

§2.9 Implicit and Logarithmic Differentiation

To this point we've done quite a few derivatives, but they have all been derivatives of functions of the form $y = f(x)$. Unfortunately, not all the functions that we're going to look at will fall into this form.

Let's take a look at an example of a function like this.

Example 2.9.1

Find $\frac{dy}{dx}$ for $xy = 1$.

Solution:

There are actually two solution methods for this problem.

Solution 1:

This is the simple way of doing the problem. Just solve for y to get the function in the form that we're used to dealing with and then differentiate.

$$y = \frac{1}{x} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

So, that's easy enough to do. However, there are some functions for which this can't be done. That's where the second solution technique comes into play.

Solution 2:

In this case we're going to leave the function in the form that we were given and work with it in that form. However, let's recall from the first part of this solution that if we could solve for y then we will get y as a function of x . In other words, if we could solve for y (as we could in this case but won't always be able to do) we get $y = y(x)$. Let's rewrite the equation to note this.

$$xy = x y(x) = 1$$

Be careful here and note that when we write $y(x)$ we don't mean y times x . What we are noting here is that y is some (probably unknown) function of x . This is important to recall when doing this solution technique.

The next step in this solution is to differentiate both sides with respect to x as follows,

$$\frac{d}{dx} (x y(x)) = \frac{d}{dx} (1).$$

The right side is easy. It's just the derivative of a constant. The left side is also easy, but we've got to recognize that we've actually got a product here, the x and the $y(x)$. So, to do the derivative of the left side we'll need to do the product rule. Doing this gives,

$$(1) y(x) + x \frac{d}{dx} (y(x)) = 0.$$

Now, recall that we have the following notational way of writing the derivative.

$$\frac{d}{dx}(y(x)) = \frac{dy}{dx}$$

Using this we get the following,

$$y + x \frac{dy}{dx} = 0.$$

Note that we dropped the " (x) " on the y as it was only there to remind us that the y was a function of x and now that we've taken the derivative it's no longer really needed. We just wanted it in the equation to recognize the product rule when we took the derivative.

So, let's now recall just what were we after. We were after the derivative, $\frac{dy}{dx}$, and notice that there is now a $\frac{dy}{dx}$ in the equation. So, to get the derivative all that we need to do is solve the equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{y}{x}$$

There it is. Using the second solution technique this is our answer. This is not what we got from the first solution however. Or at least it doesn't look like the same derivative that we got from the first solution. Recall however, that we really do know what y is in terms of x and if we plug that in we will get,

$$\frac{dy}{dx} = -\frac{1/x}{x} = -\frac{1}{x^2}$$

which is what we got from the first solution. Regardless of the solution technique used we should get the same derivative.

The process that we used in the second solution to the previous example is called **implicit differentiation** and that is the subject of this section. In the previous example we were able to just solve for y and avoid implicit differentiation. However, in the remainder of the examples in this section we either won't be able to solve for y or, as we'll see in one of the examples below, the answer will not be in a form that we can deal with.

We have actually done a small amount of implicit differentiation in previous sections, but without calling it as such. In the section on exponential and logarithm derivatives, we used implicit differentiation to determine the derivative of inverse functions (the Inverse Function Theorem).

In the second solution above we replaced the y with $y(x)$ and then did the derivative. Recall that we did this to remind us that y is in fact a function of x . We'll be doing this quite a bit in these problems, although we rarely actually write $y(x)$. So, before we actually work anymore implicit differentiation problems let's do a quick set of "simple" derivatives that will hopefully help us with doing derivatives of functions that also contain a $y(x)$.

Example 2.9.2

Differentiate each of the following.

- a. $(5x^3 - 7x + 1)^5$, $[f(x)]^5$, $[y(x)]^5$
 b. $\sin(3 - 6x)$, $\sin(y(x))$
 c. e^{x^2-9x} , $e^{y(x)}$

Solutions:

- a. With the first function here we're being asked to do the following,

$$\begin{aligned}\frac{d}{dx} [(5x^3 - 7x + 1)^5] &= 5(5x^3 - 7x + 1)^4 \cdot \overbrace{\frac{d}{dx}(5x^3 - 7x + 1)}^{\text{Chain Rule}} \\ &= 5(5x^3 - 7x + 1)^4 (15x^2 - 7)\end{aligned}$$

and this is just the chain rule. We differentiated the outside function (the exponent of 5) and then multiplied that by the derivative of the inside function (the stuff inside the parenthesis).

For the second function we're going to do basically the same thing. We're going to need to use the chain rule. The outside function is still the exponent of 5 while the inside function this time is simply $f(x)$. We don't have a specific function here, but that doesn't mean that we can't at least write down the chain rule for this function. Here is the derivative for this function,

$$\frac{d}{dx} [f(x)]^5 = 5[f(x)]^4 f'(x).$$

We don't actually know what $f(x)$ is so when we do the derivative of the inside function all we can do is write down notation for the derivative, i.e. $f'(x)$.

With the final function here we simply replaced the f in the second function with a y since most of our work in this section will involve y 's instead of f 's. Outside of that this function is identical to the second, though we will use the notation $\frac{dy}{dx}$ for the derivative of $y(x)$. So, the derivative is,

$$\frac{d}{dx} [y(x)]^5 = 5[y(x)]^4 \frac{dy}{dx}.$$

- b. The first function to differentiate here is just a quick chain rule problem again so here is it's derivative,

$$\frac{d}{dx} [\sin(3 - 6x)] = \cos(3 - 6x) \cdot \overbrace{\frac{d}{dx}(3 - 6x)}^{\text{Chain Rule}}$$

$$= -6 \cos(3 - 6x).$$

For the second function we didn't bother this time with using $f(x)$ and just jumped straight to $y(x)$ for the general version. This is still just a general version of what we did for the first function. The outside function is still the sine and the inside is given by $y(x)$ and while we don't have a formula for $y(x)$ and so we can't actually take its derivative we do have a notation for its derivative. Here is the derivative for this function,

$$\begin{aligned} \frac{d}{dx} [\sin(y(x))] &= \cos(y(x)) \cdot \overbrace{\frac{dy}{dx}}^{\text{Chain Rule}} \\ &= \frac{dy}{dx} \cos(y(x)). \end{aligned}$$

- c. In this part we'll just give the answers for each and leave out the explanation that we had in the first two parts.

$$\frac{d}{dx} (e^{x^2-9x}) = (2x-9)e^{x^2-9x} \qquad \frac{d}{dx} (e^{y(x)}) = \frac{dy}{dx} e^{y(x)}$$

These are written a little differently from what we're used to seeing. This is because we want to match up these problems with what we'll be doing in this section. Also, each of these parts has several functions to differentiate starting with a specific function followed by a general function. This again, is to help us with some specific parts of the implicit differentiation process that we'll be doing.

So, in this set of examples we were just doing some chain rule problems where the inside function was $y(x)$ instead of a specific function. This kind of derivative shows up all the time in doing implicit differentiation so we need to make sure that we can do them. Also note that we only did this for three kinds of functions but there are many more kinds of functions that we could have used here.

So, it's now time to do our first problem where implicit differentiation is required, unlike the first example where we could actually avoid implicit differentiation by solving for y .

What we're doing in this section is the Chain Rule, but with $y(x)$ as the "inner" function instead of a specific function.

Example 2.9.3

Find $\frac{dy}{dx}$ for the following function.

$$x^2 + y^2 = 9$$

Solution:

Now, this is just a circle and we can solve for y which would give,

$$y = \pm\sqrt{9 - x^2}$$

Prior to starting this problem, we stated that we had to do implicit differentiation here because we couldn't just solve for y and yet that's what we just did. So, why can't we use "normal" differentiation here? The problem is the " $\pm$ ". With this in the "solution" for y we see that y is in fact two different functions. Which should we use? Should we use both? We only want a single function for the derivative and at best we have two functions here.

So, in this example we really are going to need to do implicit differentiation so we can avoid this. In this example we'll do the same thing we did in the first example and remind ourselves that y is really a function of x and write y as $y(x)$. Once we've done this all we need to do is differentiate each term with respect to x .

$$\frac{d}{dx} (x^2 + [y(x)]^2) = \frac{d}{dx} (9)$$

As with the first example the right side is easy. The left side is also pretty easy since all we need to do is take the derivative of each term and note that the second term will be similar the previous example. All we need to do for the second term is use the chain rule.

After taking the derivative we have,

$$2x + 2[y(x)]^1 \frac{dy}{dx} = 0.$$

At this point we can drop the (x) part as it was only in the problem to help with the differentiation process. The final step is to simply solve the resulting equation for $\frac{dy}{dx}$.

$$\begin{aligned} 2x + 2y \frac{dy}{dx} &= 0 \\ 2y \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Unlike the first example we can't just plug in for y since we wouldn't know which of the two functions to use. Most answers from implicit differentiation will involve both x and y so don't get excited about that when it happens.

As always, we can't forget our interpretations of derivatives.

Example 2.9.4

Find the equation of the tangent line to $x^2 + y^2 = 9$ at the point $(2, \sqrt{5})$.

Solution:

First note that unlike all the other tangent line problems we've done in previous sections we need to be given both the x and the y values of the point. Notice as well that this point does lie on the graph of the circle (you can check by plugging the points into the equation) and so it's okay to talk about the tangent line at this point.

Recall that to write down the tangent line all we need is the slope of the tangent line and this is nothing more than the derivative evaluated at the given point. We've got the derivative from the previous example so all we need to do is plug in the given point.

$$m = \left. \frac{dy}{dx} \right|_{x=2, y=\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

The tangent line is then

$$y = \sqrt{5} - \frac{2}{\sqrt{5}}(x - 2).$$

Now, let's work some more examples. In the remaining examples we will no longer write $y(x)$ for y . This is just something that we were doing to remind ourselves that y is really a function of x to help with the derivatives. Seeing the $y(x)$ reminded us that we needed to do the chain rule on that portion of the problem. From this point on we'll leave the y 's written as y 's and in our head we'll need to remember that they really are $y(x)$ and that we'll need to do the chain rule.

There is an easy way to remember how to do the chain rule in these problems. The chain rule really tells us to differentiate the function as we usually would, except we need to add on a derivative of the inside function. In implicit differentiation this means that every time we are differentiating a term with y in it the inside function is the y and we will need to multiply a $\frac{dy}{dx}$ onto the term since that will be the derivative of the inside function.

In implicit differentiation, when we differentiate anything involving a y , we'll need to multiply by $\frac{dy}{dx}$. This is multiplying by the derivative of the "inner", per the Chain Rule.

Example 2.9.5: Using implicit differentiation

Find $\frac{dy}{dx}$ for each of the following

- $x^3y^5 + 3x = 8y^3 + 1$
- $x^2 \tan(y) + y^{10} \sec(x) = 2x$
- $e^{2x+3y} = x^2 - \ln(xy^3)$

Solutions:

- a. First differentiate both sides with respect to x and remember that each y is really $y(x)$ we just aren't going to write it that way anymore. This means that the first term on the left will be a product rule.

We differentiated these kinds of functions involving y 's to a power with the chain rule previously. Also, recall the discussion prior to the start of this problem. When doing this kind of chain rule problem all that we need to do is differentiate the y 's as normal and then add on a $\frac{dy}{dx}$, which is nothing more than the derivative of the "inside function".

Here is the differentiation of each side for this function.

$$3x^2y^5 + 5x^3y^4 \frac{dy}{dx} + 3 = 24y^2 \frac{dy}{dx}$$

Now all that we need to do is solve for the derivative, $\frac{dy}{dx}$. This is just basic solving algebra that you are capable of doing. The main problem is that it's liable to be messier than what you're used to doing. All we need to do is get all the terms with $\frac{dy}{dx}$ in them on one side and all the terms without $\frac{dy}{dx}$ in them on the other. Then factor $\frac{dy}{dx}$ out of all the terms containing it and divide both sides by the "coefficient" of the $\frac{dy}{dx}$.

$$3x^2y^5 + 3 = 24y^2 \frac{dy}{dx} - 5x^3y^4 \frac{dy}{dx}$$

$$3x^2y^5 + 3 = (24y^2 - 5x^3y^4) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2y^5 + 3}{24y^2 - 5x^3y^4}$$

The algebra in these problems can be quite messy so be careful with that.

- b. We've got two product rules to deal with this time. Here is what we get when we take the derivative of both sides.

$$2x \tan(y) + x^2 \sec^2(y) \frac{dy}{dx} + 10y^9 \frac{dy}{dx} \sec(x) + y^{10} \sec(x) \tan(x) = 2$$

Notice the derivative tacked onto the secant! Again, this is just a chain rule problem similar as before.

Now, solve for the derivative, $\frac{dy}{dx}$.

$$\begin{aligned} (x^2 \sec^2(y) + 10y^9 \sec(x)) \frac{dy}{dx} &= 2 - y^{10} \sec(x) \tan(x) - 2x \tan(y) \\ \frac{dy}{dx} &= \frac{2 - y^{10} \sec(x) \tan(x) - 2x \tan(y)}{x^2 \sec^2(y) + 10y^9 \sec(x)} \end{aligned}$$

- c. We're going to need to be careful with this problem. We've got a couple chain rules that we're going to need to deal with here that are a little different from those that we've dealt with prior to this problem.

In both the exponential and the logarithm we've got a "standard" chain rule in that there is something other than just an x or y inside the exponential and logarithm. So, this means we'll do the chain rule as usual here and then when we do the derivative of the inside function for each term we'll have to deal with differentiating y 's.

Here is the derivative of this equation.

$$e^{2x+3y} \left(2 + 3 \frac{dy}{dx} \right) = 2x - \frac{y^3 + 3xy^2 \frac{dy}{dx}}{xy^3}$$

In both of the chain rules note that the $\frac{dy}{dx}$ didn't get tacked on until we actually differentiated the y 's in that term.

Now we need to solve for the derivative and this is liable to be somewhat messy. In order to get the $\frac{dy}{dx}$ on one side we'll need to multiply the exponential through the parenthesis and break up the quotient.

$$\begin{aligned} 2e^{2x+3y} + 3 \frac{dy}{dx} e^{2x+3y} &= 2x - \frac{y^3}{xy^3} - \frac{3xy^2 \frac{dy}{dx}}{xy^3} \\ 2e^{2x+3y} + 3 \frac{dy}{dx} e^{2x+3y} &= 2x - \frac{1}{x} - \frac{3 \frac{dy}{dx}}{y} \\ (3e^{2x+3y} + 3y^{-1}) \frac{dy}{dx} &= 2x - x^{-1} - 2e^{2x+3y} \\ \frac{dy}{dx} &= \frac{2x - x^{-1} - 2e^{2x+3y}}{3e^{2x+3y} + 3y^{-1}} \end{aligned}$$

Note that to make the derivative at least look a little nicer we converted all the fractions to negative exponents.

Check Point 2.9.1: Practice implicit differentiation

Find $\frac{dy}{dx}$ given

$$e^x y^2 + x^5 = e^y$$

Okay, we've seen one application of implicit differentiation in the tangent line example above. However, there is another application that we will be seeing in every problem in the next section.

In some cases we will have two (or more) functions all of which are functions of a third variable. So, we

might have $x(t)$ and $y(t)$, for example and in these cases, we will be differentiating with respect to t . This is just implicit differentiation like we did in the previous examples, but there is a difference however.

In the previous examples we have functions involving x 's and y 's and thinking of y as $y(x)$. In these problems we differentiated with respect to x and so when faced with x 's in the function we differentiated as normal and when faced with y 's we differentiated as normal except we then added a $\frac{dy}{dx}$ onto that term because we were really doing a chain rule.

In the new example we want to look at we're assuming that $x = x(t)$ and that $y = y(t)$ and differentiating with respect to t . This means that every time we are faced with an x or a y we'll be doing the chain rule. This in turn means that when we differentiate an x we will need to add on an $\frac{dx}{dt}$ and whenever we differentiate a y we will add on a $\frac{dy}{dt}$.

These new types of problems are really the same kind of problem we've been doing in this section. They are just expanded out a little to include more than one function that will require a chain rule.

Let's take a look at an example of this kind of problem.

Example 2.9.6

Assume that $x = x(t)$ and $y = y(t)$ and differentiate the following equation with respect to t .

$$x^3 y^6 + e^{1-x} - \cos(5y) = y^2$$

Solution:

So, just differentiate as normal and add on an appropriate derivative at each step. Note as well that the first term will be a product rule since both x and y are functions of t .

$$3x^2 \frac{dx}{dt} y^6 + 6x^3 y^5 \frac{dy}{dt} - \frac{dx}{dt} e^{1-x} + 5 \frac{dy}{dt} \sin(5y) = 2y \frac{dy}{dt}$$

There really isn't all that much to this problem. Since there are two derivatives in the problem we won't be bothering to solve for one of them. When we do this kind of problem in the next section the problem will imply what we need to solve for.

At this point there doesn't seem to be any real reason for doing this kind of problem, but as we'll see in the next section every problem that we'll be doing there will involve this kind of implicit differentiation.

Higher Order Implicit Derivatives

We can also use implicit differentiation to find higher order derivatives of implicit functions. As with regular higher order derivatives, we'll first start by finding the first derivative. Then if we want to find the second derivative, we'll take the derivative again on both sides with respect to the appropriate variable.

The process will still work the same, but we'll do a bit of extra work in the "simplifying" steps. It is best to see this with an example.

Example 2.9.7Find $\frac{d^2y}{dx^2}$ for

$$x^2 + y^4 = 10$$

Solution:

We know that in order to get the second derivative we need the first derivative and in order to get that we'll need to do implicit differentiation. Here is the work for that.

$$\begin{aligned} 2x + 4y^3 \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{2y^3} \end{aligned}$$

Now, this is the first derivative. We get the second derivative by differentiating this, which will require implicit differentiation again. Of course, on the left side, the derivative of $\frac{dy}{dx}$ with respect to x will just be $\frac{d^2y}{dx^2}$. And again, any time we get to the derivative of just y , it will be $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left[-\frac{x}{2y^3} \right] \\ &= -\frac{2y^3 - x \left(6y^2 \frac{dy}{dx} \right)}{(2y^3)^2} \\ &= -\frac{2y^3 - 6xy^2 \frac{dy}{dx}}{4y^6} \\ &= -\frac{y - 3x \frac{dy}{dx}}{2y^4} \end{aligned}$$

This is fine as far as it goes. However, we would like there to be no derivatives in the answer. We don't, generally, mind having x 's and/or y 's in the answer when doing implicit differentiation, but we really don't like derivatives in the answer. We can get rid of the derivative however by acknowledging that we know what the first derivative is and substituting this into the second derivative equation. Doing this gives,

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{y - 3x \frac{dy}{dx}}{2y^4} \\ &= -\frac{y - 3x \left(-\frac{x}{2y^3} \right)}{2y^4} \\ &= -\frac{y - 3x \left(-\frac{x}{2y^3} \right)}{2y^4} \cdot \frac{2y^3}{2y^3} \end{aligned}$$

$$= -\frac{2y^4 + 3x^2}{4y^7}$$

Check Point 2.9.2: Practice implicit differentiation for higher order derivatives

Find $\frac{d^2y}{dx^2}$ for

$$\sqrt{x} + \sqrt{y} = 12$$

Logarithmic Differentiation

Taking the derivatives of some complicated functions can be simplified by using logarithms. This is called logarithmic differentiation. It's easiest to see how this works in an example.

Take the function

$$y = \frac{x^5}{(1 - 10x)\sqrt{x^2 + 2}}.$$

Differentiating this function could be done with a product rule and a quotient rule. However, that would be a fairly messy process. We can simplify things somewhat by taking logarithms of both sides. We use the natural logarithm as its derivative is more simple than other based logarithms, but we technically could use any based logarithm.

$$\ln(y) = \ln\left(\frac{x^5}{(1 - 10x)\sqrt{x^2 + 2}}\right)$$

Of course, this isn't really simpler. What we need to do is use the properties of logarithms to expand the right side. Let's remind ourselves of those properties. We give these properties for the natural logarithm, but the properties are the same for any based logarithm.

- (1) $\ln(ab) = \ln(a) + \ln(b)$
- (2) $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
- (3) $\ln(a^b) = b\ln(a)$

We can apply these properties to our equation above, step by step until we can no longer expand further.

$$\begin{aligned}
\ln(y) &= \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right) \\
\ln(y) &= \ln(x^5) - \ln\left((1-10x)\sqrt{x^2+2}\right) && \text{Logarithm Property (2)} \\
\ln(y) &= \ln(x^5) - \left(\ln(1-10x) + \ln(\sqrt{x^2+2})\right) && \text{Logarithm Property (1)} \\
\ln(y) &= \ln(x^5) - \ln(1-10x) - \ln(\sqrt{x^2+2}) \\
\ln(y) &= 5\ln(x) - \ln(1-10x) - \frac{1}{2}\ln(x^2+2) && \text{Logarithm Property (3)}
\end{aligned}$$

Note that there is no property to simplify logarithms with addition or subtraction on the inside, thus all terms above are expanded as much as possible.

This doesn't look all that simple. However, the differentiation process will be simpler. What we need to do at this point is differentiate both sides with respect to x . Note that this is really implicit differentiation. So we will need to be careful as the left hand side is $\ln(y)$, not just y .

$$\begin{aligned}
\frac{1}{y} \cdot \overbrace{\frac{dy}{dx}}^{\text{Chain}} &= 5 \cdot \frac{1}{x} - \frac{1}{1-10x} \cdot \overbrace{\frac{d}{dx}(1-10x)}^{\text{Chain}} - \frac{1}{2} \cdot \frac{1}{x^2+2} \cdot \overbrace{\frac{d}{dx}(x^2+2)}^{\text{Chain Rule}} \\
\frac{1}{y} \cdot \frac{dy}{dx} &= 5 \cdot \frac{1}{x} - \frac{1}{1-10x} \cdot (-10) - \frac{1}{2} \cdot \frac{1}{x^2+2} \cdot (2x) \\
\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}
\end{aligned}$$

Now we are not done yet. Remember, the goal is to figure out what $\frac{dy}{dx}$ is equal to. So we need to take this equation and solve for $\frac{dy}{dx}$. To finish the problem all that we need to do is multiply both sides by y to isolate the $\frac{dy}{dx}$ and then plug in the original equation for y since we do know $y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$ from the original equation.

$$\begin{aligned}
\frac{dy}{dx} &= y \left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2} \right) \\
&= \frac{x^5}{(1-10x)\sqrt{x^2+2}} \left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2} \right)
\end{aligned}$$

Depending upon the person, doing this would probably be slightly easier than doing both the product and quotient rule. The answer is almost definitely simpler than what we would have gotten using the product and quotient rule.

For problems such as these, logarithmic differentiation is just a tool or method to help with derivatives of certain complicated functions. It is not a differentiation rule such as others we have learned previously. This

method is particularly helpful when your function has products, quotients, and powers, as these related to the properties of logarithms.

Example 2.9.8: Using logarithmic differentiation

Use logarithmic differentiation to find the derivative of

$$f(x) = \sqrt{\frac{x^5 \sin(4x)}{3 - x^2}}.$$

Solution:

Start by taking the natural logarithm on both sides.

$$\ln(f(x)) = \ln\left(\sqrt{\frac{x^5 \sin(4x)}{3 - x^2}}\right)$$

Now use the properties of logarithms to expand the right side. Note that we will start with the property involving powers as the main inside is a square root, or power of $\frac{1}{2}$. We show the expanding step by step.

$\ln(f(x)) = \frac{1}{2} \ln\left(\frac{x^5 \sin(4x)}{3 - x^2}\right)$	Logarithm Property 3
$\ln(f(x)) = \frac{1}{2} (\ln(x^5 \sin(4x)) - \ln(3 - x^2))$	Logarithm Property 2
$\ln(f(x)) = \frac{1}{2} (\ln(x^5) + \ln(\sin(4x)) - \ln(3 - x^2))$	Logarithm Property 1
$\ln(f(x)) = \frac{1}{2} (5 \ln(x) + \ln(\sin(4x)) - \ln(3 - x^2))$	Logarithm Property 3
$\ln(f(x)) = \frac{5}{2} \ln(x) + \frac{1}{2} \ln(\sin(4x)) - \frac{1}{2} \ln(3 - x^2)$	

Note that the $\frac{1}{2}$ from the first step needs to be multiplied by everything that comes after it. Be very careful with using those logarithm properties and use parentheses to help keep things straight. Since we can't expand any further, we can go ahead and take the derivative on both sides. Again, we are really using implicit differentiation with the left side.

$$\begin{aligned} \frac{1}{f(x)} f'(x) &= \frac{5}{2} \frac{1}{x} + \frac{1}{2} \frac{1}{\sin(4x)} 4 \cos(4x) - \frac{1}{2} \frac{1}{3 - x^2} (-2x) \\ \frac{1}{f(x)} f'(x) &= \frac{5}{2x} + 2 \cot(4x) + \frac{x}{3 - x^2} \end{aligned}$$

Finally, we solve for $f'(x)$ by multiplying both sides by $f(x)$ and substituting the original function back.

$$f'(x) = \left(\frac{5}{2x} + 2 \cot(4x) + \frac{x}{3 - x^2} \right) \sqrt{\frac{x^5 \sin(4x)}{3 - x^2}}$$

Check Point 2.9.3: Practice logarithmic differentiation

Let $f(x) = \frac{x^5 \sqrt{4x-3}}{(x^2+3)^5}$. Use logarithmic differentiation to find the derivative.

While logarithmic differentiation can be a helpful tool to avoid messy product and quotient rules, it is sometimes necessary to use. For example, take functions of the form

$$y = (f(x))^{g(x)}.$$

This is not a power function as the exponent is not constant, so we can not use the power rule. Nor is this an exponential function as the base is not a constant, so we can not use the derivative for exponential functions. However, logarithms can help simplify expressions with exponents, so let's see how logarithmic differentiation can help.

Let's take a quick look at a simple example of this.

Example 2.9.9

Differentiate $y = x^x$.

Solution:

First take the logarithm of both sides as we did in the first example and use the logarithm properties to simplify things a little.

$$\begin{aligned}\ln y &= \ln x^x \\ \ln y &= x \ln x\end{aligned}$$

Differentiate both sides using implicit differentiation.

$$\frac{\frac{dy}{dx}}{y} = \ln x + x \left(\frac{1}{x} \right) = \ln x + 1$$

Multiply by y and substitute back in for y .

$$\begin{aligned}\frac{dy}{dx} &= y(1 + \ln x) \\ \frac{dy}{dx} &= x^x(1 + \ln x)\end{aligned}$$

Now let's take a look at a more complicated example.

Example 2.9.10

Differentiate $y = (1 - 3x)^{\cos(x)}$.

Solution:

Now, this looks much more complicated than the previous example, but is in fact only slightly more complicated. The process is pretty much identical, so we first take the log of both sides and then simplify the right side.

$$\ln y = \ln \left[(1 - 3x)^{\cos(x)} \right] = \cos(x) \ln(1 - 3x)$$

Next, do some implicit differentiation.

$$\frac{\frac{dy}{dx}}{y} = -\sin(x) \ln(1 - 3x) + \cos(x) \frac{-3}{1 - 3x} = -\sin(x) \ln(1 - 3x) - \cos(x) \frac{3}{1 - 3x}$$

Finally, solve for $\frac{dy}{dx}$ and substitute back in for y .

$$\begin{aligned} \frac{dy}{dx} &= -y \left(\sin(x) \ln(1 - 3x) + \cos(x) \frac{3}{1 - 3x} \right) \\ &= -(1 - 3x)^{\cos(x)} \left(\sin(x) \ln(1 - 3x) + \cos(x) \frac{3}{1 - 3x} \right) \end{aligned}$$

Check Point 2.9.4

If $y = (4x)^{\sin(3x)}$, find $\frac{dy}{dx}$

We'll close this section out with a quick recap of all the various ways we've seen of differentiating functions with exponents. It is important to not get all of these confused.

Definition 1

$$\frac{d}{dx} (a^b) = 0$$

Constant Rule

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

Power Rule

$$\frac{d}{dx} (a^x) = a^x \ln a$$

Derivative of Exponentials

$$\frac{d}{dx} (x^x) = x^x (1 + \ln x)$$

Logarithmic Differentiation

Practice Problems

For problems 1 – 3 do each of the following.

- Find $\frac{dy}{dx}$ by solving the equation for y and differentiating directly.
- Find $\frac{dy}{dx}$ by implicit differentiation.
- Check that the derivatives in (a) and (b) are the same.

1. $\frac{x}{y^3} = 1$

2. $x^2 + y^3 = 4$

3. $x^2 + y^2 = 2$

For problems 4 – 9 find $\frac{dy}{dx}$ by implicit differentiation.

4. $2y^3 + 4x^2 - y = x^6$

7. $4x^2y^7 - 2x = x^5 + 4y^3$

5. $7y^2 + \sin(3x) = 12 - y^4$

8. $\cos(x^2 + 2y) + x e^{y^2} = 1$

6. $e^x - \sin(y) = x$

9. $\tan(x^2y^4) = 3x + y^2$

For problems 10–11 find the equation of the tangent line at the given point.

10. $x^4 + y^2 = 3$ at $(1, -\sqrt{2})$

11. $y^2e^{2x} = 3y + x^2$ at $(0, 3)$

For problems 12 & 13 assume that $x = x(t)$, $y = y(t)$, and $z = z(t)$ then differentiate the given equation with respect to t .

12. $x^2 - y^3 + z^4 = 1$

13. $x^2 \cos(y) = \sin(y^3 + 4z)$

For problems 14 & 15 determine $\frac{d^2y}{dx^2}$ with the given equation.

14. $2x^3 + y^2 = 1 - 4y$

15. $6y - xy^2 = 1$

For problems 16 – 18 use logarithmic differentiation to find the first derivative of the given function.

$$16. f(x) = (5 - 3x^2)^7 \sqrt{6x^2 + 8x - 12}$$

$$17. y = \frac{\sin(3z + z^2)}{(6 - z^4)^3}$$

$$18. h(t) = \frac{\sqrt{5t+8} \sqrt[3]{1-9\cos(4t)}}{\sqrt[4]{t^2+10t}}$$

For problems 19 & 20 find the first derivative of the given function.

$$19. g(w) = (3w - 7)^{4w}$$

$$20. f(x) = (2x - e^{8x})^{\sin(2x)}$$

Assignment Problems

For problems 1 – 6 do each of the following.

a. Find $\frac{dy}{dx}$ by solving the equation for y and differentiating directly.

b. Find $\frac{dy}{dx}$ by implicit differentiation.

c. Check that the derivatives in (a) and (b) are the same.

$$1. x^2 y^9 = 2$$

$$4. 8x - y^2 = 3$$

$$2. \frac{6x}{y^7} = 4$$

$$5. 4x - 6y^2 = xy^2$$

$$3. 1 = x^4 + 5y^3$$

$$6. \ln(xy) = x$$

For problems 7 – 21 find $\frac{dy}{dx}$ by implicit differentiation.

$$7. y^2 - 12x^3 = 8y$$

$$12. x + \ln(y) = \sec(y)$$

$$8. 3y^7 + x^{10} = y^{-2} - 6x^3 + 2$$

$$13. y^2(4 - x^2) = y^7 + 9x$$

$$9. y^{-3} + 4x^{-1} = 8y^{-1}$$

$$14. 6x^{-2} - x^3 y^2 + 4x = 0$$

$$10. 10x^4 - y^{-6} = 7y^3 + 4x^{-3}$$

$$15. 8xy + 2x^4 y^{-3} = x^3$$

$$11. \sin(x) + \cos(y) = e^{4y}$$

$$16. yx^3 - \cos(x) \sin(y) = 7x$$

17. $e^x \cos(y) + \sin(xy) = 9$

20. $e^{x^2+y^2} = e^{x^2y^2} + 1$

18. $x^2 + \sqrt{x^3 + 2y} = y^2$

21. $\sin\left(\frac{x}{y}\right) + x^3 = 2 - y^4$

19. $\tan(3x + 7y) = 6 - 4x^{-1}$

For problems 22 - 24 find the equation of the tangent line at the given point.

22. $3x + y^2 = x^2 - 19$ at $(-4, 3)$

23. $x^2y = y^2 - 6x$ at $(2, 6)$

24. $2 \sin(x) \cos(y) = 1$ at $\left(\frac{\pi}{4}, -\frac{\pi}{4}\right)$

For problems 25 - 27 determine if the function is increasing, decreasing or not changing at the given point.

25. $x^2 - y^3 = 4y + 9$ at $(2, -1)$

26. $e^{1-x}e^{y^2} = x^3 + y$ at $(1, 0)$

27. $\sin(\pi - x) + y^2 \cos(x) = y$ at $\left(\frac{\pi}{2}, 1\right)$

For problems 28 - 31 assume that $x = x(t)$, $y = y(t)$, and $z = z(t)$ then differentiate the given equation with respect to t .

28. $x^4 - 6z = 3 - y^2$

30. $z^7 e^{6y} = (y^2 - 8x)^{10} + z^{-4}$

29. $xy^4 = y^2z^3$

31. $\cos(z^2x^3) + \sqrt{y^2 + x^2} = 0$

For problems 32 - 35 determine $\frac{d^2y}{dx^2}$ with the given equation.

32. $6y - y^2 = 3x^4 + 9x$

34. $e^y + 4x = y^3 - 1$

33. $y^3 - 4x^2 = 11x - 2y^2$

35. $y \cos(x) = 3 + 4y^2$

For problems 36 - 41 use logarithmic differentiation to find the first derivative of the given function.

36. $h(x) = x^8 \cos(3x) (6 + 3x^2)^4$

38. $h(z) = \frac{(1 + 7z^2)^3}{(2 + 3z + 4z^2)^4}$

37. $f(w) = \sqrt{4 + 2w - 9w^2} \sqrt[5]{7w + 2w^3 + w^5}$

39. $g(x) = \frac{\sqrt{1 + \sin(2x)}}{2x - \tan(x)}$

40. $h(t) = \frac{(9-3t)^{10}}{t^2 \sin(7t)}$

41. $y = \frac{3+8x}{(1+2x^2)^4} \frac{\cos(1-x)}{(5x+x^2)^7}$

For problems 42 – 45 find the first derivative of the given function.

42. $y = x^{\ln(x)}$

44. $h(w) = (6-w^2)^{2+8w+w^3}$

43. $R(t) = [\sin(4t)]^{6t}$

45. $g(z) = z^2[3+z]^{1-z^2}$

Section 2.9 Solutions

1. a. First, we just need to solve the equation for y .

$$y^3 = x \quad \Rightarrow \quad y = x^{\frac{1}{3}}$$

Now differentiate with respect to x .

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

- b. First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y . Also, prior to taking the derivative a little rewrite might make this a little easier.

$$x y^{-3} = 1$$

Now take the derivative and don't forget that we actually have a product of functions of x here and so we'll need to use the Product Rule when differentiating the left side.

$$y^{-3} - 3x y^{-4} \frac{dy}{dx} = 0$$

Finally, all we need to do is solve this for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{y^{-3}}{3xy^{-4}} = \frac{y}{3x}$$

- c. From (a) we have a formula for y written explicitly as a function of x so plug that into the derivative we found in (b) and, with a little simplification/work, show that we get the same derivative as we got in (a).

$$\frac{dy}{dx} = \frac{y}{3x} = \frac{x^{\frac{1}{3}}}{3x} = \frac{1}{3}x^{-\frac{2}{3}}$$

So, we got the same derivative as we should.

2. a. First, we just need to solve the equation for y .

$$y^3 = 4 - x^2 \quad \Rightarrow \quad y = (4 - x^2)^{\frac{1}{3}}$$

Now differentiate with respect to x .

$$\frac{dy}{dx} = -\frac{2}{3}x(4 - x^2)^{-\frac{2}{3}}$$

- b. First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y . Taking the derivative gives,

$$2x + 3y^2 \frac{dy}{dx} = 0.$$

Finally, all we need to do is solve this for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{2x}{3y^2}$$

- c. From (a) we have a formula for y written explicitly as a function of x so plug that into the derivative we found in (b) and, with a little simplification/work, show that we get the same derivative as we got in (a).

$$\frac{dy}{dx} = -\frac{2x}{3y^2} = -\frac{2x}{3(4 - x^2)^{\frac{2}{3}}} = -\frac{2}{3}x(4 - x^2)^{-\frac{2}{3}}$$

So, we got the same derivative as we should.

3. a. First, we just need to solve the equation for y .

$$y^2 = 2 - x^2 \quad \Rightarrow \quad y = \pm(2 - x^2)^{\frac{1}{2}}$$

Note that because we have no restriction on y (i.e. we don't know if y is positive or negative) we really do need to have the " $\pm$ " there and that does lead to issues when taking the derivative.

Now, because there are two formulas for y we will also have two formulas for the derivative, one for each formula for y . The derivatives are then,

$$\begin{aligned} y = (2 - x^2)^{\frac{1}{2}} &\Rightarrow \frac{dy}{dx} = -x(2 - x^2)^{-\frac{1}{2}} & (y > 0) \\ y = -(2 - x^2)^{\frac{1}{2}} &\Rightarrow \frac{dy}{dx} = x(2 - x^2)^{-\frac{1}{2}} & (y < 0) \end{aligned}$$

As noted above the first derivative will hold for $y > 0$ while the second will hold for $y < 0$ and we can use either for $y = 0$ as the plus/minus won't affect that case.

- b. First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y . Taking the derivative gives,

$$2x + 2y \frac{dy}{dx} = 0$$

Finally, all we need to do is solve this for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{x}{y}$$

- c. From (a) we have a formula for y written explicitly as a function of x so plug that into the derivative we found in (b) and, with a little simplification/work, show that we get the same derivative as we got in (a).

Also, because we have two formulas for y we will have two formulas for the derivative.

First, if $y > 0$ we will have,

$$y = (2 - x^2)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{(2 - x^2)^{\frac{1}{2}}} = -x(2 - x^2)^{-\frac{1}{2}}.$$

Next, if $y < 0$ we will have,

$$y = -(2 - x^2)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-(2 - x^2)^{\frac{1}{2}}} = x(2 - x^2)^{-\frac{1}{2}}.$$

So, in both cases, we got the same derivative as we should.

4. First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y . Differentiating with respect to x gives,

$$6y^2 \frac{dy}{dx} + 8x - \frac{dy}{dx} = 6x^5.$$

Finally, all we need to do is solve this for $\frac{dy}{dx}$.

$$(6y^2 - 1) \frac{dy}{dx} = 6x^5 - 8x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{6x^5 - 8x}{6y^2 - 1}$$

5. First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y . Differentiating with respect to x gives,

$$14y \frac{dy}{dx} + 3 \cos(3x) = -4y^3 \frac{dy}{dx}$$

Finally, all we need to do is solve this for $\frac{dy}{dx}$.

$$(14y + 4y^3) \frac{dy}{dx} = -3 \cos(3x) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-3 \cos(3x)}{14y + 4y^3}$$

6. First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y . Differentiating with respect to x gives,

$$e^x - \cos(y) \frac{dy}{dx} = 1.$$

Don't forget the $\frac{dy}{dx}$ on the cosine after differentiating. Again, y is really $y(x)$ and so when differentiating $\sin(y)$ we really differentiating $\sin[y(x)]$ and so we are differentiating using the Chain Rule!

Finally, all we need to do is solve this for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1 - e^x}{-\cos(y)} = (e^x - 1) \sec(y)$$

7. First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y . This also means that the first term on the left side is really a product of functions of x and hence we will need to use the Product Rule when differentiating that term. Differentiating with respect to x gives,

$$8xy^7 + 28x^2y^6 \frac{dy}{dx} - 2 = 5x^4 + 12y^2 \frac{dy}{dx}.$$

Finally, all we need to do is solve this for $\frac{dy}{dx}$.

$$8xy^7 - 5x^4 - 2 = (12y^2 - 28x^2y^6) \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{8xy^7 - 5x^4 - 2}{12y^2 - 28x^2y^6}$$

8. First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y . This also means that the second term on the left side is really a product of functions of x and hence we will need to use the Product Rule when differentiating that term. Differentiating with respect to x gives,

$$-\left(2x + 2\frac{dy}{dx}\right) \sin(x^2 + 2y) + e^{y^2} + 2y \frac{dy}{dx} x e^{y^2} = 0.$$

Finally, all we need to do is solve this for $\frac{dy}{dx}$ (with some potentially messy algebra).

$$\begin{aligned} -2x \sin(x^2 + 2y) - 2\frac{dy}{dx} \sin(x^2 + 2y) + e^{y^2} + 2y \frac{dy}{dx} x e^{y^2} &= 0 \\ \left(2yx e^{y^2} - 2 \sin(x^2 + 2y)\right) \frac{dy}{dx} &= 0 + 2x \sin(x^2 + 2y) - e^{y^2} \\ \frac{dy}{dx} &= \frac{2x \sin(x^2 + 2y) - e^{y^2}}{2yx e^{y^2} - 2 \sin(x^2 + 2y)} \end{aligned}$$

9. First, we just need to take the derivative of everything with respect to x and we'll need to recall that y is really $y(x)$ and so we'll need to use the Chain Rule when taking the derivative of terms involving y . This also means that the when doing Chain Rule on the first tangent on the left side we will need to do Product Rule when differentiating the "inside term". Differentiating with respect to x gives,

$$\left(2x y^4 + 4x^2 y^3 \frac{dy}{dx}\right) \sec^2(x^2 y^4) = 3 + 2y \frac{dy}{dx}.$$

Finally, all we need to do is solve this for $\frac{dy}{dx}$ (with some potentially messy algebra).

$$\begin{aligned}
2x y^4 \sec^2(x^2 y^4) + 4x^2 y^3 \frac{dy}{dx} \sec^2(x^2 y^4) &= 3 + 2y \frac{dy}{dx} \\
(4x^2 y^3 \sec^2(x^2 y^4) - 2y) \frac{dy}{dx} &= 3 - 2x y^4 \sec^2(x^2 y^4) \\
\frac{dy}{dx} &= \frac{3 - 2x y^4 \sec^2(x^2 y^4)}{4x^2 y^3 \sec^2(x^2 y^4) - 2y}
\end{aligned}$$

10. The first thing to do is use implicit differentiation to find $\frac{dy}{dx}$ for this function.

$$4x^3 + 2y \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2x^3}{y}$$

Evaluating the derivative at the point in question to get the slope of the tangent line gives,

$$m = \left. \frac{dy}{dx} \right|_{x=1, y=-\sqrt{2}} = -\frac{2}{-\sqrt{2}} = \sqrt{2}.$$

Now, we just need to write down the equation of the tangent line.

$$y - (-\sqrt{2}) = \sqrt{2}(x - 1) \quad \text{OR} \quad y = \sqrt{2}(x - 1) - \sqrt{2} = \sqrt{2}(x - 2)$$

11. The first thing to do is use implicit differentiation to find $\frac{dy}{dx}$ for this function.

$$2y \frac{dy}{dx} e^{2x} + 2y^2 e^{2x} = 3 \frac{dy}{dx} + 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x - 2y^2 e^{2x}}{2y e^{2x} - 3}$$

Evaluating the derivative at the point in question to get the slope of the tangent line gives,

$$m = \left. \frac{dy}{dx} \right|_{x=0, y=3} = \frac{-18}{3} = -6.$$

Now, we just need to write down the equation of the tangent line.

$$y - 3 = -6(x - 0) \quad \text{OR} \quad y = -6x + 3$$

12. Differentiating with respect to t gives,

$$2x \frac{dx}{dt} - 3y^2 \frac{dy}{dt} + 4z^3 \frac{dz}{dt} = 0.$$

Note that because we were not asked to give the formula for a specific derivative we don't need to go any farther. We could however, if asked, solve this for any of the three derivatives that are present.

13. Differentiating with respect to t gives,

$$2x \frac{dx}{dt} \cos(y) - x^2 \sin(y) \frac{dy}{dt} = \left(3y^2 \frac{dy}{dt} + 4 \frac{dz}{dt} \right) \cos(y^3 + 4z)$$

Note that because we were not asked to give the formula for a specific derivative we don't need to go any farther. We could however, if asked, solve this for any of the three derivatives that are present.

14. Not much to this problem other than to take two derivatives so each step will show each successive derivative until we get to the second. Note however that we are going to have to do implicit differentiation to do each derivative. Here is the work for the first derivative.

$$\begin{aligned} 6x^2 + 2y \frac{dy}{dx} &= -4 \frac{dy}{dx} \\ (2y + 4) \frac{dy}{dx} &= -6x^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-6x^2}{2y + 4} = \frac{-3x^2}{y + 2} \end{aligned}$$

Now, the second derivative will also need implicit differentiation. Note as well that we can work with the first derivative in its present form which will require the quotient rule or we can rewrite it as,

$$\frac{dy}{dx} = -3x^2(y + 2)^{-1}$$

and use the product rule.

These get messy enough as it is so we'll go with the product rule to try and keep the "mess" down a little. Using implicit differentiation to take the derivative of first derivative gives,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = -6x(y + 2)^{-1} + 3x^2(y + 2)^{-2} \frac{dy}{dx}.$$

Finally, recall that we don't want a $\frac{dy}{dx}$ in the second derivative so to finish this out we need to plug in the formula for $\frac{dy}{dx}$ (which we know) and do a little simplifying to get the final answer.

$$\frac{d^2y}{dx^2} = -6x(y+2)^{-1} + 3x^2(y+2)^{-2} \left(-3x^2(y+2)^{-1} \right) = -6x(y+2)^{-1} - 9x^4(y+2)^{-3}$$

15. Not much to this problem other than to take two derivatives so each step will show each successive derivative until we get to the second. Note however that we are going to have to do implicit differentiation to do each derivative. Here is the work for the first derivative.

$$\begin{aligned} 6\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} &= 0 \\ (6 - 2xy)\frac{dy}{dx} &= y^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y^2}{6 - 2xy} \end{aligned}$$

Now, the second derivative will also need implicit differentiation. Note as well that we can work with the first derivative in its present form which will require the quotient rule or we can rewrite it as,

$$\frac{dy}{dx} = y^2(6 - 2xy)^{-1}$$

and use the product rule.

These get messy enough as it is so we'll go with the product rule to try and keep the "mess" down a little. Using implicit differentiation to take the derivative of first derivative gives,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = 2y \frac{dy}{dx} (6 - 2xy)^{-1} - y^2 (6 - 2xy)^{-2} \left(-2y - 2x \frac{dy}{dx} \right)$$

Finally, recall that we don't want a $\frac{dy}{dx}$ in the second derivative. So, to finish this out let's do a little "simplifying" of the to make it "easier" to plug in the formula for $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2y \frac{dy}{dx} (6 - 2xy)^{-1} + 2y^3 (6 - 2xy)^{-2} + 2xy^2 \frac{dy}{dx} (6 - 2xy)^{-2} \\ &= 2y \frac{dy}{dx} (6 - 2xy)^{-1} \left(1 + xy(6 - 2xy)^{-1} \right) + 2y^3 (6 - 2xy)^{-2} \end{aligned}$$

The point of all of this was to get down to a single $\frac{dy}{dx}$ in the formula for the second derivative, which won't always be possible to do, and a little factoring to try and make things a little easier to deal with.

Finally, all we need to do is plug in the formula for $\frac{dy}{dx}$ to get the final answer.

$$\begin{aligned}\frac{d^2y}{dx^2} &= 2y \left[y^2(6 - 2xy)^{-1} \right] (6 - 2xy)^{-1} \left(1 + xy(6 - 2xy)^{-1} \right) + 2y^3(6 - 2xy)^{-2} \\ &= 2y^3(6 - 2xy)^{-2} \left(1 + xy(6 - 2xy)^{-1} \right) + 2y^3(6 - 2xy)^{-2}\end{aligned}$$

Note that for a further simplification step, if we wanted to go further, we could factor a

$$2y^3(6 - 2xy)^{-2}$$

out of both terms to get,

$$\frac{d^2y}{dx^2} = 2y^3(6 - 2xy)^{-2} \left(2 + xy(6 - 2xy)^{-1} \right).$$

16. Take the logarithm of both sides and do a little simplifying.

$$\begin{aligned}\ln[f(x)] &= \ln \left[(5 - 3x^2)^7 \sqrt{6x^2 + 8x - 12} \right] \\ &= \ln \left[(5 - 3x^2)^7 \right] + \ln \left[(6x^2 + 8x - 12)^{\frac{1}{2}} \right] \\ &= 7 \ln(5 - 3x^2) + \frac{1}{2} \ln(6x^2 + 8x - 12)\end{aligned}$$

Use implicit differentiation to differentiate both sides with respect to x .

$$\frac{f'(x)}{f(x)} = 7 \frac{-6x}{5 - 3x^2} + \frac{1}{2} \frac{12x + 8}{6x^2 + 8x - 12} = \frac{-42x}{5 - 3x^2} + \frac{6x + 4}{6x^2 + 8x - 12}$$

Finally, solve for the derivative and plug in the equation for $f(x)$.

$$\begin{aligned}f'(x) &= f(x) \left(\frac{-42x}{5 - 3x^2} + \frac{6x + 4}{6x^2 + 8x - 12} \right) \\ &= (5 - 3x^2)^7 \sqrt{6x^2 + 8x - 12} \left(\frac{-42x}{5 - 3x^2} + \frac{6x + 4}{6x^2 + 8x - 12} \right)\end{aligned}$$

17. Take the logarithm of both sides and do a little simplifying.

$$\begin{aligned}\ln(y) &= \ln \left[\frac{\sin(3z + z^2)}{(6 - z^4)^3} \right] = \ln[\sin(3z + z^2)] - \ln[(6 - z^4)^3] \\ &= \ln[\sin(3z + z^2)] - 3 \ln[6 - z^4]\end{aligned}$$

Use implicit differentiation to differentiate both sides with respect to z .

$$\frac{\frac{dy}{dx}}{y} = \frac{(3 + 2z) \cos(3z + z^2)}{\sin(3z + z^2)} - 3 \left[\frac{-4z^3}{6 - z^4} \right] = (3 + 2z) \cot(3z + z^2) + \frac{12z^3}{6 - z^4}$$

Finally, solve for the derivative and plug in the equation for y .

$$\frac{dy}{dx} = y \left((3 + 2z) \cot(3z + z^2) + \frac{12z^3}{6 - z^4} \right) = \frac{\sin(3z + z^2)}{(6 - z^4)^3} \left((3 + 2z) \cot(3z + z^2) + \frac{12z^3}{6 - z^4} \right)$$

18. Take the logarithm of both sides and do a little simplifying.

$$\begin{aligned}\ln[h(t)] &= \ln \left[\frac{\sqrt{5t+8} \sqrt[3]{1-9\cos(4t)}}{\sqrt[4]{t^2+10t}} \right] \\ &= \ln \left[\sqrt{5t+8} \sqrt[3]{1-9\cos(4t)} \right] - \ln \left[\sqrt[4]{t^2+10t} \right] \\ &= \ln \left[(5t+8)^{\frac{1}{2}} \right] + \ln \left[(1-9\cos(4t))^{\frac{1}{3}} \right] - \ln \left[(t^2+10t)^{\frac{1}{4}} \right] \\ &= \frac{1}{2} \ln(5t+8) + \frac{1}{3} \ln(1-9\cos(4t)) - \frac{1}{4} \ln(t^2+10t)\end{aligned}$$

Note that the logarithm simplification work was a little complicated for this problem, but if you know your logarithm properties you should be okay with that.

Use implicit differentiation to differentiate both sides with respect to t .

$$\frac{h'(t)}{h(t)} = \frac{1}{2} \frac{5}{5t+8} + \frac{1}{3} \frac{36 \sin(4t)}{1-9\cos(4t)} - \frac{1}{4} \frac{2t+10}{t^2+10t}$$

Finally, solve for the derivative and plug in the equation for $h(t)$.

$$\begin{aligned}
 h'(t) &= h(t) \left(\frac{\frac{5}{2}}{5t+8} + \frac{12 \sin(4t)}{1-9 \cos(4t)} - \frac{\frac{1}{2}t + \frac{5}{2}}{t^2+10t} \right) \\
 &= \frac{\sqrt{5t+8} \sqrt[3]{1-9 \cos(4t)}}{\sqrt[4]{t^2+10t}} \left(\frac{5}{10t+16} + \frac{12 \sin(4t)}{1-9 \cos(4t)} - \frac{t+5}{2t^2+20t} \right)
 \end{aligned}$$

19. We just need to do some logarithmic differentiation so take the logarithm of both sides and do a little simplifying.

$$\ln[g(w)] = \ln[(3w-7)^{4w}] = 4w \ln(3w-7)$$

Use implicit differentiation to differentiate both sides with respect to w . Don't forget to product rule the right side.

$$\frac{g'(w)}{g(w)} = 4 \ln(3w-7) + 4w \frac{3}{3w-7} = 4 \ln(3w-7) + \frac{12w}{3w-7}$$

Finally, solve for the derivative and plug in the equation for $g(w)$.

$$\begin{aligned}
 g'(w) &= g(w) \left(4 \ln(3w-7) + \frac{12w}{3w-7} \right) \\
 &= (3w-7)^{4w} \left(4 \ln(3w-7) + \frac{12w}{3w-7} \right)
 \end{aligned}$$

20. We just need to do some logarithmic differentiation so take the logarithm of both sides and do a little simplifying.

$$\ln[f(x)] = \ln[(2x - e^{8x})^{\sin(2x)}] = \sin(2x) \ln(2x - e^{8x})$$

Use implicit differentiation to differentiate both sides with respect to w . Don't forget to product rule the right side.

$$\begin{aligned}
 \frac{f'(x)}{f(x)} &= 2 \cos(2x) \ln(2x - e^{8x}) + \sin(2x) \frac{2 - 8e^{8x}}{2x - e^{8x}} \\
 &= 2 \cos(2x) \ln(2x - e^{8x}) + \sin(2x) \frac{2 - 8e^{8x}}{2x - e^{8x}}
 \end{aligned}$$

Finally, solve for the derivative and plug in the equation for $f(x)$.

$$\begin{aligned} f'(x) &= f(x) \left(2 \cos(2x) \ln(2x - e^{8x}) + \sin(2x) \frac{2 - 8e^{8x}}{2x - e^{8x}} \right) \\ &= (2x - e^{8x})^{\sin(2x)} \left(2 \cos(2x) \ln(2x - e^{8x}) + \sin(2x) \frac{2 - 8e^{8x}}{2x - e^{8x}} \right) \end{aligned}$$