let's practice taking derivatives involving exponential functions. Remember, we still have all other rules to follow, including product, quotient, and chain rules.

Example 2.5.1: Using derivative formulas of exponential functions

Differentiate each of the following functions.

- $R(v) = 3^v + v^3$
- $h(w) = e^{w^4 - 3w^2 + 9}$
- $y = \frac{5e^x}{3e^x + 1}$
- $f(z) = \sin(ze^z)$

Solutions:

- Notice that the two terms must follow different derivative rules. The first is an exponential (the base is a number while the exponent is a variable) and must follow the derivative rules for exponential functions. The second is a power function (the base is a variable and the exponent is a number) and so must follow the power rule.

Thus the derivative is

$$R'(v) = 3^v \ln(3) + 3v^2.$$

- Overall, this is an exponential function with base e . So we will be following the derivative of exponential functions (specifically for base e). However, the exponent is not just the variable w , thus we need to be a bit careful.

$h(w)$ is actually a composite function: $h(w) = e^{f(w)}$ where $f(w) = w^4 - 3w^2 + 9$. So $f(w)$ is our 'inner function', and we will need to use the chain rule. Separating this work into separate steps, we get

$$\begin{aligned} h'(w) &= e^{w^4-3w^2+9} \frac{d}{dw} [w^4 - 3w^2 + 9] \\ &= e^{w^4-3w^2+9} (4w^3 - 6w) \\ &= (4w^3 - 6w) e^{w^4-3w^2+9}. \end{aligned}$$

Make sure to use parentheses as the derivative of the inside function has more than one term.

c. We'll need to use the quotient rule on this one.

$$\begin{aligned} \frac{dy}{dx} &= \frac{5e^x(3e^x + 1) - (5e^x)(3e^x)}{(3e^x + 1)^2} \\ &= \frac{15e^{2x} + 5e^x - 15e^{2x}}{(3e^x + 1)^2} \\ &= \frac{5e^x}{(3e^x + 1)^2} \end{aligned}$$

d. In this problem we will first need to apply the chain rule and when we go to differentiate the inside function we'll need to use the product rule.

Here is the chain rule portion of the problem.

$$f'(z) = \cos(ze^z) \frac{d}{dz} [ze^z]$$

In this case we did not actually do the derivative of the inside yet. We just left it in the derivative notation to make it clear that in order to do the derivative of the inside function we now have a product rule.

Here is the rest of the work for this problem.

$$f'(z) = \cos(ze^z) (e^z + ze^z)$$

There's an important concept in Example 1b that we don't want to miss.

Note: In an exponential function, the entire exponent is the "inner" function (for Chain Rule purposes).

For example, in 1b, when we differentiated $h(w) = e^{w^4-3w^2+9}$, we treated the $w^4 - 3w^2 + 9$ as the inside function when we applied the chain rule.

See if you can practice some on your own.

Check Point 2.5.1: Practice Exponential Derivatives

Differentiate each function.

a. $f(y) = 4^{-9y+8}$

b. $y = \cos\left(\frac{1-e^{7x}}{1+e^{7x}}\right)$

Before moving on we need to go back over a couple of derivatives to make sure that we don't confuse the two. The two derivatives are,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Power Rule

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Derivative of an exponential function

It is important to note that with the Power rule the exponent MUST be a constant and the base MUST be a variable while we need exactly the opposite for the derivative of an exponential function. For an exponential function the exponent MUST be a variable and the base MUST be a constant.

It is easy to get locked into one of these formulas and just use it for both of these. We also haven't even talked about what to do if both the exponent and the base involve variables. We'll see this situation in a later section.

When the **variable is in the base**, the rule is $\frac{d}{dx}(x^n) = nx^{n-1}$.

When the **variable is in the exponent**, the rule is $\frac{d}{dx}(a^x) = a^x \ln(a)$.

When the variable is in the base *and* the exponent, we don't know what to do (yet)!

Inverse Functions

Now we will give some quick reminders from algebra about inverse functions, then see how we apply derivatives to inverse functions. This will then help us with logarithmic functions as well as inverse trigonometric functions later on.

Firstly, we say that a function, $y = f(x)$, is one-to-one if no two values of x produce the same y . This is a fairly simple definition of one-to-one but it takes an example of a function that isn't one-to-one to show just what it means. Before doing that however we should note that this definition of one-to-one is not really the mathematically correct definition of one-to-one. It is identical to the mathematically correct definition it just doesn't use all the notation from the formal definition, written below.

Definition 3: One-to-one

Suppose $f(x)$ is a function. We say $f(x)$ is one-to-one if $f(a) = f(b)$ implies $a = b$ for all values a and b in the domain of f .

Showing that a function is one-to-one is often a tedious and difficult process. Let's see an example of a function that isn't one-to-one. The function $f(x) = x^2$ is not one-to-one because both $f(2) = 4$ and $f(-2) = 4$. In other words, there are two different values of x that produce the same value of y . Note that we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \leq x < \infty$. This can sometimes be done with functions.

Inverse functions deal with composition. We will get to the formal definition of inverse functions, but it is important to note that only one-to-one functions have inverses. This is particularly important for the inverse trigonometric functions as the basic trigonometric functions are not one-to-one unless we restrict their domain.

Definition 4: Inverse Functions

Given two one-to-one functions $f(x)$ and $g(x)$, if

$$(f \circ g)(x) = x \quad \text{AND} \quad (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are inverses of each other. More specifically we will say that $g(x)$ is the inverse of $f(x)$ and may denote it by

$$g(x) = f^{-1}(x).$$

Likewise, we could also say that $f(x)$ is the inverse of $g(x)$ and denote it by

$$f(x) = g^{-1}(x).$$

Note that the -1 is NOT a power.

Hopefully this has been a bit of a reminder about inverse functions. What we care about now is how we take derivatives of inverse functions. This is summarized in the following theorem.

Definition 5: Inverse Function Theorem

Suppose $f(x)$ is a function which has an inverse, $f^{-1}(x)$, and both f and f^{-1} are differentiable. Then for all x such that $f'(f^{-1}(x)) \neq 0$,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

We can show where this formula comes from by using the chain rule. Now, we will be thinking about derivatives slightly differently here; just know we are still using all the same rules we have covered so far.

We start with the functions $f(x)$ and $f^{-1}(x)$. Since they are inverses of one another, we know that

$$x = f(f^{-1}(x)).$$

What we are going to do now is take the derivative of this equation with respect to x **on both sides of the equation**. This may seem like something new, but really we have been doing this all along. When we had $y = \dots$ and we took the derivative, that left hand side became $y' = \dots$ or $\frac{dy}{dx} = \dots$. So we did take the derivative of y , but since it is with respect to x , there is no way to "simplify" $\frac{dy}{dx}$ and so it is left as is.

So take the derivative of the equation above on both sides with respect to x . Note that on the right side we will need to use the chain rule, which we separate into steps here.

$$1 = f'(f^{-1}(x)) \cdot \frac{d}{dx} [f^{-1}(x)]$$

$$1 = f'(f^{-1}(x)) \cdot (f^{-1})'(x).$$

We can then take this last equation and solve for $(f^{-1})'(x)$, giving

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

This will be used below for logarithmic functions and later for inverse trigonometric function.

Logarithmic Functions

Recall that logarithmic functions are inverses of exponential functions. Specifically, for a positive real number $a \neq 1$,

$$a^{\log_a(x)} = x \quad \text{AND} \quad \log_a(a^x) = x.$$

Let's start with $f(x) = e^x$; note that $f'(x) = e^x$. So $f^{-1}(x) = \ln(x)$, the natural logarithm function (where the base of this logarithm is e). According to the inverse function theorem,

$$\frac{d}{dx} [\ln(x)] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

Note that this is for when $x > 0$ as that is required for the domain of $\ln(x)$. It can also be shown that,

$$\frac{d}{dx} (\ln |x|) = \frac{1}{x} \quad x \neq 0.$$

Using this all we need to avoid is $x = 0$. We can actually find the derivative of the general logarithm function quite simply. All that we need is the derivative of the natural logarithm, which we just found, and the change of base formula. Supposing a positive real number and not 1,

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Differentiation is then fairly simple.

$$\begin{aligned}
 \frac{d}{dx} [\log_a(x)] &= \frac{d}{dx} \left[\frac{\ln(x)}{\ln(a)} \right] \\
 &= \frac{1}{\ln(a)} \frac{d}{dx} (\ln(x)) \\
 &= \frac{1}{x \ln(a)}
 \end{aligned}$$

We took advantage of the fact that a was a constant and so $\ln(a)$ is also a constant and can be factored out of the derivative. Here is a summary of the derivatives of logarithms. Keep in mind the the derivative for $\ln(x)$ still follows the rule for the general logarithm, understanding that $\ln(e) = 1$.

Definition 6: Derivatives of Logarithmic Functions

Assuming a is positive real number which is not 1,

$$\frac{d}{dx} [\ln(x)] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx} [\log_a(x)] = \frac{1}{x \ln(a)}.$$

Okay, now that we have the derivations of the logarithms out of the way let's compute a couple of derivatives.

Example 2.5.2: Using derivatives of logarithmic functions

Differentiate each of the following functions.

- $R(w) = 4^w - 5\log_9(w)$
- $f(x) = 3e^x + 10x^3 \ln(x)$
- $g(x) = \ln(x^{-4} + x^4)$

Solutions:

- Here we just follow the derivative rules. We will need to use the exponential derivative rules for the first term and the logarithmic derivative rules for the second term.

$$R'(w) = 4^w \ln(4) - \frac{5}{w \ln(9)}$$

- Not much to this one. Just remember to use the product rule on the second term.

$$\begin{aligned}
 f'(x) &= 3e^x + 30x^2 \ln(x) + 10x^3 \left(\frac{1}{x} \right) \\
 &= 3e^x + 30x^2 \ln(x) + 10x^2
 \end{aligned}$$

- We will need the chain rule here. The outside function is the natural logarithm and the inside function is stuff on the inside of the logarithm.

$$g'(x) = \frac{1}{x^{-4} + x^4} (-4x^{-5} + 4x^3) = \frac{-4x^{-5} + 4x^3}{x^{-4} + x^4}$$

Again remember to leave the inside function alone when differentiating the outside function. So, upon differentiating the logarithm we end up not with $1/x$ but instead with $1/(\text{inside function})$.

Now practice some on your own.

Check Point 2.5.2: Practice with logarithmic derivatives

Differentiate each function.

a. $f(z) = \frac{\ln(5z)}{-9z-6}$

b. $f(w) = \log_5(\csc(w))$

Practice Problems

For problems 1 – 17 differentiate the given function.

1. $f(x) = 2e^x - 8^x$

2. $g(t) = 4\log_3(t) - \ln(t)$

3. $R(w) = 3^w \log(w)$

4. $y = z^5 - e^z \ln(z)$

5. $h(y) = \frac{y}{1 - e^y}$

6. $f(t) = \frac{1 + 5t}{\ln(t)}$

7. $f(t) = 5 + e^{4t+t^7}$

8. $g(x) = e^{1-\cos(x)}$

9. $H(z) = 2^{1-6z}$

10. $F(y) = \ln(1 - 5y^2 + y^3)$

11. $V(x) = \ln(\sin(x) - \cot(x))$

12. $f(x) = \ln(\sin(x)) - (x^4 - 3x)^{10}$

13. $q(t) = t^2 \ln(t^5)$

14. $K(x) = \frac{1 + e^{-2x}}{x + \tan(12x)}$

15. $f(x) = \cos(x^2 e^x)$

16. $f(t) = (e^{-6t} + \sin(2-t))^3$

17. $g(x) = (\ln(x^2 + 1) - \tan(6x))^{10}$

18. Find the tangent line to $f(x) = 7^x + 4e^x$ at $x = 0$.

19. Find the tangent line to $f(x) = \ln(x) \log_2(x)$ at $x = 2$.

20. Find the tangent line to $f(x) = 4\sqrt{2x} - 6e^{2-x}$ at $x = 2$.

21. Determine if $V(t) = \frac{t}{e^t}$ is increasing or decreasing at the following points.

a. $t = -4$

b. $t = 0$

c. $t = 10$

22. Determine if $G(z) = (z - 6) \ln(z)$ is increasing or decreasing at the following points.

a. $z = 1$

b. $z = 5$

c. $z = 20$

23. Determine where $A(t) = t^2 e^{5-t}$ is increasing and decreasing.

24. Determine where in the interval $[-1, 20]$ the function $f(x) = \ln(x^4 + 20x^3 + 100)$ is increasing and decreasing.
25. Determine the second derivative of $z = \ln(7 - x^3)$.
26. Determine the fourth derivative of $y = e^{-5z} + 8 \ln(2z^4)$.

Assignment Problems

For problems 1 – 35 differentiate the given function.

1. $g(z) = 10^z - 9^z$
2. $f(x) = 9\log_4(x) + 12\log_{11}(x)$
3. $h(t) = 6^t - 4e^t$
4. $R(x) = 20\ln(x) + \log_{123}(x)$
5. $Q(t) = (t^2 - 6t + 3)e^t$
6. $y = v + 8^v 9^v$
7. $U(z) = \log_4(z) - z^6 \ln(z)$
8. $h(x) = \log_3(x) \log(x)$
9. $f(w) = \frac{1 - e^w}{1 + 7e^w}$
10. $f(t) = \frac{1 + 4\ln(t)}{5t^3}$
11. $g(r) = \frac{r^2 + \log_7(r)}{7r}$
12. $V(t) = \frac{t^4 e^t}{\ln(t)}$
13. $f(t) = e^{1-t^2}$
14. $J(z) = e^{12z-z^6}$
15. $f(z) = e^{z+\ln(z)}$
16. $B(x) = 7^{\cos(x)}$
17. $z = 3^{x^2-9x}$
18. $R(z) = \ln(6z + e^z)$
19. $h(w) = \ln(w^7 - w^5 + w^3 - w)$
20. $g(t) = \ln(1 - \csc(t))$
21. $H(z) = \ln(6z) - 4\sec(z)$
22. $f(u) = e^{4u} - 6e^{-u} + 7e^{u^2-8u}$
23. $f(x) = \sqrt{6-x^4} \ln(10x+3)$
24. $h(v) = \sqrt{5v} + \ln(v^4) e^{6+9v}$
25. $f(x) = \frac{e^{x^2+8x}}{\sqrt{x^4+7}}$
26. $g(t) = \frac{\csc(1-t)}{1+e^{-t}}$
27. $U(w) = \ln(e^w \cos(w))$
28. $h(t) = \tan((5-t^2)\ln(t))$
29. $z = \ln\left(\frac{3+x}{2-x^2}\right)$
30. $g(v) = \sqrt{\frac{e^v}{7+2v}}$
31. $h(z) = (7z - z^2 + e^{5z^2+z})^{-4}$
32. $A(y) = \ln(7y^3 + \sin^2(y))$
33. $V(w) = \sqrt[4]{\cos(9-w^2) + \ln(6w+5)}$

34. $h(t) = \sin(t^3 e^{-6t})$
35. $B(r) = (e^{\sin(r)} - \sin(e^r))^8$
36. Find the tangent line to $f(x) = (1 - 8x)e^x$ at $x = -1$.
37. Find the tangent line to $f(x) = 3x^2 \ln(x)$ at $x = 1$.
38. Find the tangent line to $f(x) = 3e^x + 8 \ln(x)$ at $x = 2$.
39. Find the tangent line to $f(x) = e^{2x+4} - 8 \ln(x^2 - 3)$ at $x = -2$.
40. Determine if $U(y) = 4^y - 3e^y$ is increasing or decreasing at the following points.
- a. $y = -2$ b. $y = 0$ c. $y = 3$
41. Determine if $y(z) = \frac{z^2}{\ln(z)}$ is increasing or decreasing at the following points.
- a. $z = \frac{1}{2}$ b. $z = 2$ c. $z = 6$
42. Determine if $h(x) = x^2 e^x$ is increasing or decreasing at the following points.
- a. $x = -1$ b. $x = 0$ c. $x = 2$
43. Determine where $H(w) = (w^2 - 1)e^{2-w^2}$ is increasing and decreasing.
44. What percentage of $[3, 5]$ is the function $g(z) = e^{z^2-8} + 3e^{1-2z^2}$ decreasing?
45. The position of an object traveling horizontally in a straight line is given by $s(t) = \ln(2t^3 - 21t^2 + 36t + 200)$. During the first 10 hours of motion (assuming the motion starts at $t = 0$) what percentage of the time is the object moving to the right?
46. For the function $f(x) = 1 - \frac{x}{2} - \ln(2 + 9x - x^2)$ determine each of the following.
- a. The interval on which the function is defined.
- b. Where the function is increasing and decreasing.

For problems 47 – 48 determine the fourth derivative of the given function.

47. $R(x) = 2e^{-x} - 3e^{1+8x} + 9 \ln(6x)$
48. $f(t) = \ln(t^6) - \cos(4t) + 9 \sin(2t) + e^{7t}$

For problems 49 – 52 determine the fourth derivative of the given function.

49. $f(z) = \sin(1 + e^{2x})$

50. $f(u) = e^{4u^2+9u}$

51. $h(x) = \ln(x^2 - 3x)$

52. $g(z) = \ln(3 + \cos(z))$

53. Determine the third derivative of $y = e^{1-2t^3}$

Section 2.6 Solutions

1. Not much to do here other than take the derivative using the derivative formulas.

$$f'(x) = 2e^x - 8^x \ln(8)$$

2. Not much to do here other than take the derivative using the derivative formulas.

$$g'(t) = \frac{4}{t \ln(3)} - \frac{1}{t}$$

3. Note that we will need to start with the product rule.

$$R'(w) = 3^w \ln(3) \log(w) + \frac{3^w}{w \ln(10)}$$

Recall that $\log(x)$ is the "common logarithm" and so is really $\log_{10}(x)$.

4. We will need the product rule for the second term.

$$\frac{dy}{dx} = 5z^4 - e^z \ln(z) - \frac{e^z}{z}$$

5. We will need to use the quotient rule.

$$h'(y) = \frac{(1)(1 - e^y) - y(-e^y)}{(1 - e^y)^2} = \frac{1 - e^y + y e^y}{(1 - e^y)^2}$$

6. We will need to use the quotient rule.

$$f'(t) = \frac{5 \ln(t) - (1 + 5t) \left(\frac{1}{t}\right)}{[\ln(t)]^2} = \frac{5 \ln(t) - \frac{1}{t} - 5}{[\ln(t)]^2}$$

7. Note that we only need to use the Chain Rule on the second term as we can differentiate the first term without the Chain Rule.

Now, recall that for exponential functions outside function is the exponential function itself and the inside function is the exponent. The derivative is then,

$$f'(t) = (4 + 7t^6) e^{4t+t^7}.$$

8. For exponential functions remember that the outside function is the exponential function itself and the inside function is the exponent. The derivative is then,

$$g'(x) = \sin(x) e^{1-\cos(x)}.$$

9. For exponential functions remember that the outside function is the exponential function itself and the inside function is the exponent. The derivative is then,

$$H'(z) = -6(2^{1-6z}) \ln(2).$$

10. For this problem the outside function is (hopefully) clearly the logarithm and the inside function is the stuff inside of the logarithm. The derivative is then,

$$F'(y) = \frac{1}{1 - 5y^2 + y^3} (-10y + 3y^2) = \frac{-10y + 3y^2}{1 - 5y^2 + y^3}.$$

With logarithm problems remember that after differentiating the logarithm (i.e. the outside function) you need to substitute the inside function into the derivative. So, instead of getting just, $\frac{1}{y}$, we get the following (i.e. we plugged the inside function into the derivative),

$$\frac{1}{1 - 5y^2 + y^3}.$$

Then, we can't forget of course to multiply by the derivative of the inside function as we follow the chain rule.

11. For this problem the outside function is the logarithm and the inside function is the stuff inside of the logarithm. The derivative is then,

$$V'(x) = \frac{1}{\sin(x) - \cot(x)} (\cos(x) + \csc^2(x)) = \frac{\cos(x) + \csc^2(x)}{\sin(x) - \cot(x)}.$$

12. For this problem each term will require a separate application of the Chain Rule. The derivative is then,

$$f'(x) = \frac{\cos(x)}{\sin(x)} - 10(4x^3 - 3)(x^4 - 3x)^9 = \cot(x) - 10(4x^3 - 3)(x^4 - 3x)^9.$$

13. For this problem we'll need to do the Product Rule to start off the derivative. In the process we'll need to use the Chain Rule when we differentiate the second term.

The derivative is then,

$$q'(t) = 2t \ln(t^5) + t^2 \left(\frac{5t^4}{t^5} \right) = 2t \ln(t^5) + 5t.$$

14. For this problem we'll need to do the Quotient Rule to start off the derivative. In the process we'll need to use the Chain Rule when we differentiate both the numerator and the denominator. The derivative is then,

$$K'(x) = \frac{-2e^{-2x}(x + \tan(12x)) - (1 + e^{-2x})(1 + 12\sec^2(12x))}{(x + \tan(12x))^2}.$$

15. For this problem we'll start off using the Chain Rule, however when we differentiate the inside function we'll need to do the Product Rule. The derivative is then,

$$f'(x) = -(2xe^x + x^2e^x) \sin(x^2e^x).$$

16. This problem will require multiple uses of the Chain Rule and so we'll step through the derivative process to make each use clear. Here is the first step of the derivative and we'll need to use the Chain Rule in this step.

$$f'(t) = 3(e^{-6t} + \sin(2-t))^2 \frac{d}{dt} [e^{-6t} + \sin(2-t)]$$

In this step we can see that we'll need to use the Chain Rule on each of the terms. The derivative is then,

$$f'(t) = 3(e^{-6t} + \sin(2-t))^2 (-6e^{-6t} - \cos(2-t)).$$

17. This problem will require multiple uses of the Chain Rule and so we'll step through the derivative process to make each use clear. Here is the first step of the derivative and we'll need to use the Chain Rule in this step.

$$g'(x) = 10(\ln(x^2 + 1) - \tan(6x))^9 \frac{d}{dx} [\ln(x^2 + 1) - \tan(6x)]$$

In this step we can see that we'll need to use the Chain Rule on each of the terms. The derivative is then,

$$g'(x) = 10(\ln(x^2 + 1) - \tan(6x))^9 \left(\frac{2x}{x^2 + 1} - 6\sec^2(6x) \right).$$

18. We know that the derivative of the function will give us the slope of the tangent line so we'll need the derivative of the function.

$$f'(x) = 7^x \ln(7) + 4e^x$$

Now all we need to do is evaluate the function and the derivative at the point in question.

$$f(0) = 5 \qquad f'(0) = \ln(7) + 4$$

Lastly, we need to do write down the equation of the tangent line.

$$y - 5 = (\ln(7) + 4)(x - 0) \quad \text{OR} \quad y = (\ln(7) + 4)x + 5$$

19. We know that the derivative of the function will give us the slope of the tangent line so we'll need the derivative of the function.

$$f'(x) = \frac{\log_2(x)}{x} + \frac{\ln(x)}{x \ln(2)}$$

Now all we need to do is evaluate the function and the derivative at the point in question.

$$f(2) = \ln(2) \log_2(2) = \ln(2) \qquad f'(2) = \frac{\log_2(2)}{2} + \frac{\ln(2)}{2 \ln(2)} = 1$$

Lastly, we need to do write down the equation of the tangent line.

$$y - \ln(2) = (1)(x - 2) \quad \text{OR} \quad y = x - 2 + \ln(2)$$

20. We know that the derivative of the function will give us the slope of the tangent line so we'll need the derivative of the function. Differentiating each term will require the Chain Rule as well.

$$f(x) = 4(2x)^{\frac{1}{2}} - 6e^{2-x}$$

$$f'(x) = 4\left(\frac{1}{2}\right)(2x)^{-\frac{1}{2}}(2) - 6e^{2-x}(-1) = 4(2x)^{-\frac{1}{2}} + 6e^{2-x} = \frac{4}{\sqrt{2x}} + 6e^{2-x}$$

Now all we need to do is evaluate the function and the derivative at the point in question.

$$f(2) = 4(2) - 6e^0 = 2 \qquad f'(2) = \frac{4}{2} + 6e^0 = 8$$

Lastly, we need to do write down the equation of the tangent line.

$$y - 2 = (8)(x - 2) \quad \text{OR} \quad y = 8x - 14$$

21. a. We know that the derivative of the function will give us the rate of change for the function and so we'll need that.

$$V'(t) = \frac{(1)e^t - t(e^t)}{(e^t)^2} = \frac{e^t - te^t}{(e^t)^2} = \frac{1-t}{e^t}$$

Now, all we need to do is evaluate the derivative at the point in question. So,

$$V'(-4) = \frac{5}{e^{-4}} \approx 272.991 > 0.$$

$V'(-4) > 0$ and so the function must be increasing at $t = 4$.

Note that we did not need to find a decimal approximation to answer this question. $V'(-4) = \frac{5}{e^{-4}}$ and e^t is positive for all values of t , including -4 . And certainly 5 is positive. So a positive number divided by a positive number is always positive. Thus $V'(-4) > 0$.

- b. We found the derivative of the function in the first part so here all we need to do is the evaluation.

$$V'(0) = \frac{1}{e^0} = 1 > 0$$

$V'(0) > 0$ and so the function must be increasing at $t = 0$.

c. We found the derivative of the function in the first part so here all we need to do is the evaluation.

$$V'(10) = \frac{-9}{e^{10}} \approx -0.0004086 < 0$$

$V'(10) < 0$ and so the function must be decreasing at $t = 10$.

Again, we did not need to find a decimal approximation to answer this question. $V'(10) = \frac{-9}{e^{10}}$ and e^{10} is positive. So a negative number divided by a positive number is always negative. Thus $V'(10) < 0$.

22. a. We know that the derivative of the function will give us the rate of change for the function and so we'll need that. Note that we will need to use the product rule.

$$G'(z) = \ln(z) + \frac{z-6}{z}$$

Now, all we need to do is evaluate the derivative at the point in question. So,

$$G'(1) = \ln(1) - 5 = -5 < 0.$$

$G'(1) < 0$ and so the function must be decreasing at $z = 1$.

b. We found the derivative of the function in the first part so here all we need to do is the evaluation.

$$G'(5) = \ln(5) - \frac{1}{5} \approx 1.40944 > 0$$

$G'(5) > 0$ and so the function must be increasing at $z = 5$.

c. We found the derivative of the function in the first part so here all we need to do is the evaluation.

$$G'(20) = \ln(20) + \frac{7}{10} \approx 3.69573$$

$G'(20) > 0$ and so the function must be increasing at $z = 20$.

23. We'll first need the derivative because we know that the derivative will give us the rate of change of the function. Here is the derivative.

$$A'(t) = 2te^{5-t} - t^2e^{5-t} = te^{5-t}(2-t)$$

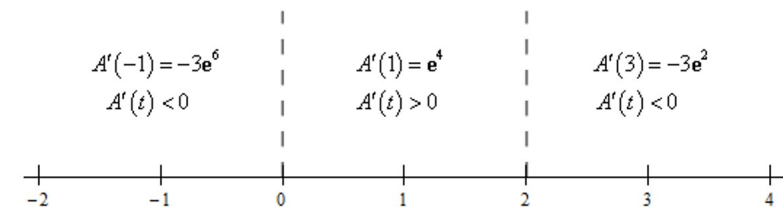
Note that we factored the derivative to help with the next step. In general, we don't need to do this.

Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve. In this case it's pretty easy to spot where the derivative will be zero since we have our derivative factored. Also note that e^{5-t} is positive for all values of t , thus it is never zero.

$$te^{5-t}(2-t) = 0 \quad \Rightarrow \quad t = 0, \quad t = 2$$

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. A quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.



From this we get the following increasing/decreasing information.

Increasing : $0 < t < 2$

Decreasing : $-\infty < t < 0, \quad 2 < t < \infty$

24. We'll first need the derivative because we know that the derivative will give us the rate of change of the function. Here is the derivative.

$$f'(x) = \frac{4x^3 + 60x^2}{x^4 + 20x^3 + 100} = \frac{4x^2(x+15)}{x^4 + 20x^3 + 100}$$

Note that we factored the numerator to help with the next step. In general, we don't need to do this.

Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve.

$$\frac{4x^2(x+15)}{x^4+20x^3+100} = 0 \quad \rightarrow \quad 4x^2(x+15) = 0 \quad \Rightarrow \quad x = 0, \quad x = -15$$

Note, that in general, we also need to be concerned with where the derivative is not defined as well. Functions can (and often do) change sign where they are not defined. In this case however we've restricted the interval down to a range where the function exists and is continuous on the given interval and so this is something we need to worry about for this problem.

Note as well that we really should at this point step back and recall that we are working on the interval $[1, 20]$. We are only interested in what is happening on this interval and so we should make sure that the points found above are inside the interval.

In this case only $x = 0$ is in the interval and so we'll need to exclude $x = 15$ from our work for the rest of this problem.

To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. A quick number line will give us the sign of the derivative for the various intervals.

So, we can see that, in this case function is increasing everywhere in the interval $[1, 20]$ except $x = 0$. Recall that at this point the derivative was zero and hence the function is not changing (and therefore can't be increasing).

So, the formal answer for this problem is,

$$\text{Increasing : } -1 \leq x < 0, \quad 0 < x \leq 20$$

Note that because we've only looked at what is happening in the interval $[1, 20]$ we can't say anything about the increasing/decreasing nature of the function outside of this interval.

25. Not much to this problem other than to take two derivatives so each step will show each successive derivative until we get to the second. The first derivative is then,

$$\frac{dz}{dx} = \frac{-3x^2}{7-x^3}.$$

Do not forget that often we will end up needing to do a quotient rule in the second derivative even though we did not need to do that in the first derivative. The second derivative is then,

$$\frac{d^2z}{dx^2} = \frac{-6x(7-x^3) - (-3x^2)(-3x^2)}{(7-x^3)^2} = \frac{-42x-3x^4}{(7-x^3)^2}.$$

26. Not much to this problem other than to take four derivatives so each step will show each successive derivative until we get to the fourth. The first derivative is then,

$$\frac{dy}{dz} = -5e^{-5z} + 8 \left(\frac{8z^3}{2z^4} \right) = -5e^{-5z} + 32z^{-1}.$$

The second derivative is,

$$\frac{d^2y}{dz^2} = 25e^{-5z} - 32z^{-2}.$$

The third derivative is,

$$\frac{d^3y}{dz^3} = -125e^{-5z} + 64z^{-3}.$$

The fourth, and final derivative for this problem, is,

$$\frac{d^4y}{dz^4} = 625e^{-5z} - 192z^{-4}.$$