

§2.8 Hyperbolic Functions and Their Derivatives

The last set of functions that we're going to be looking in this chapter at are the hyperbolic functions. In many physical situations combinations of e^x and e^{-x} arise fairly often. Because of this these combinations are given names. There are six hyperbolic functions and they are defined as follows.

Definition 1: Definition of the Hyperbolic Functions

$$\begin{aligned}\sinh(x) &= \frac{e^x - e^{-x}}{2} \\ \tanh(x) &= \frac{\sinh(x)}{\cosh(x)} \\ \operatorname{sech}(x) &= \frac{1}{\cosh(x)}\end{aligned}$$

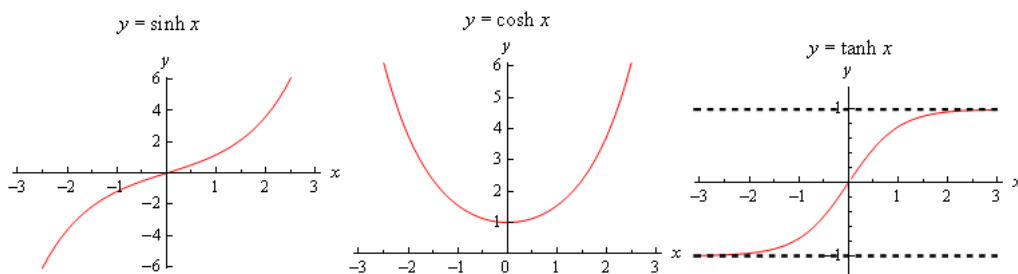
$$\begin{aligned}\cosh(x) &= \frac{e^x + e^{-x}}{2} \\ \coth(x) &= \frac{\cosh(x)}{\sinh(x)} = \frac{1}{\tanh(x)} \\ \operatorname{csch}(x) &= \frac{1}{\sinh(x)}\end{aligned}$$

Notice that the hyperbolic tangent, cotangent, secant, and cosecant are defined in a similar ways as with the standard trigonometric functions. This can make it easier to remember.

From the definitions, which rely on exponential functions, we can easily determine the domains and ranges of these functions.

	Domain	Range
$y = \sinh(x)$	$(-\infty, \infty)$	$(-\infty, \infty)$
$y = \cosh(x)$	$(-\infty, \infty)$	$[1, \infty)$
$y = \tanh(x)$	$(-\infty, \infty)$	$(-1, 1)$
$y = \coth(x)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
$y = \operatorname{sech}(x)$	$(-\infty, \infty)$	$(0, 1]$
$y = \operatorname{csch}(x)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Below are the graphs of the hyperbolic sine, cosine, and tangent functions.



While these six new functions have many interesting relationships with one another, here are some identities that are similar, but not the same, to some of the more common trigonometric identities.

$$\begin{aligned}\sinh(-x) &= -\sinh(x) \\ \cosh^2(x) - \sinh^2(x) &= 1\end{aligned}$$

$$\begin{aligned}\cosh(-x) &= \cosh(x) \\ 1 - \tanh^2(x) &= \operatorname{sech}^2(x)\end{aligned}$$

You should be able to verify these easily with the definitions of the functions, so we leave this as an exercise.

Derivatives of Hyperbolic Functions

Because the hyperbolic functions are defined in terms of exponential functions finding their derivatives is fairly simple. Just remember to use the chain rule when taking the derivative of e^{-x} . We'll do the derivative for hyperbolic sine and leave the rest to you as an exercise.

$$\begin{aligned}\frac{d}{dx} [\sinh(x)] &= \frac{d}{dx} \left[\frac{e^x - e^{-x}}{2} \right] \\ &= \frac{1}{2} \frac{d}{dx} [e^x - e^{-x}] \\ &= \frac{1}{2} (e^x - e^{-x}(-1)) \\ &= \frac{e^x + e^{-x}}{2} \\ &= \cosh(x)\end{aligned}$$

Note that there was no need to use the quotient rule as the denominator was just a constant.

The other derivatives can be found in a similar way or by using the quotient rule where appropriate. Here are all six derivatives.

Definition 2: Derivatives of Hyperbolic Functions

$$\begin{aligned}\frac{d}{dx} [\sinh(x)] &= \cosh(x) & \frac{d}{dx} [\cosh(x)] &= \sinh(x) \\ \frac{d}{dx} [\tanh(x)] &= \operatorname{sech}^2(x) & \frac{d}{dx} [\coth(x)] &= -\operatorname{csch}^2(x) \\ \frac{d}{dx} [\operatorname{sech}(x)] &= -\operatorname{sech}(x) \tanh(x) & \frac{d}{dx} [\operatorname{csch}(x)] &= -\operatorname{csch}(x) \coth(x)\end{aligned}$$

We see some similarities to the derivatives of the basic trigonometric functions, but be careful as they are not the same.

Let's do some practice.

Example 2.8.1: Using the derivatives of hyperbolic functions

Determine the derivative of each.

a. $f(x) = 2x^5 \cosh(x)$

b. $h(t) = \frac{\sinh(t)}{t+1}$

Solutions:

a. Here we use the product rule and our new formula for the derivative of $\cosh(x)$.

$$f'(x) = 10x^4 \cosh(x) + 2x^5 \sinh(x)$$

b. For this function, we'll need the new formula for the derivative of $\sinh(t)$ as well as the quotient rule.

$$h'(t) = \frac{(t+1) \cosh(t) - \sinh(t)}{(t+1)^2}$$

Check Point 2.8.1: Practice the derivatives of hyperbolic functions

Differentiate each function.

a. $f(x) = (9x)$

b. $g(x) = e^{\cosh(9x)}$

Inverse Hyperbolic Functions

From¹ the graphs of the hyperbolic functions, we see that all of them are one-to-one except $\cosh(x)$ and $\operatorname{sech}(x)$. If we restrict the domains of these two functions to the interval $[0, \infty)$, then all the hyperbolic functions are one-to-one, and we can define the inverse hyperbolic functions. Since the hyperbolic functions themselves involve exponential functions, the inverse hyperbolic functions involve logarithmic functions. Included are the alternate notations for these functions.

Definition 3: Definitions of the Inverse Hyperbolic Functions

$$\begin{aligned} \sinh^{-1}(x) &= \operatorname{arsinh}(x) = \ln \left(x + \sqrt{x^2 + 1} \right) & \cosh^{-1}(x) &= \operatorname{arcosh} = \ln \left(x + \sqrt{x^2 - 1} \right) \\ \tanh^{-1}(x) &= \operatorname{artanh} = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) & \coth^{-1}(x) &= \operatorname{arcoth} = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \\ \operatorname{sech}^{-1}(x) &= \operatorname{arsech} = \ln \left(\frac{1 + \sqrt{1-x^2}}{x} \right) & \operatorname{csch}^{-1}(x) &= \operatorname{arcsch} = \ln \left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|} \right) \end{aligned}$$

Before explaining how these come about, it is important to understand the domains and ranges of these functions. For example, the definition for the inverse hyperbolic tangent and cotangent look identical, but they are defined on entirely different domains.

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Access for free at <https://openstax.org/books/calculus-volume-1/pages/1-introduction>.

	Domain	Range
$y = \sinh^{-1}(x)$	$(-\infty, \infty)$	$(-\infty, \infty)$
$y = \cosh^{-1}(x)$	$[1, \infty)$	$[0, \infty)$
$y = \tanh^{-1}(x)$	$(-1, 1)$	$(-\infty, \infty)$
$y = \coth^{-1}(x)$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \operatorname{sech}^{-1}(x)$	$(0, 1]$	$[0, \infty)$
$y = \operatorname{csch}^{-1}(x)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Let's look at how to derive the formula for the inverse hyperbolic sine function. The others follow similarly.

Suppose $y = \sinh^{-1}(x)$. Then $x = \sinh(y)$ and, by the definition of the hyperbolic sine function,

$$x = \frac{e^y - e^{-y}}{2}.$$

Therefore,

$$e^y - 2x - e^{-y} = 0.$$

Multiplying this equation by e^y , which is always positive, we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

This can be solved like a quadratic equation in terms of e^y , with the solutions

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since $e^y > 0$, the only solution possible is the one with the addition. Finally, solving for y , we conclude that

$$\sinh^{-1}(x) = y = \ln\left(x + \sqrt{x^2 + 1}\right).$$

We leave the remaining definitions as exercises.

Derivatives of Inverse Hyperbolic Functions

We can use the definitions of the inverse hyperbolic functions to easily compute their derivatives. We will need to remember how to take the derivative of the natural logarithm function as well as using the chain rule.

Example 2.8.2: The derivative of the inverse hyperbolic sine function

Use the definition of the inverse hyperbolic sine function to determine the derivative of $y = \sinh^{-1}(x)$. Simplify your derivative.

Solution:

$$\begin{aligned}\frac{d}{dx} [\sinh^{-1}(x)] &= \frac{d}{dx} \left[\ln \left(x + \sqrt{x^2 + 1} \right) \right] \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{2x}{2\sqrt{x^2 + 1}} \right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \\ &= \frac{1}{\sqrt{x^2 + 1}}\end{aligned}$$

Here are the derivatives of all six inverse hyperbolic functions. Remember that the domains of the inverse hyperbolic tangent and cotangent are different, even though they look identical in their definitions.

Definition 4: Derivatives of Inverse Hyperbolic Functions

$$\begin{array}{ll}\frac{d}{dx} [\sinh^{-1}(x)] = \frac{1}{\sqrt{x^2 + 1}} & \frac{d}{dx} [\cosh^{-1}(x)] = \frac{1}{\sqrt{x^2 - 1}} \\ \frac{d}{dx} [\tanh^{-1}(x)] = \frac{1}{1 - x^2} & \frac{d}{dx} [\coth^{-1}(x)] = \frac{1}{1 - x^2} \\ \frac{d}{dx} [\operatorname{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1 - x^2}} & \frac{d}{dx} [\operatorname{csch}^{-1}(x)] = \frac{-1}{|x|\sqrt{x^2 + 1}}\end{array}$$

As always, we should do lots of practice problems with these new derivatives.

Example 2.8.3: Using the inverse hyperbolic derivatives

Evaluate the following derivatives.

- $\frac{d}{dx} \left[\cosh^{-1} \left(\frac{x}{3} \right) \right]$
- $\frac{d}{dx} \left[(\tanh^{-1}(x))^2 \right]$

Solutions:

- We need to use the chain rule along with the new derivative rules.

$$\frac{d}{dx} \left[\cosh^{-1} \left(\frac{x}{3} \right) \right] = \frac{1}{\sqrt{\left(\frac{x}{3} \right)^2 - 1}} \cdot \frac{1}{3}$$

$$= \frac{1}{\sqrt{\frac{x^2}{9} - 1}} \cdot \frac{1}{3}$$

$$= \frac{1}{\sqrt{x^2 - 9}}$$

b. We need to use the chain rule along with the new derivative rules.

$$\frac{d}{dx} \left[(\tanh^{-1}(x))^2 \right] = 2 \tanh^{-1}(x) \frac{1}{1-x^2} = \frac{2 \tanh^{-1}(x)}{1-x^2}$$

Check Point 2.8.2: Practice the inverse hyperbolic derivatives

Differentiate each function.

a. $y = \sinh^{-1}(4)$

b. $f(x) = 5 \cosh^{-1}(6xe^x)$.

Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a catenary. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries.

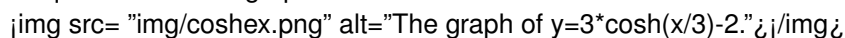
Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a \cosh\left(\frac{x}{a}\right)$ are catenaries.

Example 2.8.4

A chain hangs from two posts 2 m apart to form a catenary described by the equation $y = 3 \cosh\left(\frac{x}{3}\right) - 2$. Find the slope of the catenary at the left fence post.

Solution:

First, let's take a quick look at the graph of this function.



Since the shape of the hanging chain will be symmetric through the center, and that the posts are located 2 meters apart, we will imagine that the posts are located at the $x = -1$ and $x = 1$ locations.

So to find the slope of the catenary at the left post, we simply need to compute $\left. \frac{dy}{dx} \right|_{x=-1}$. Let's start by finding the derivative.

$$\frac{dy}{dx} = 3 \sinh\left(\frac{x}{3}\right) \cdot \frac{1}{3} = \sinh\left(\frac{x}{3}\right)$$

Now we evaluate this at $x = -1$.

$$\left. \frac{dy}{dx} \right|_{x=-1} = \sinh\left(\frac{-1}{3}\right) = \frac{1 - e^{2/3}}{2\sqrt[3]{e}} \approx -0.3395$$

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Practice Problems

For problems 1 – 7 differentiate the given function.

$$1. f(x) = \sinh(x) + 2 \cosh(x) - \operatorname{sech}(x)$$

$$2. R(t) = \tan(t) + t^2 \operatorname{csch}(t)$$

$$3. g(z) = \frac{z+1}{\tanh(z)}$$

$$4. k(w) = \cot(3w) + \coth(3w)$$

$$5. A(x) = \operatorname{sech}(\ln(x))$$

$$6. P(y) = \cosh^{-1}(\sinh(y))$$

$$7. v(t) = t^2 \tanh^{-1}(e^t)$$

Assignment Problems

For problems 1 – 12 differentiate the given function.

$$1. h(w) = w^2 - 3 \sinh(w)$$

$$2. g(x) = \cos(x) + \cosh(x)$$

$$3. H(t) = 3 \operatorname{csch}(t) + 7 \sinh(t)$$

$$4. A(r) = \tan(r) \tanh(r)$$

$$5. f(x) = e^x \cosh(x)$$

$$6. f(z) = \frac{\operatorname{sech}(z) + 1}{1 - z}$$

$$7. Q(w) = \frac{\coth(w)}{w + \sinh(w)}$$

$$8. m(t) = \operatorname{csch}(t \ln(t))$$

$$9. y = \frac{\tanh(4x)}{x + e^x}$$

$$10. g(z) = z \sinh^{-1}(3z) + z \sinh(3z)$$

$$11. p(x) = \operatorname{sech}^{-1}\left(\frac{x}{5}\right)$$

$$12. y = \ln\left(\tanh^{-1}\left(\frac{w}{2}\right)\right)$$

Section 2.8 Solutions

1.

$$f'(x) = \cosh(x) + 2 \sinh(x) + \operatorname{sech}(x) \tanh(x)$$

2. Remember to use the product rule for the second term.

$$R'(t) = \sec^2(t) + 2t \operatorname{csch}(t) - t^2 \operatorname{csch}(t) \coth(t)$$

3. Remember to use the quotient rule.

$$g'(z) = \frac{\tanh(z) - (z+1) \operatorname{sech}^2(z)}{\tanh^2(z)}$$

4. Not that the first term is the regular cotangent function.

$$k'(w) = -3 \csc^2(3w) - 3 \operatorname{csch}^2(3w)$$

5. We will use the chain rule.

$$A'(x) = \frac{-\operatorname{sech}(\ln(x)) \tanh(\ln(x))}{x}$$

6. We will use the chain rule.

$$P'(y) = \frac{1}{\sqrt{\sinh^2(y) - 1}} \cdot \cosh(y) = \frac{\cosh(y)}{\sqrt{\sinh^2(y) - 1}}$$

7. Make sure to use the product rule as well as the chain rule.

$$\begin{aligned} v'(t) &= t^2 \cdot \frac{1}{1 - (e^t)^2} \cdot e^t + 2t \tanh^{-1}(e^t) \\ &= \frac{t^2 e^t}{1 - e^{2t}} + 2t \tanh^{-1}(e^t) \end{aligned}$$