

§3.9 Linear Approximation and Differentials

In this section we discuss using the derivative to compute a linear approximation to a function. We can use the linear approximation to a function to approximate values of the function at certain points. While it might not seem like a useful thing to do with when we have the function there really are reasons that one might want to do this. We give two ways this can be useful in the examples.

Also, we will compute the differential for a function. We will give an application of differentials in this section. However, one of the more important uses of differentials will come in the next chapter and unfortunately we will not be able to discuss it until then.

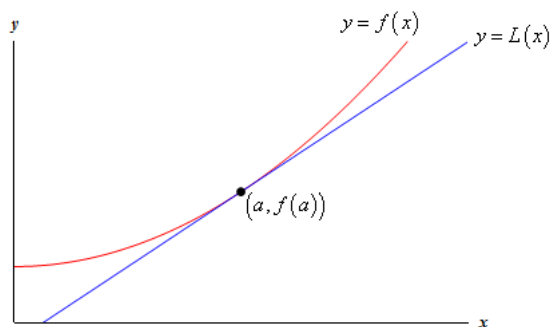
Linear Approximation

Linear approximation is really an application of the tangent line. Of course, to get the tangent line we do need to take derivatives, so in some way this is an application of derivatives as well.

Recall that the tangent line to the graph of $y = f(x)$ at $x = a$ is the line which best approximates $f(x)$ when x is *near* a . The equation of the tangent line, which we'll call $L(x)$ for this discussion, is,

$$L(x) = f(a) + f'(a)(x - a).$$

Again, this is just the tangent line, but we've given it a new name. Take a look at the following graph of a function and its tangent line.



From this graph we can see that near $x = a$ the tangent line and the function have nearly the same graph. On occasion we will use the tangent line, $L(x)$, as an approximation to the function, $f(x)$, near $x = a$. In these cases we call the tangent line the **linear approximation** to the function at $x = a$.

So, why would we do this? Let's take a look at an example.

Example 3.9.1

Determine the linear approximation for $f(x) = \sqrt[3]{x}$ at $x = 8$. Use this to approximate the values of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

Solution:

Since this is just the tangent line there really isn't a whole lot to finding the linear approximation.

$$f(8) = 2 \qquad f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}} \qquad f'(8) = \frac{1}{12}$$

The linear approximation is then,

$$L(x) = 2 + \frac{1}{12}(x - 8).$$

As for approximating values of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$, we recognize that $f(x) \approx L(x)$. Thus,

$$\begin{aligned} \sqrt[3]{8.05} &= f(8.05) \approx L(8.05) \\ &= 2 + \frac{1}{12}(8.05 - 8) \\ &= 2 + \frac{1}{12}(0.05) \\ &= 2 + \frac{1}{240} \\ &= 2.0041\bar{6} \end{aligned}$$

and

$$\begin{aligned} \sqrt[3]{25} &= f(25) \approx L(25) \\ &= 2 + \frac{1}{12}(25 - 8) \\ &= 2 + \frac{1}{12}(17) \\ &= 2 + 1.41\bar{6} \\ &= 3.41\bar{6}. \end{aligned}$$

So we have $\sqrt[3]{8.05} \approx 2.0041\bar{6}$ and $\sqrt[3]{25} \approx 3.41\bar{6}$ using $L(x)$.

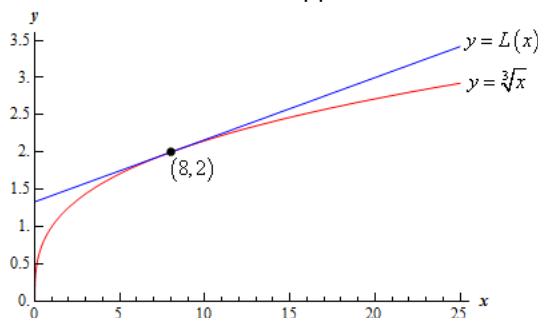
Just for fun, let's compare these to the actual values (still rounded slightly):

$$\begin{array}{ll} L(8.05) = 2.0041\bar{6} & \sqrt[3]{8.05} = 2.00415802 \\ L(25) = 3.41\bar{6} & \sqrt[3]{25} = 2.92401774 \end{array}$$

So, at $x = 8.05$ this linear approximation does a very good job of approximating the actual value. However, at $x = 25$ it doesn't do such a good job.

This shouldn't be too surprising if you think about it. Near $x = 8$ both the function and the linear approximation have nearly the same slope and since they both pass through the point $(8, 2)$ they should have nearly the same value as long as we stay close to $x = 8$. However, as we move away from $x = 8$ the linear approximation is a line and so will always have the same slope while the function's slope will change as x changes and so the function will, in all likelihood, move away from the linear approximation.

Here's a quick sketch of the function and its linear approximation at $x = 8$.



As noted above, the farther from $x = 8$, the more $f(x)$ and $L(x)$ differ in y -values. Thus $L(x)$ does not produce good approximations for values of $f(x)$.

Linear approximations do a very good job of approximating values of $f(x)$ as long as we stay "near" $x = a$. However, the farther away from $x = a$ we get the worse the approximation is liable to be. The main problem here is that how near we need to stay to $x = a$ in order to get a good approximation will depend upon both the function we're using and the value of $x = a$ that we're using. Also, there will often be no easy way of predicting how far away from $x = a$ we can get and still have a "good" approximation.

How good an approximation is will depend on how close we are to $x = a$, the function we're using, and the value of $x = a$ itself.

Check Point 3.9.1: Practice Linear Approximation

Use linear approximation to estimate $\sqrt{49.4}$.

Let's take a look at another example that is actually used fairly heavily in some places.

Example 3.9.2

Determine the linear approximation for $\sin(\theta)$ at $\theta = 0$.

Solution: Again, there really isn't a whole lot to this example. All that we need to do is compute the tangent line to $\sin(\theta)$ at $\theta = 0$.

$$\begin{aligned} f(\theta) &= \sin(\theta) \\ f(0) &= 0 \end{aligned}$$

$$\begin{aligned} f'(\theta) &= \cos(\theta) \\ f'(0) &= 1 \end{aligned}$$

The linear approximation is

$$\begin{aligned}
 L(\theta) &= f(0) + f'(0)(\theta - 0) \\
 &= 0 + 1 \cdot (\theta - 0) \\
 &= \theta.
 \end{aligned}$$

So, as long as θ stays small, we can say that $\sin(\theta) \approx \theta$.

This is actually a somewhat important linear approximation. In optics this linear approximation is often used to simplify formulas. This linear approximation is also used to help describe the motion of a pendulum and vibrations in a string.

In practice, one might not be given an actual function and "nice" value to use in an approximation. For example, say you wanted to find an approximation of the value of $\sqrt{10}$. One way to do that could be to use linear approximation, allowing $f(x) = \sqrt{x}$ and $x = 9$. Then $\sqrt{10} = f(10) \approx L(10)$. We might suspect that this approximation is fairly good since 10 is near $x = 9$ and choosing to do our approximation with $x = 9$ is nice since it gives nice values for $f(x)$ and $f'(x)$ while still being close to 10.

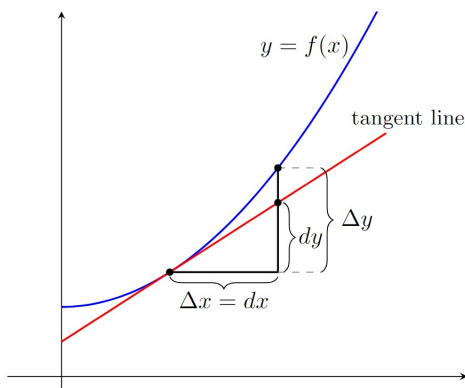
Differentials

Recall in the section on the definition of the derivative, we introduced a possible way to denote derivatives with the limit definition:

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

where Δx and Δy can be thought of as just the "change in x " and "change in y ", respectively. We also have typically treated $\frac{dy}{dx}$ as one big variable. In fact, we can think about dy and dx separately.

While Δy represents the difference in the y -values of a secant line for a certain difference of x -values (Δx), dy represents the different in y -values of the tangent line for the same difference in x -values ($dx = \Delta x$). See the graph below to get a better idea of these values:



To reiterate the idea of the derivative of a function, we would expect the values of $\frac{\Delta y}{\Delta x}$ to get close to the values of $\frac{dy}{dx}$ as the difference in the x -values gets very small ($\Delta x \rightarrow 0$). Thus we know that for a function $y = f(x)$,

$$\frac{dy}{dx} = f'(x).$$

With the idea of treating dy and dx separately, we can rewrite the above as

$$dy = f'(x) dx.$$

We call dy and dx **differentials**. Note that if we are just given $f(x)$ then the differentials are df and dx and we compute them in the same manner:

$$df = f'(x) dx.$$

While we are really just doing derivatives here, this new way of writing it out is quite important, (hopefully) becoming apparent in the next chapter. Let's work on computing some of these differentials.

Example 3.9.3

Compute the differential for each of the following.

a. $y = t^3 - 4y^2 + 7t$

b. $w = x^2 \sin(2x)$

c. $f(z) = e^{3-z^4}$

Hint:

Before working any of these we should first discuss just what we're being asked to find here. We defined two differentials earlier and here we're being asked to compute a differential.

So, which differential are we being asked to compute? In this kind of problem we're being asked to compute the differential of the function. In other words, dy for the first problem, dw for the second problem and df for the third problem.

Solutions:

a. This is just like finding $\frac{dy}{dt}$, the derivative, and "solving" for dy . In fact, let's do that for this example.

$$\begin{aligned} \frac{dy}{dt} &= 3t^2 - 8t + 7 \\ dt \cdot \left(\frac{dy}{dt} \right) &= (3t^2 - 8t + 7) \cdot dt \end{aligned}$$

$$dy = (3t^2 - 8t + 7)dt$$

- b. For this example, we'll skip the part of keeping $\frac{dw}{dx}$ on one side and go right to where we would have multiplied by dx . Keep in mind that we'll need the product rule to take the derivative of $x^2 \sin(2x)$.

$$dw = (2x \sin(2x) + 2x^2 \cos(2x)) dx$$

- c. Let's also jump right to the differentials for this one.

$$df = -4z^3 e^{3-z^4} dz$$

Check Point 3.9.2: Practice Differentials

Let $y = 2x^2 + 3x + 4$. If $\Delta x = 0.1$ at $x = 3$, estimate the value of Δy .

We can use differentials, such as dy and dx , and their relationships with Δy and Δx to do a few applications. Remembering that Δy and Δx are really just the change, or difference, in y and x , respectively, we can say $\Delta y = f(x + \Delta x) - f(x)$ for some function $f(x)$. As noted before, when $\Delta x = dx$ is small, we can assume that $\Delta y \approx dy$.

Example 3.9.4

Compute dy and Δy if $y = \cos(x^2 + 1) - x$ and x changes from $x = 2$ to $x = 2.03$.

Solution: First, let's compute the actual change in y , Δy .

$$\Delta y = [\cos((2.03)^2 + 1) - 2.03] - [\cos((2)^2 + 1) - 2] \approx 0.08358$$

To determine dy , we should first compute the differential, as we have done before.

$$dy = (-2x \sin(x^2 + 1) - 1) dx$$

Recall that $dx = \Delta x$. So we have $dx = 0.03$. Also, as x is starting at 2, then we'll use $x = 2$ to evaluate dy .

$$dy = (-2(2) \sin(2^2 + 1) - 1) \cdot (0.03) \approx 0.08507$$

We can see that, in fact, $\Delta y \approx dy$, provided we keep Δx small.

Check Point 3.9.3: Practice Differentials

Let $y = 2x^2 + 5x + 2$. Find the differential dy .

Let's see another example where we use the fact that $\Delta y \approx dy$.

Example 3.9.5

A sphere was measured and its radius was found to be 45 inches with a possible error of no more than 0.01 inches. What is the maximum possible error in the volume if we use this value of the radius?

Solution:

First, recall the equation for the volume of a sphere.

$$V = \frac{4}{3}\pi r^3$$

We can think of the error of the radius and volume as $\Delta r = 0.01$ and ΔV , respectively. So if we want to find ΔV , we can use dV to get a decent approximation (since $\Delta r = dr$ is small). Let's first find the differential dV .

$$dV = 4\pi r^2 dr$$

Now we can evaluate this with $r = 45$ (the "initial" measurement of the radius) and $dr = \Delta r = 0.01$. We'll also include our variables here.

$$\Delta V \approx dV = 4\pi(45)^2(0.01) \approx 254.47 \text{ in}^3$$

Thus the maximum error in the volume (the difference between the actual volumes of spheres with radius 45 and 45.01 in) is approximately 254.47 in³.

Be careful to not assume this is a large error. On the surface it looks large, however if we compute the actual volume for $r = 45$ we get $V = 381,703.51 \text{ in}^3$. So, in comparison the error in the volume is,

$$\frac{254.47}{381703.51} \times 100 = 0.067\%,$$

which is not much possible error at all!

§3.9.1 Practice Problems

For problems 1 and 2 find a linear approximation to the function at the given point.

1. $f(x) = 3xe^{2x-10}$ at $x = 5$

2. $h(t) = t^4 - 6t^3 + 3t - 7$ at $t = -3$

3. Find the linear approximation to $g(z) = \sqrt[4]{z}$ at $z = 2$. Use the linear approximation to approximate the value of $\sqrt[4]{3}$ and $\sqrt[4]{10}$. Compare the approximated values to the exact values.

4. Find the linear approximation to $f(t) = \cos(2t)$ at $t = \frac{1}{2}$. Use the linear approximation to approximate the value of $\cos(2)$ and $\cos(18)$. Compare the approximated values to the exact values.

5. Without using any kind of computational aid use a linear approximation to estimate the value of $e^{0.1}$.

For problems 6 – 8 compute the differential of the given function.

6. $f(x) = x^2 - \sec(x)$

7. $w = e^{x^4 - x^2 + 4x}$

8. $h(z) = \ln(2z) \sin(2z)$

9. Compute dy and Δy for $y = e^{x^2}$ as x changes from 3 to 3.01.
10. Compute dy and Δy for $y = x^5 - 2x^3 + 7x$ as x changes from 6 to 5.9.
11. The sides of a cube are found to be 6 feet in length with a possible error of no more than 1.5 inches. What is the maximum possible error in the volume of the cube if we use this value of the length of the side to compute the volume?

Section 3.9 Solutions

1. We'll need the derivative (which requires the product rule) as well as both the function and derivative evaluated at $x = 5$.

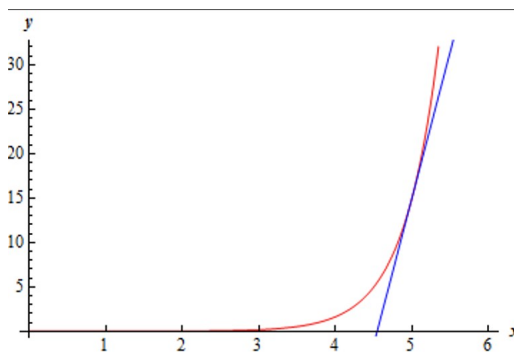
$$f'(x) = 3e^{2x-10} + 6xe^{2x-10} \quad f(5) = 15 \quad f'(5) = 33$$

s

There really isn't much to do at this point other than write down the linear approximation.

$$L(x) = 15 + 33(x - 5) = 33x - 150$$

While it wasn't asked for, here is a quick sketch of the function and the linear approximation.



2. We'll need the derivative first as well as a couple of function evaluations.

$$h'(t) = 4t^3 - 18t^2 + 3 \quad h(-3) = 227 \quad h'(-3) = -267$$

There really isn't much to do at this point other than write down the linear approximation.

$$L(t) = 227 - 267(t + 3) = -267t - 574$$

3. We'll need the derivative first as well as a couple of function evaluations.

$$g'(z) = \frac{1}{4}z^{-\frac{3}{4}} \quad g(2) = 2^{\frac{1}{4}} \quad g'(2) = \frac{1}{4}\left(2^{-\frac{3}{4}}\right)$$

Here is the linear approximation.

$$L(z) = 2^{1/4} + \frac{1}{4}\left(2^{-3/4}\right)(z - 2)$$

Finally, here are the approximations of the values along with the exact values.

So, as we might have expected the farther from $z = 2$ we got the worse the approximation is. Recall that the approximation will generally be more accurate the closer to the point of the linear approximation.

4. We'll need the derivative first as well as a couple of function evaluations.

$$f'(t) = -2 \sin(2t) \quad f\left(\frac{1}{2}\right) = \cos(1) \quad f'\left(\frac{1}{2}\right) = -2 \sin(1)$$

Here is the linear approximation.

$$L(t) = \cos(1) - 2 \sin(1) \left(t - \frac{1}{2}\right) = 0.5403 - 1.6829 \left(t - \frac{1}{2}\right)$$

Make sure you are using radians!

Now, if we want to approximate $\cos(2)$, that is equivalent to evaluating $f(1) = \cos(2)$, we need to evaluate the linear approximation at $t = 1$. Likewise, to approximate $\cos(18)$ we need to evaluate the linear approximation at $t = 9$.

So, here are the approximations of the values along with the exact values.

So, as we might have expected the farther from $t = \frac{1}{2}$ we got the worse the approximation is. Recall that the approximation will generally be more accurate the closer to the point of the linear approximation.

5. This is really the same problem as Problems 3 and 4 from this section. The difference is that we need to determine the function and point for the linear approximation.

Given the value we are being asked to estimate it should be fairly clear that the function should be $f(x) = e^x$.

The point for the linear approximation should also be somewhat clear. With the function in hand it's now clear that we are being asked to use a linear approximation to estimate $f(0.1)$. So, we'll need a point that is close to $x = 0.1$ and one that we can evaluate in the function without a calculator. It therefore seems fairly clear that $x = 0$ would be a really nice point use for the linear approximation.

At this point finding the linear approximation shouldn't be too bad so here is the work for that.

$$f'(x) = e^x \quad f(0) = 1 \quad f'(0) = 1$$

The linear approximation is then,

$$L(t) = 1 + (1)(x - 0) = x + 1.$$

The estimation of $e^{0.1}$ is then,

$$e^{0.1} \approx L(0.1) = 1.1$$

For comparison purposes the exact value is $f(0.1) = 1.10517$ and so we have an error of 0.467884%.

6. There is not really a whole lot to this problem. We need to simply differentiate each side, keeping the differentials df and dx on their respective sides.

$$df = (2x - \sec(x) \tan(x)) dx$$

7. There is not really a whole lot to this problem. Make sure you're using the chain rule!

$$dw = (4x^3 - 2x + 4) e^{x^4 - x^2 + 4x} dx$$

8. Let's get right to the differentials. Make sure you're using the product rule (as well as chain rule).

$$dh = \left(\frac{1}{z} \sin(2z) + 2 \ln(2z) \cos(2z) \right) dz$$

9. First let's get the actual change, Δy .

$$\Delta y = e^{3.01^2} - e^{3^2} = 501.927$$

Next, we'll need the differential.

$$dy = 2xe^{x^2} dx$$

As x changes from 3 to 3.01 we have $\Delta x = 3.013 - 3 = 0.013 \approx 0.01$ and $dx = \Delta x = 0.01$. The approximate change, dy , is then,

$$dy = 2(3) e^{3^2} (0.01) = 486.185$$

Don't forget to use the "starting" value of x (i.e. $x = 3$) for all the x 's in the differential.

10. First let's get the actual change, Δy .

$$\Delta y = (5.9^5 - 2(5.9^3) + 7(5.9)) - (6^5 - 2(6^3) + 7(6)) = -606.215$$

Next, we'll need the differential.

$$dy = (5x^4 - 6x^2 + 7) dx$$

As x changes from 6 to 5.9 we have $\Delta x = 5.96 - 6 = -0.04 \approx -0.1$ and $dx = \Delta x = -0.1$. The approximate change, dy , is then,

$$dy = (5(6^4) - 6(6^2) + 7)(-0.1) = -627.1$$

Don't forget to use the "starting" value of x (i.e. $x = 6$) for all the x 's in the differential

11. Let's get everything set up first. If we let the side of the cube be denoted by x the volume is then,

$$V(x) = x^3.$$

We are told that $x = 6$ and we can assume that $dx = \Delta x = \frac{1.5}{12} = 0.125$ (don't forget to convert the inches to feet!).

We want to estimate the maximum error in the volume and so we can again assume that $\Delta V \approx dV$.

The differential is then,

$$dV = 3x^2 dx$$

The maximum error in the volume is then,

$$\Delta V \approx dV = 3(6^2)(0.125) = 13.5 \text{ ft}^3$$