§3.2 Absolute Extrema

Definitions

Many of the applications in this chapter involve the minimum and maximum values of a function. While we can all visualize minimum and maximum values on a graph, there are some subtleties that we need to pay attention to when talking about them. In particular, we want to differentiate between what we will call **relative** and **absolute** minimum and maximum points.

Before we state the definitions, here is a quick note on terminology. The plurals of ”minimum” and ”maximum” are ”minima” and ”maxima,” respectively. Also, we will call all of the minimum and maximum points of a function the **extrema** of the function. So ”relative extrema” will refer to the relative minima and maxima, while ”absolute extrema” will refer to the absolute minima and maxima. A single extreme point is called an ”extremum” of the function.

Definition 1: Extrema

1. We say that $f(x)$ has an **absolute (or global) maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in the domain in which we're working.
2. We say that $f(x)$ has a **relative (or local) maximum** at $x = c$ if $f(x) \leq f(c)$ for every x in some open interval around $x = c$.
3. We say that $f(x)$ has an **absolute (or global) minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in the domain in which we're working.
4. We say that $f(x)$ has a **relative (or local) minimum** at $x = c$ if $f(x) \geq f(c)$ for every x in some open interval around $x = c$.

When we say an ”open interval around $x = c$” in the definition, we mean that we can find some interval (a, b) that does not include the endpoints and such that acb. That is, c is in the interval but is not one of the endpoints.

So what is the difference between relative and absolute extrema? This can be a little subtle.

A function may have an absolute maximum (or minimum) at $x = c$ provided that $f(c)$ is the largest (or smallest) value that the function will ever take **on the domain that we're working on.** When we say, ”the domain we are working on,” we mean the set of x values we have chosen to work with in a given problem. It may not be the entire domain of the function because we may have chosen to restrict the domain for some reason.

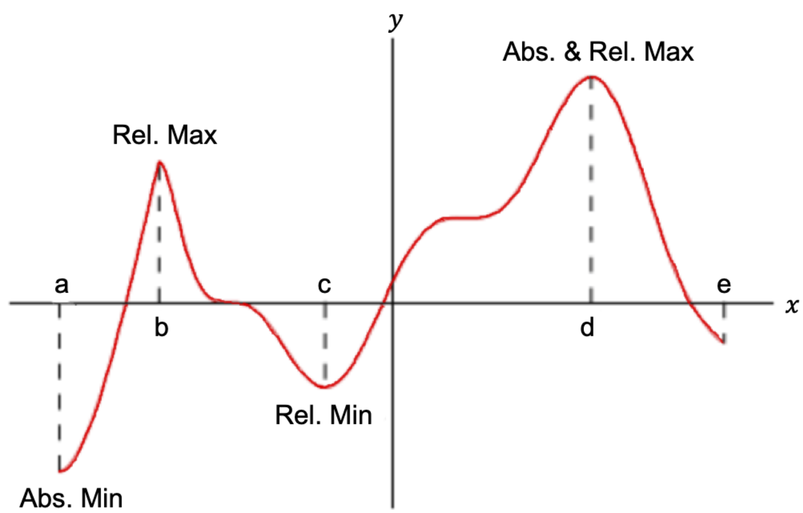
An **absolute** max or min is the largest or smallest output value on a specified domain.

The situation with a relative maximum or minimum is slightly different. To be a relative maximum (or minimum), a point just needs to be the largest (or smallest) value of the function in some open interval of x values that contains $x = c$. There may be larger (or smaller) values of the function at some other place, but $f(c)$ is the largest (or smallest) "in its neighborhood."

Note as well that in order for a point to be a relative extremum, we must be able to look at function values at x's on **both** sides of $x = c$. This means that relative extrema do not occur at the endpoints of a domain. This will be discussed in a little more detail at the end of the section.

A **relative** max or min is the largest or smallest output value in its neighborhood, which must include x-values on both sides of $x = c$.

To get a feel for the definitions, let's look at the graph below.



The function shown in this graph has relative maxima at $x = b$ and $x = d$. Both of these points are relative maxima because they are interior to the domain being considered and their y values are the largest on the graph in some interval around the corresponding x values.

The graph also has a relative minimum at $x = c$ since this point is interior to the domain under consideration and its y value is the lowest on the graph in an interval around its x value.

This function has an absolute maximum at $x = d$ and an absolute minimum at $x = a$. The y values at these two points are the largest and smallest that the function will ever be on the domain shown. We'll observe here that the absolute extrema for a function will occur at either the endpoints of the domain or at relative extrema. This idea will be important for us later in the chapter.

Finally, we observe that the point at $x = e$ on the graph is not a relative minimum because it is not interior to the domain under consideration, it occurs at an endpoint.

Examples

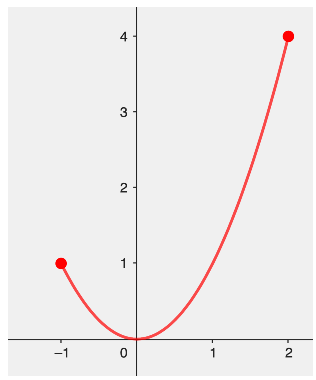
Let's take a look at some examples to make sure that we have the definitions of absolute extrema and relative extrema straight.

Example 3.2.1

Identify the absolute extrema and relative extrema for the function $f(x) = x^2$ on the domain $[-1, 2]$.

Solution:

Let's take a look at the graph. Since the domain is restricted to $[-1, 2]$, that is all we'll include. We will use dots at the ends of the graph to indicate that the graph ends at those points.



We can now identify the extrema from the graph. There is a relative and absolute minimum of zero at $x = 0$ and an absolute maximum of four at $x = 2$.

Note that there is not a relative maximum at $x = -1$ since this is an endpoint of the domain interval. This function does not have any relative maxima.

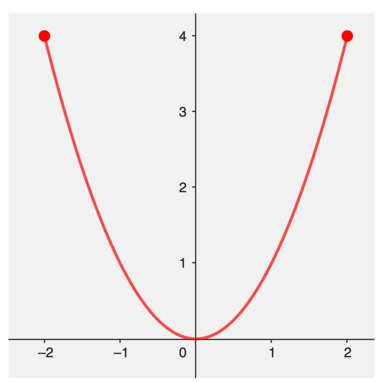
This example shows us that functions do not have to have any relative extrema.

Example 3.2.2

Identify the absolute extrema and relative extrema for the function $f(x) = x^2$ on the domain $[-2, 2]$.

Solution:

Here is the graph of this function on this domain.



In this case there is still a relative and absolute minimum of zero at $x = 0$, and there is still an absolute maximum of four. However, for this graph, the absolute maximum occurs at both $x = 2$ and $x = -2$.

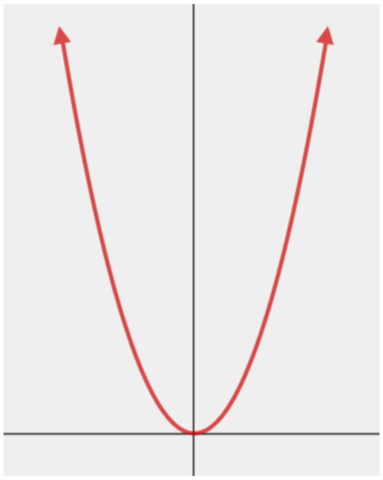
Again, the function does not have any relative maxima.

As this example has shown, there can only be a single absolute maximum and a single absolute minimum **value**, but those can occur at more than one place in the domain.

Example 3.2.3

Identify the absolute extrema and relative extrema for the function $f(x) = x^2$.

Solution:



This time, we were not given a domain interval, so we will take the largest possible domain. For this function, that is all real numbers. Here is the graph:

In this case the graph increases at both ends without stopping, so there are no maxima of either kind for this function. In other words, no matter which point we choose on the graph, there will be a point that is higher than it on one side or the other.

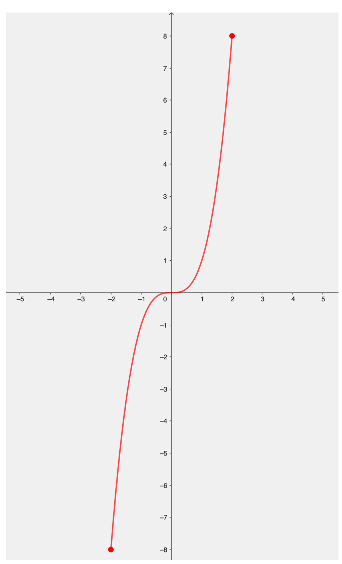
There is still a relative and absolute minimum value of zero at $x = 0$.

So, some graphs have a minimum but no maximum. Others might have a maximum but no minimum.

Example 3.2.4

Identify the absolute extrema and relative extrema for the function $f(x) = x^3$ on the domain $[-2, 2]$.

Solution: Here is the graph of this function on this domain:



This function has an absolute maximum of eight at $x = 2$ and an absolute minimum of negative eight at $x = -2$. This function has no relative extrema.

Example 3.2.5

Identify the absolute extrema and relative extrema for the function $f(x) = x^3$.

Solution:

There is no restriction on the domain, so we will consider the function on its entire domain, all real numbers. Here is the graph:

This function has no relative extrema or absolute extrema.

This last example shows that a function may have no extrema of any kind.

Example 3.2.6

Identify the absolute extrema and relative extrema for the function $f(x) = \cos(x)$.

Solution:

There is no restriction on the domain for this function, so we will use its entire domain, all real numbers. Here is the graph:

Cosine has both relative and absolute maxima of one at $x = \ldots -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

Cosine also has both relative and absolute minima of negative one at $x = \ldots -3\pi, -\pi, \pi, 3\pi, \ldots$

This example shows that a function can have extrema at a large number of points (infinitely-many, in this case).

The Extreme Value Theorem

We've worked quite a few examples now, and we can use these examples to see a nice fact about absolute extrema, which we'll state as a theorem in a moment. First notice that all the functions in the examples above were continuous functions. Next notice that every time the domain was restricted to a closed interval (that is, an interval that contains its endpoints), the function had absolute maxima and absolute minima on that interval. Finally, in only one of the three examples in which the domain was not restricted did we find both an absolute maximum and an absolute minimum.

These observations lead us to the following theorem:

Definition 2: Extreme Value Theorem

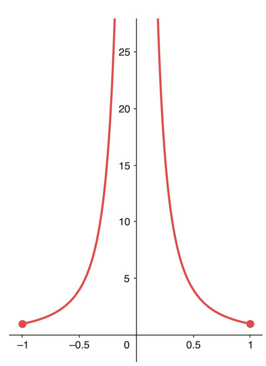
If $f(x)$ is continuous on the closed interval $[a, b]$, then there are two numbers c and d with $a \leq c \leq b$ and $a \leq d \leq b$ such that $f(c)$ is an absolute maximum and $f(d)$ is an absolute minimum for the function.

This theorem guarantees that if we have a function that is continuous on a closed interval $[a, b]$, then we will be able to find both an absolute maximum and an absolute minimum for the function somewhere in the interval. The theorem doesn't tell us where they will occur or if they will occur more than once, but at least it tells us that they do exist somewhere. We will develop a strategy for finding the absolute extrema in the next part of this section.

This theorem also doesn't say anything about absolute extrema if we are not working on a closed interval. In fact, we saw examples of functions above that had both kinds of absolute extrema, had just one absolute extremum, or had no absolute extrema when we didn't restrict the domain to an interval.

The requirement that the function be continuous is necessary for us to use the theorem. Let's look at an example to see why.

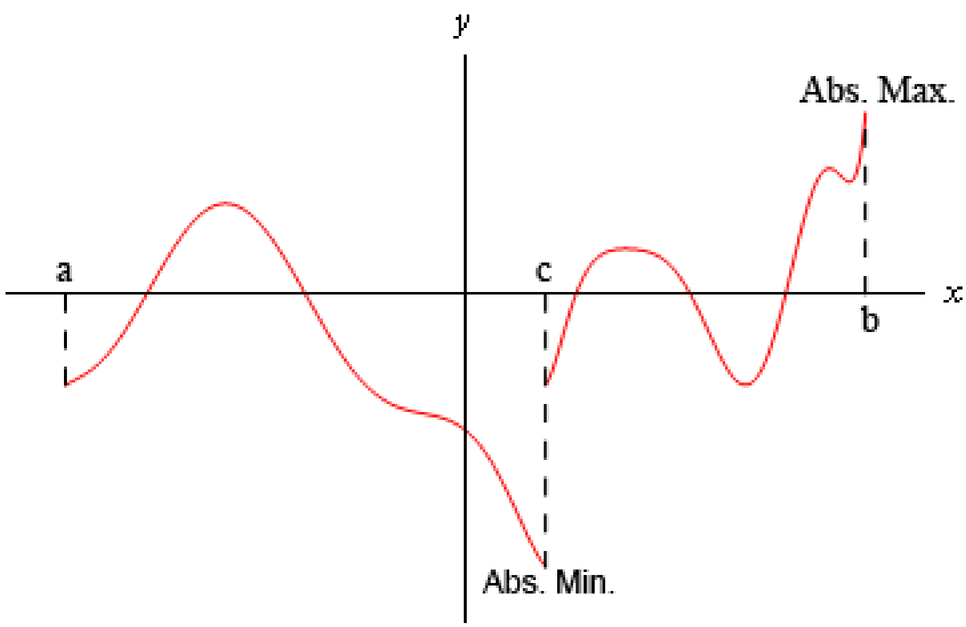
Consider the graph of $f(x) = \dfrac{1}{x^2}$ on the interval $[-1, 1]$.



We can see on the graph that there is no absolute maximum for this function on this interval since the function approaches infinity as x approaches zero from either side. It does have an absolute minimum, though. It occurs twice, at both $x = 1$ and at $x = -1$. Since this function has a discontinuity at $x = 0$, and since $x = 0$ is contained in the interval $[-1, 1]$, the function does not meet the hypothesis of the theorem.

Consider the same function on the interval $\left[\frac{1}{2}, 1\right]$. This function has both absolute extrema. Since this interval doesn't contain the point of discontinuity, the theorem applies to the function over this domain.

We should also point out that just because a function is not continuous at a point doesn't mean that it won't have both absolute extrema in an interval that contains that point. As an example, consider the graph below.



This function is not continuous at $x = c$, yet it has both an absolute maximum (at $x = b$) and an absolute minimum (at $x = c$). In this example, one of the absolute extrema occurs at the point of discontinuity, but it could appear anywhere in the domain or at one of the endpoints.

These graphs illustrate a point that can be made about all theorems, especially the Extreme Value Theorem. We can only draw the conclusion stated by the theorem if all of its hypotheses are met and we should be careful not to misinterpret what happens if the hypotheses are not met.

We can only use theorems to draw a conclusion if ALL of the hypotheses of the theorem are met!

In order to use the Extreme Value Theorem, we must have an interval that includes its endpoints (a closed interval), and the function must be continuous everywhere on that interval. If we don't have a closed interval and/or if the function is not continuous on the interval, then the function may or may not have absolute extrema.

Fermat's Theorem

We need to discuss one final topic before moving onto the first major application of the derivative that we'll look at in this chapter.

Definition 3: Fermat's Theorem

If $f(x)$ has a relative extremum at $x = c$ and if $f'(c)$ exists, then $x = c$ is a critical number of $f(x)$. In fact, it will be a critical number such that $f'(c) = 0$.

Note: The proof can be found in Appendix B.

The list of critical numbers gives us a list of **possible** relative extrema.

This theorem tells us that there is a nice relationship between relative extrema and critical numbers. In fact, it allows us to create a list of all possible relative extrema. Since a relative extremum must occur at a critical number, the list of all critical numbers gives us a list of all possible relative extrema.

To illustrate this, consider the function $f(x) = x^2$. We saw that this function has a relative minimum at $x = 0$ in several of our earlier examples. According to Fermat's Theorem, $x = 0$ should be a critical number. Since the derivative is

$$f'(x) = 2x$$

it is clear that $x = 0$ is indeed a critical number.

It is **not** true that every critical number will give the location of a max or min.

As with every theorem, we have to be careful that we are using this one correctly. It says that every relative extremum occurs at a critical number. It **does not** say that every critical number will correspond to a relative extremum.

To see this, consider the function $f(x) = x^3$. Its derivative is $f'(x) = 3x^2$, and clearly $x = 0$ is a critical number. However, we saw in an earlier example that this function has no relative extrema of any kind. Thus $x = 0$ is a critical number that does not correspond to a relative extremum.

We note further that Fermat's Theorem says nothing about absolute extrema. An absolute extremum may or may not occur at a critical number.

Before we move on, there are two more points we need to make.

First, Fermat's Theorem only applies to values $x = c$ at which $f'(c)$ exists. This doesn't mean, though, that relative extrema won't occur at critical numbers where the derivative does not exist. To see this, consider $f(x) = |x|$. This function has a relative minimum at $x = 0$ and yet in the section on the definition of the derivative, we showed in an example that $f'(0)$ does not exist.

Note: To locate relative extrema we only need to look at the critical numbers, as those are the only places where relative extrema may exist.

Finally, recall that at the beginning of this section we stated that if the domain is an interval, relative extrema cannot be located at the endpoints of the interval. If we allowed relative extrema to occur at endpoints, this may well (and frequently does) violate Fermat's Theorem. There is no reason to expect the endpoints of intervals to be critical numbers of either kind.

Finding Absolute Extrema

Our first major application of the derivative in this chapter is to determine the absolute extrema of a function that is continuous on a closed interval. First, let's observe that this is exactly the kind of function that the Extreme Value Theorem applies to, so we know that the absolute extrema exist.

Next, we saw above that absolute extrema can occur at endpoints or at relative extrema. We also know that the list of critical numbers is a list of all possible relative extrema. So, the endpoints plus the list of critical numbers is a list of all possible absolute extrema.

Now we just need to recall that the absolute extrema are simply the largest and smallest values that a function will take. Putting this all together gives us the following strategy for finding absolute extrema.

Definition 4: Finding Absolute Extrema of $f(x)$ on $[a, b]$

1. Verify that the function is continuous on the interval $[a, b]$.
2. Find all critical numbers of $f(x)$ that are in the interval $[a, b]$.
3. Evaluate the function at the critical numbers found in step 2, and at the end points.
4. Compare the values found in the previous step and identify the absolute maxima and absolute minima.

It's worth emphasizing step 2 before we do any examples. Since we are only interested in the function over the interval $[a, b]$, any critical numbers we find that are not in this interval can and should be ignored.

Example 3.2.7

Determine the absolute extrema for the following function on the given interval.

$$g(t) = 2t^3 + 3t^2 - 12t + 4 \text{ on } [-4, 2]$$

Solution:
First we notice that this function is a polynomial, and is therefore continuous everywhere. Therefore, it is continuous on the given interval.

We need to find the critical numbers of this function, so we need its derivative:

$$g'(t) = 6t^2 + 6t - 12 = 6(t+2)(t-1)$$

Since the derivative is a polynomial, there are no points at which it is undefined. We see from the factored form of $g'(t)$ that there are two values of t for which the derivative is zero: $t = -2$ and $t = 1$. To find the absolute extrema of a function on a given interval, we only need to consider critical numbers that fall in the interval. In this case, both critical numbers are in the given interval, so we will consider both of them.

Now we evaluate the function at the critical numbers and at the endpoints of the interval.

$$g(-2) = 24 \qquad g(1) = -3$$
$$g(-4) = -28 \qquad g(2) = 8$$

These four points are the only points in the interval where the absolute extrema can occur. From this list, we see that the absolute maximum of $g(t)$ on this interval is 24 and it occurs at $t = -2$ (a critical number). The absolute minimum of $g(t)$ on this interval is -28 and it occurs at $t = -4$ (an endpoint).

In this example, we saw that absolute extrema can and will occur at both endpoints and at critical numbers. One of the mistakes students frequently make on this kind of problem is to forget to check the endpoints of the interval.

Example 3.2.8

Determine the absolute extrema for the following function on the given interval.

$$g(t) = 2t^3 + 3t^2 - 12t + 4 \text{ on } [0, 2]$$

Solution:
This example is almost identical to the last example, we've only changed the interval. This small change will completely change the answer, though. Notice that this interval excludes both of the answers from the last example.

Again, we need the derivative and the critical numbers. Recall that

$$g'(t) = 6(t+2)(t-1)$$

and that the critical numbers of g are $t = -2$ and $t = 1$. Since we only consider critical numbers that fall in the interval, we will only consider $t = 1$ this time.

Now evaluate the function at this one critical number and the two endpoints.

$$g(1) = -3 \qquad g(0) = 4 \qquad g(2) = 8$$

From this list of function values, we see that the absolute maximum of $g(t)$ on this interval is 8 and it occurs at $t = 2$ (an endpoint). The absolute minimum is -3 and it occurs at $t = 1$ (a critical number).

As we saw in this last example, a simple change in the interval can completely change the answer. It also has shown us that we do need to be careful to exclude critical numbers that are not in the interval. Had we forgotten this and included $t = -2$, we would have found the wrong absolute maximum!

Example 3.2.9

Suppose that the population (in thousands) of a certain kind of insect after t months is given by the function

$$P(t) = 3t + \sin(4t) + 100$$

Determine the minimum and maximum population in the first 4 months.

Solution:
We are being asked to find the absolute extrema of $P(t)$ on the interval $[0, 4]$. Since this function is continuous everywhere, the Extreme Value Theorem tells us that we can do this.

We need the critical numbers of the function, so let's find the derivative.

$$P'(t) = 3 + 4\cos(4t)$$

The derivative exists everywhere, so we just need to determine where the derivative is zero.

$$\begin{aligned} 3 + 4\cos(4t) &= 0 \\ \cos(4t) &= -\frac{3}{4} \end{aligned}$$

The solutions to this equation are

$$\begin{array}{lcl} 4t \approx 2.4189 + 2\pi n, \quad n = 0, \pm 1, \pm 2, ... & & t \approx 0.6047 + \dfrac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, ... \\ & \Rightarrow & \\ 4t \approx 3.8643 + 2\pi n, \quad n = 0, \pm 1, \pm 2, ... & & t \approx 0.9661 + \dfrac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, ... \end{array}$$

So there are the critical numbers of the function. We need to determine which of these fall in the interval $[0, 4]$, though. The easiest way to do this is to start listing values of t for different n until we find all the ones we need.

$n = 0$:

$$t \approx 0.6047 \qquad t \approx 0.9661$$

Both of these are in the interval.

$n = 1$:

$$t \approx 0.6047 + \pi 2 \approx 2.1755 \qquad t \approx 0.9661 + \frac{\pi}{2} \approx 2.5359$$

These are as well.

$n = 2$:

$$t \approx 0.6047 + \pi \approx 3.7463 \qquad t \approx 0.9661 + \pi \approx 4.1077$$

The first of these is in the interval, but the second falls outside of it. So there are five critical numbers in the interval. They are (approximately)

$$0.6047, 0.9661, 2.1755, 2.5369, 3.7463$$

To determine the absolute minimum and maximum population in the first four months, we evaluate $P(t)$ at these values and at the two endpoints.

$$\begin{aligned} P(0) &= 100.0 & P(4) &\approx 111.7121 \\ P(0.6047) &\approx 102.4756 & P(0.9661) &\approx 102.2368 \\ P(2.1755) &\approx 107.1880 & P(2.5369) &\approx 106.9492 \\ P(3.7463) &\approx 111.9004 \end{aligned}$$

From this list, we see that the minimum population is $100,000$ (remember that P gives the population in thousands) and it occurs at $t = 0$. In other words, the minimum population is the starting population. The maximum population is about $111,900$ and it occurs at $t \approx 3.7463$.

This example illustrates the importance of solving trigonometric equations correctly. If we had forgotten the $2\pi n$, we would have missed the last three critical numbers in the interval and hence would have missed the correct absolute maximum.

Also, we should be careful with rounding answers. If we had rounded to the nearest integer, for instance, it would have appeared that the maximum population occurred at two different times instead of only one.

Example 3.2.10

Suppose that the amount of money in a bank account after t years is given by

$$A(t) = 2000 - 10t\mathrm{e}^{5-\frac{t^2}{8}}$$

Determine the minimum and maximum amount of money in the account during the first 10 years that it is open.

¡button onclick=”myFunction1(’ex10’)”¿ Click for solution ¡/button¿
¡div id =”ex10” class=”solution”¿ We are being asked to find the absolute extrema of $A(t)$ on the interval $[0, 10]$. This function is continuous everywhere, so the Extreme Value Theorem guarantees that these values exist.

First we’ll need the derivative so that we can find the critical numbers.

$$\begin{aligned} A'(t) &= -10\mathrm{e}^{5-\frac{t^2}{8}} - 10t\mathrm{e}^{5-\frac{t^2}{8}}\left(-\frac{t}{4}\right) \\ &= 10\mathrm{e}^{5-\frac{t^2}{8}}\left(-1+\frac{t^2}{4}\right) \end{aligned}$$

This derivative exists everywhere, so we just need to determine where it equals zero. The exponential function is never zero, so the derivative is only zero where

$$-1+\frac{t^2}{4} = 0 \qquad \Rightarrow \qquad t^2 = 4 \qquad \Rightarrow \qquad t = \pm 2$$

There are two critical numbers, however only $t = 2$ is in the interval, so that is the only critical number we’ll use.

Let’s evaluate the function at this critical number and the two endpoints.

$$A(0) = 2000 \qquad A(2) \approx 199.66 \qquad A(10) \approx 1999.94$$

So, the maximum amount in the account is $\$2000$, which occurs at $t = 0$, and the minimum amount in the account will be $\$199.66$ which occurs at the two-year mark.

There are two important things to note about this example. First, if we had included the second critical number, we would have found an incorrect value for the maximum amount. It’s worth repeating: we need to be careful with which critical numbers we include and which we exclude.

All of the examples we have done to this point had derivatives that existed everywhere, so the only critical numbers we needed to look for were those at which the derivative is zero. This will be the case with many problems, but not all of them. Don’t forget to look for **both** types of critical numbers.

Let's work one more example to make this point.

Example 3.2.11

Determine the absolute extrema of the following function on the given interval.

$$Q(y) = 3y(y+4)^{\frac{2}{3}} \text{ on } [-5, -1]$$

Solution:
Again, since this function is continuous on the given interval, the Extreme Value Theorem tells us that it has an absolute minimum and an absolute maximum value.

To find the critical numbers, let's first take the derivative and simplify it to make the algebra easier.

$$\begin{aligned} Q'(y) &= 3(y+4)^{\frac{2}{3}} + 3y\left(\frac{2}{3}\right)(y+4)^{-\frac{1}{3}} \\ &= 3(y+4)^{\frac{2}{3}} + \frac{2y}{(y+4)^{\frac{1}{3}}} \\ &= \frac{3(y+4)+2y}{(y+4)^{\frac{1}{3}}} \\ &= \frac{5y+12}{(y+4)^{\frac{1}{3}}} \end{aligned}$$

So it looks like we have two critical numbers:

$$\begin{aligned} (y+4)^{\frac{1}{3}} &= 0 \qquad \Rightarrow y = -4 \qquad \text{(The derivative does not exist here.)} \\ 5y+12 &= 0 \qquad \Rightarrow y = -\frac{12}{5} \qquad \text{(The derivative is zero here.)} \end{aligned}$$

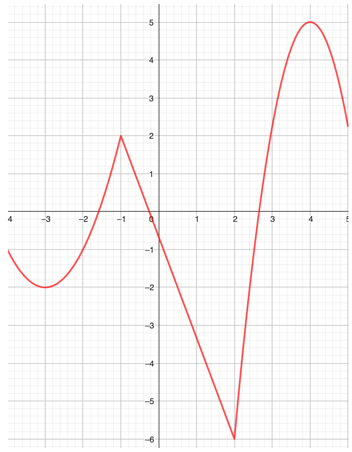
Both of these fall within the given interval, $[-5, -1]$, so we will evaluate the function at these points and at the two endpoints.

$$\begin{aligned} Q(-4) &= 0 \qquad & Q\left(-\frac{12}{5}\right) &\approx -9.85 \\ Q(-5) &= -15 \qquad & Q(-1) &\approx -6.24 \end{aligned}$$

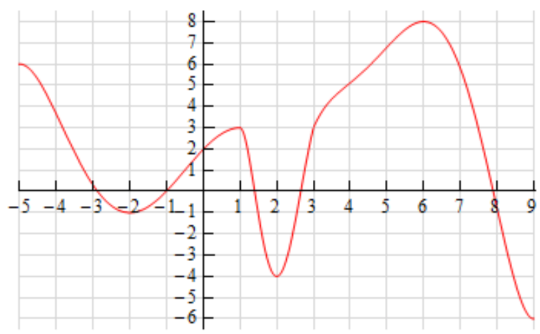
This function has an absolute maximum of zero at $y = -4$ and an absolute minimum of -15 at $y = -5$.

§3.2.1 Practice Problems

1. Below is the graph of a function, $y = f(x)$. Use the graph to identify all of the relative and absolute extrema of the function.



2. Below is the graph of a function, $y = f(x)$. Use the graph to identify all of the relative and absolute extrema of the function.



3. Sketch the graph of the function $g(x) = x^2 - 4x$, then use the graph to identify all the relative and absolute extrema of the function on each of the following intervals.

   a. $(-\infty, \infty)$

   b. $[-1, 4]$

   c. $[1, 3]$

   d. $[3, 5]$

   e. $(-1, 5]$

4. Sketch the graph of the function $h(x) = -(x+4)^3$, then use the graph to identify all the relative and absolute extrema of the function on each of the following intervals.

   a. $(-\infty, \infty)$

   b. $[-5.5, -2]$

   c. $[-4, -3)$

   d. $[-4, -3]$

5. Sketch the graph of a function on the interval $[1, 6]$ that has an absolute maximum at $x = 6$ and an absolute minimum at $x = 3$.

6. Sketch the graph of a function on the interval $[-4, 3]$ that has an absolute maximum at $x = -3$ and an absolute minimum at $x = 2$.

7. Sketch the graph of a function that meets the following conditions:

   - It is continuous.
   - It has two relative minima.
   - One of the relative minima is also an absolute minimum and the other relative minimum is not an absolute minimum.
   - It has one relative maximum.
   - It has no

In each of the following problems, determine the absolute extrema of the given function on the specified interval.

8. $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-8, 2]$

9. $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-4, 2]$

10. $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$ on $[-4.5, 4]$

11. $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$ on $[0, 7]$

12. $h(z) = 4z^3 - 3z^2 + 9z + 12$ on $[-2, 1]$

13. $g(x) = 3x^4 - 26x^3 + 60x^2 - 11$ on $[1, 5]$

14. $Q(x) = (2 - 8x)^4(x^2 - 9)^3$ on $[-3, 3]$

15. $h(w) = 2w^3(w+2)^5$ on $\left[-\frac{5}{2}, \frac{1}{2}\right]$

16. $f(z) = \dfrac{z+4}{2z^2 + z + 8}$ on $[-10, 0]$

17. $A(t) = t^2(10 - t)^{\frac{2}{3}}$ on $(2, 10.5]$

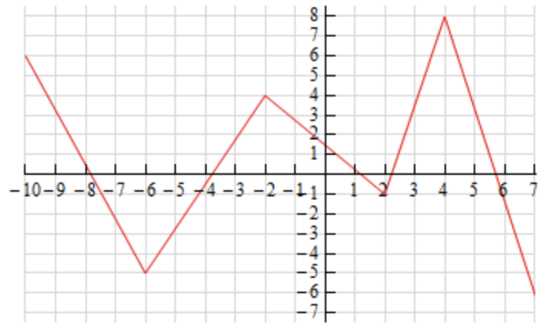
18. $f(y) = \sin\left(\dfrac{y}{3}\right) + \dfrac{2y}{9}$ on $[-10, 15]$

19. $g(w) = \mathrm{e}^{w^3 - 2w^2 - 7w}$ on $\left[-\dfrac{1}{2}, \dfrac{5}{2}\right]$

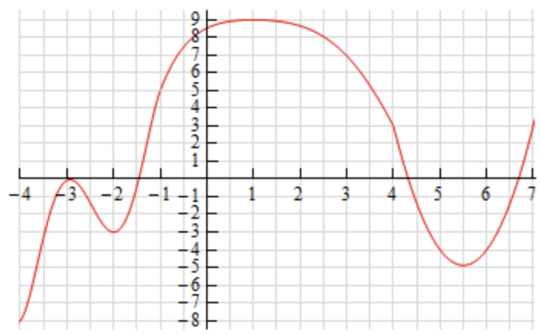
20. $R(x) = \ln(x^2 + 4x + 14)$ on $[-4, 2]$

Assignment Problems

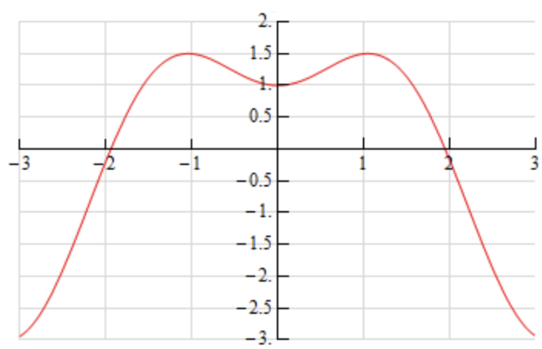
1. Below is the graph of some function, $f(x)$. Identify all of the relative and absolute extrema of the function.



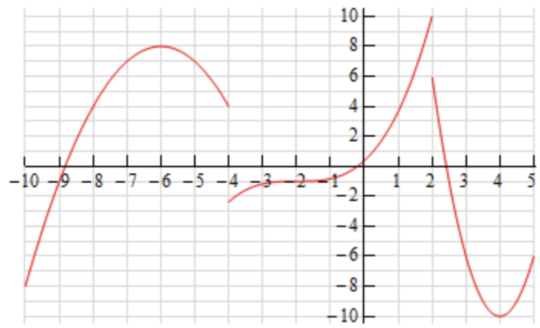
2. Below is the graph of some function, $f(x)$. Identify all of the relative and absolute extrema of the function.



3. Below is the graph of some function, $f(x)$. Identify all of the relative and absolute extrema of the function.



4. Below is the graph of some function, $f(x)$. Identify all of the relative and absolute extrema of the function.



5. Sketch the graph of $f(x) = 3 - \frac{1}{2}x$ and identify all the relative and absolute extrema on each of the following intervals.

   a. $(-\infty, \infty)$
   b. $[-3, 2]$
   c. $[-4, 1)$
   d. $(0, 5)$

6. Sketch the graph of $g(x) = (x-2)^2 + 1$ and identify all the relative and absolute extrema on each of the following intervals.

   a. $(-\infty, \infty)$
   b. $[0, 3]$
   c. $[-1, 5]$
   d. $[-1, 1]$
   e. $[1, 3)$
   f. $(2, 4)$

7. Sketch the graph of $h(x) = \mathrm{e}^{3-x}$ and identify all the relative and absolute extrema on each of the following intervals.

   a. $(-\infty, \infty)$
   b. $[-1, 3]$
   c. $[-6, -1]$
   d. $(1, 4]$

8. Sketch the graph of $h(x) = \cos(x) + 2$ and identify all the relative and absolute extrema on each of the following intervals.

   a. $(-\infty, \infty)$
   b. $[-\frac{\pi}{3}, \frac{\pi}{4}]$
   c. $[-\frac{\pi}{2}, 2\pi]$
   d. $[\frac{1}{2}, 1]$

9. Sketch the graph of a function on the interval $[3, 9]$ that has an absolute maximum at $x = 5$ and an absolute minimum at $x = 4$.

10. Sketch the graph of a function on the interval $[0, 10]$ that has an absolute minimum at $x = 5$ and absolute maxima at $x = 0$ and $x = 10$.

11. Sketch the graph of a function on the interval $(-\infty, \infty)$ that has a relative minimum at $x = -7$, a relative maximum at $x = 2$ and no absolute extrema.

12. Sketch the graph of a function that meets the following conditions:

    a. Has at least one absolute maximum.

    b. Has one relative minimum.

    c. Has no absolute minimum.

13. Sketch the graph of a function that meets the following conditions:

    a. Is graphed on the interval $[2, 9]$.

    b. Has a discontinuity at some point interior to the interval.

    c. Has an absolute maximum at the discontinuity in part (b).

    d. Has an absolute minimum at the discontinuity in part (b).

14. Sketch the graph of a function that meets the following conditions:

    a. Is graphed on the interval $[-4, 10]$.

    b. Has no relative extrema.

    c. Has an absolute maximum at one endpoint.

    d. Has an absolute minimum at the other endpoint.

15. Sketch the graph of a function that meets the following conditions:

    a. Has a discontinuity at some point.

    b. Has an absolute minimum and an absolute maximum.

    c. Neither absolute extremum occurs at the discontinuity.