

§1.1 Rates of Change

In this section we are going to take a look at two fairly important problems in the study of calculus: tangent lines and rates of change. There are two reasons for looking at these problems now.

First, both of these problems will lead us into the study of limits, which is the topic of this chapter. Looking at these problems here will allow us to start to understand just what a limit is and what it can tell us about a function.

Secondly, the rate of change problem is the foundation for studying derivatives, which you will use in virtually every remaining chapter of this book and course that has a pre-requisite of calculus. So, looking at it now will get us to start thinking about it from the very beginning.

Velocity

Let's start thinking about these problems by looking at a very common type of function used to model real-world problems.¹

Example 1.1.1

Imagine a person is standing on a platform and throwing a ball into the air. Suppose the height (in feet), f , of a ball at time t (in seconds) is given by the formula

$$f(t) = 64 - 16(t - 1)^2.$$

- Find $f(1)$. What does $f(1)$ represent in this real world situation?
- Construct a graph of $y = f(t)$ on the time interval $0 \leq t \leq 3$. Label at least six distinct points on the graph, including the three points showing when the ball was released, when the ball reaches its highest point, and when the ball lands.

- Consider the expression

$$\frac{f(1) - f(0.5)}{1 - 0.5}.$$

Compute this value. What does this value tell us about the motion of the ball?

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Solution:

- a. To find $f(1)$, we must substitute 1 in for each of the t 's in the function:

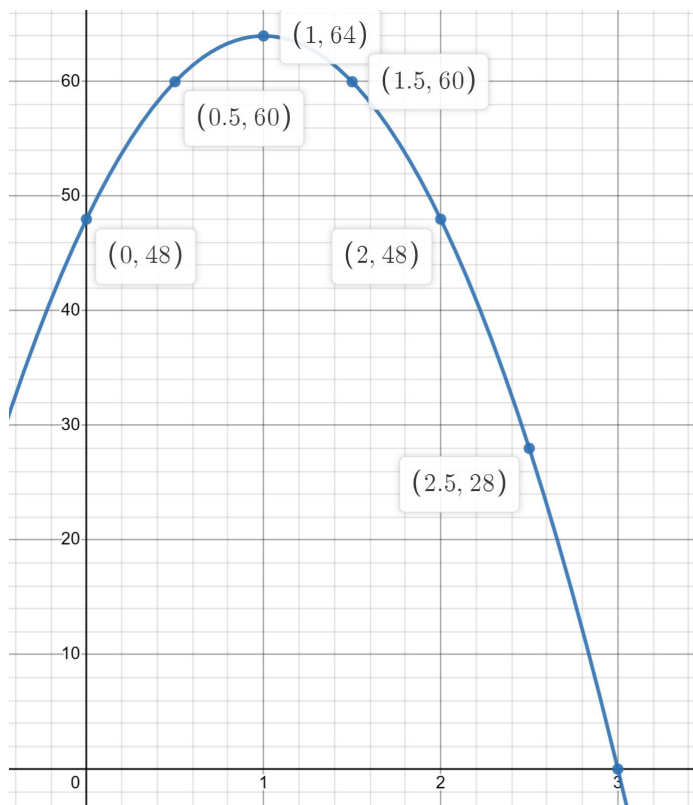
$$\begin{aligned} f(1) &= 64 - 16(1 - 1)^2 \\ &= 64 - 16(0) \\ &= 64. \end{aligned}$$

This tells us that the ball is at a height of 64 feet at $t = 1$ second.

- b. To construct our graph, we can make a table of values.

t	$f(t)$
0	48
0.5	60
1	64
1.5	60
2	48
2.5	29
3	0

Now we can plot these points (like $(0, 48)$, $(0.5, 60)$, etc) to create a sketch of the graph of our function.



We see the moment when the ball was thrown at $(0, 48)$, its highest point at $(1, 64)$, and when it lands at $(3, 0)$.

- c. To calculate $\frac{f(1) - f(0.5)}{1 - 0.5}$, we must evaluate each of $f(1)$ and $f(0.5)$ in our function first. Good news; we already found those values in the table above! Therefore, we have

$$\frac{f(1) - f(0.5)}{1 - 0.5} = \frac{64 - 60}{1 - 0.5} = \frac{4}{0.5} = 8.$$

This value gives us a change in height over a change in time; in other words, it's a measure of average speed. So the ball's speed was 8 ft per second over the time period from 0.5 to 1 seconds.

What we see from this example is that any moving object has a position that can be considered a function of time. When the motion is along a straight line, the position is given by a single variable, which we denote by $f(t)$. For example, $f(t)$ might give the mile marker of a car traveling on a straight highway at time in hours. Similarly, the function f described in Example 1 is a position function, where position is measured vertically relative to the ground.

On any time interval, a moving object also has an **average velocity**. For example, to compute a car's average velocity we divide the number of miles traveled by the time elapsed, which gives the velocity in miles per hour. Similarly, the value of part c in Example 1 gave the average velocity of the ball on the time interval $[0.5, 1]$, measured in feet per second. In general, we can make the following definition:

Definition 1: Average Velocity

For an object moving in a straight line with position function $f(t)$, the **average velocity of the object on the interval from $t = a$ to $t = b$** is given by the formula

$$\frac{f(b) - f(a)}{b - a}.$$

The units for average velocity are "units of f per unit of t , such as "miles per hour" or "feet per second."

Example 1.1.2

Suppose the height (in feet), f , of a ball at time t (in seconds) is given by the formula

$$f(t) = 64 - 16(t - 1)^2.$$

- Compute the average velocity of the ball on each of the intervals $[a, 0.8]$, using the following values of a : 0.4, 0.7, 0.79, 0.799.
- Compute the average velocity of the ball on each of the intervals $[0.8, b]$, using the following values of b : 1.2, 0.9, 0.81, 0.801.
- What do you conjecture is the velocity of the ball at the instant $t = 0.8$? Why?

Solution:

- a. For each value of
- a
- , we'll calculate

$$\frac{f(0.8) - f(a)}{0.8 - a}.$$

We summarize the results in the following table.

a	Average velocity on interval $[a, 0.8]$
0.4	12.8
0.7	8
0.79	6.56
0.799	6.416

- b. For each value of
- b
- , we'll calculate

$$\frac{f(b) - f(0.8)}{b - 0.8}.$$

We summarize the results in the following table.

b	Average velocity on interval $[0.8, b]$
1.2	0
0.9	4.8
0.81	6.24
0.801	6.384

- c. In each table above, we're choosing second values that get closer and closer to 0.8. As we move those values closer to 0.8, it looks like the average velocity is trending toward 6.4 ft/sec, so that's what we'll estimate the instantaneous velocity is at 0.8 seconds.

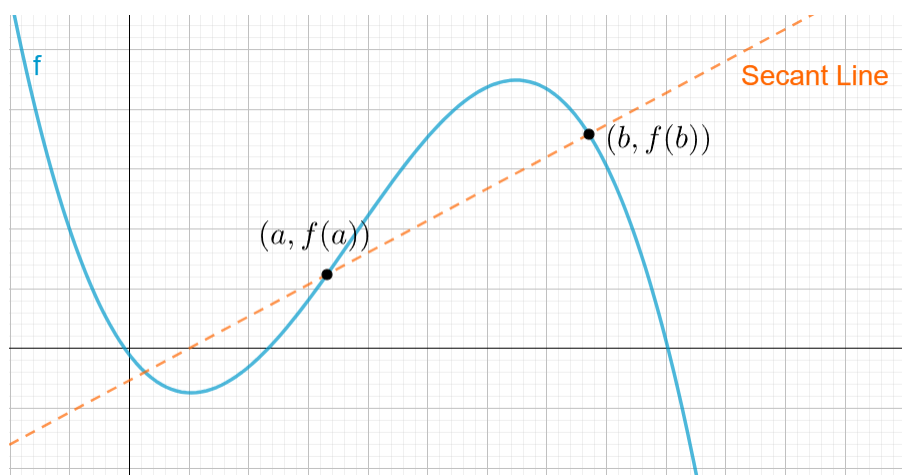
Tangent Lines

Let's take a look at the process that we're doing in Example 2. If we graph $f(t)$ and plot the two points we're using for our average velocity calculation, we can connect those points with a line that touches $f(t)$ in two places. We have a special name for that line; it's called the **secant line**.

Definition 2: Secant Line

For a function, f , a secant line connects two points on the graph of f , $(a, f(a))$ and $(b, f(b))$.

A secant line might look something like this.



Before moving on² let's do a quick review of just what we did in the above example using this new point of view where we think about lines instead of just velocities. We started with the point $P = (a, f(a))$ as a point we know is on the tangent line. Next, we took a second point that is on the graph of the function and called it $Q = (b, f(b))$. With these two points, we computed the slope of the line connecting P and Q using the formula

$$m_{PQ} = \frac{f(b) - f(a)}{b - a}.$$

We then took values of t that got closer and closer to $t = a$ (making sure to look at t -values on both sides of $t = a$), and we used this list of values to estimate the instantaneous slope, m .

We will call the line at $t = a$ with instantaneous slope m the **tangent line**. The tangent line then had equation $y = m(x - a) + f(a)$.

Definition 3: Tangent Line

The tangent line to the function $f(x)$ through the point $x = a$ has a slope found by selecting values of b that get very close to a to estimate what the exact slope, m , is at $x = a$ using the formula

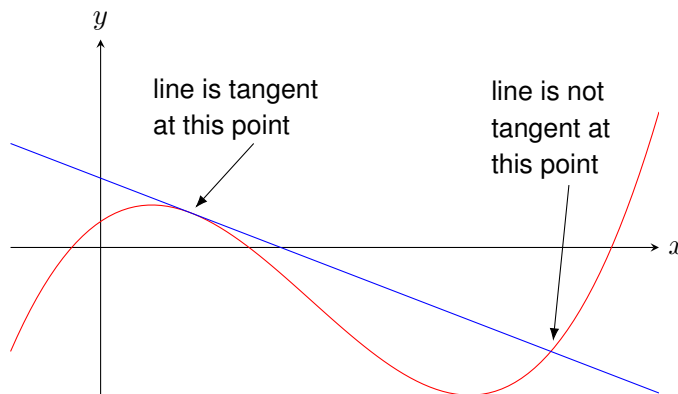
$$\frac{f(b) - f(a)}{b - a}.$$

Using this estimation, we create the equation of the tangent line

$$y = m(x - a) + f(a).$$

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Take a look at the graph below.



In this graph the line is a tangent line at the first indicated point because it just touches the graph at that point and also goes in the same direction as the graph of $f(x)$ at that point. Likewise, at the second point shown, the line goes through the same point as the graph of $f(x)$, but in a different direction than $f(x)$, so it's not a tangent line to the graph at that point.

We can think about the tangent line as being formed by creating a secant line through two points, and then moving one of those points closer and closer to the other. That

We've used the word "direction" a couple of times now and we should probably be a little careful with it. To be more precise, we will need to investigate what it means to find the slope of a curve at a given point

Now that we have defined the tangent line, let's move on to the tangent line problem with an example.

Example 1.1.3

Find an equation of the tangent line to the graph of $f(x) = 15 - 2x^2$ at $x = 1$.

Solution: To find an equation of a line we need either two points on the line or a single point on the line and the slope of the line. Since we know that we are after a tangent line we can use $f(x)$ to find the point where the line must touch the graph of the function. Specifically, the tangent line and the graph of the function must touch at $x = 1$, so we plug $x = 1$ into the formula $f(x) = 15 - 2x^2$ to get the y -value of $f(1) = 15 - 2(1)^2 = 13$. So the point $(1, 13)$ must be on the tangent line.

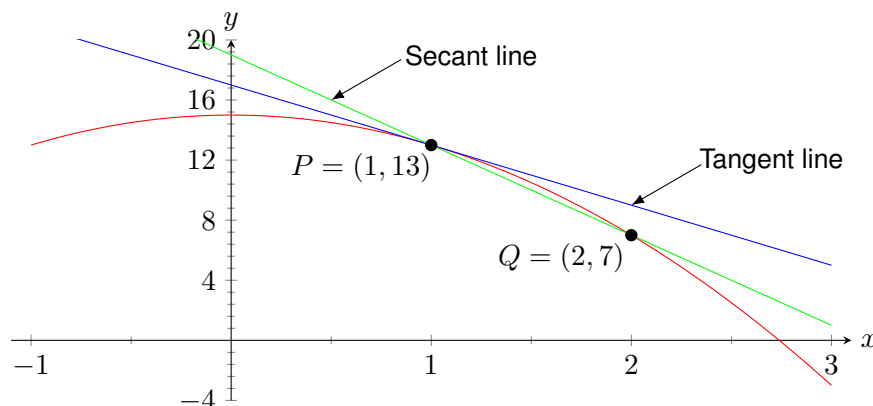
Now we reach the tangent line problem. In order to find the tangent line we need either a second point or the slope of the tangent line. Since we want the tangent line to go in the same direction as $f(x)$ at $(1, 13)$, we will concentrate on attempting to determine the slope of the tangent line.

At this point in time, all we can do with certainty is estimate the slope of the tangent line, but if we can find a pattern in our estimates, we may be able to find the actual slope of the tangent line, not just an estimate. We'll do this by starting with the point that we already have on the line, let's call it $P = (1, 13)$.

To estimate slope, we need two points, so we will then pick another point that lies on the graph of

the function; let's call that point $Q = (x, f(x))$. For example, if $x = 2$, then the second point will be $Q = (2, 7)$ since $f(2) = 15 - 2(2)^2 = 7$.

Below is a graph of the function, the tangent line, and the secant line that connects P and Q .

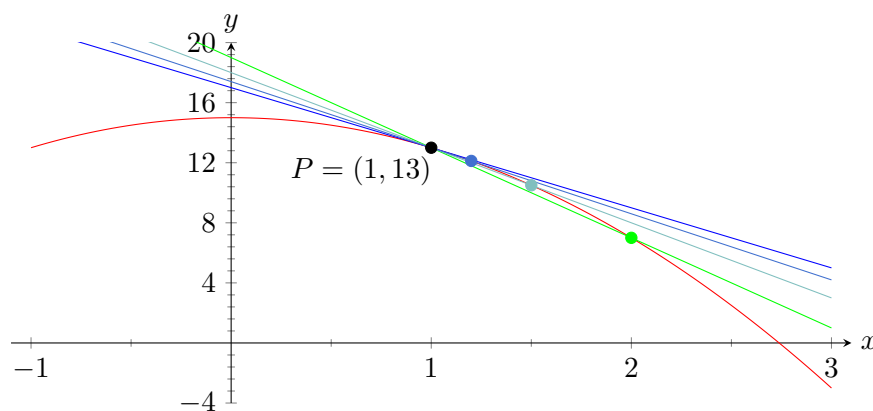


We can see from this graph that the secant and tangent lines are somewhat similar and so the slope of the secant line should be somewhat close to the actual slope of the tangent line. So, as an estimate of the slope of the tangent line we can use the slope of the secant line. Let's call the slope of the secant line m_{PQ} , which is

$$m_{PQ} = \frac{f(2) - f(1)}{2 - 1} = \frac{7 - 13}{2 - 1} = -6.$$

Now, this is one estimate for the slope of the tangent line, but we can do better. To improve the estimate, we can take x closer to $x = 1$ and redo the above work to get a new estimate on the slope. We could then repeat the process with a yet closer x value to get an even better estimate.

In other words, as we take Q closer and closer to P , the slope of the secant line connecting Q and P should be getting closer and closer to the slope of the tangent line. Below you can see many different secant lines through $(1, 13)$ whose slopes are getting closer and closer to the tangent line. You can view this graph on the web if you want to see it animated.



As we move Q closer and closer to P , the secant lines do start to look more and more like the tangent line and so the approximate slopes (i.e. the slopes of the secant lines) are getting closer and closer to the exact slope.

In this figure we only looked at points Q that were to the right of P , but we could have just as easily used points that were to the left of P and we would have received the same results. In fact, we should always take a look at points that are on both sides of P , since these could give us different results (and we will eventually see examples of this in future sections). Remember to always look at what is happening on both sides of the point in question when doing this kind of process.

Now, let's find the approximate slopes showed in the graph above, and hence an estimation of the slope of the tangent line. In order to simplify the process a little, let's write down a formula for the slope of the line between P and Q , m_{PQ} , that will work for any x that we choose to work with. Since the slope is between $P = (1, 13)$ and $Q = (x, f(x))$, the formula for m_{PQ} is

$$m_{PQ} = \frac{f(x) - f(1)}{x - 1} = \frac{15 - 2x^2 - 13}{x - 1} = \frac{2 - 2x^2}{x - 1}.$$

Now let's make a table of the slope m_{PQ} at specific x -values getting closer and closer to $x = 1$.

x	m_{PQ}	x	m_{PQ}
2	-6	0	-2
1.5	-5	0.5	-3
1.1	-4.2	0.9	-3.8
1.01	-4.02	0.99	-3.98
1.001	-4.002	0.999	-3.998
1.0001	-4.0002	0.9999	-3.9998

So, if we take x -values to the right of $x = 1$ and take them closer and closer to $x = 1$, it appears that the slope of the secant lines is approaching -4 . Likewise, if we take x -values to the left of $x = 1$ and take them closer and closer to $x = 1$, it appears the slope of the secant lines is again approaching -4 .

Based on this evidence, the slopes of the secant lines are approaching -4 as the x -values of Q approach $x = 1$ from both the left and the right sides, so we will estimate that the slope of the tangent line is also -4 . We will eventually be able to prove that this is the slope of the tangent line and not just guess from a pattern on a table.

As a reminder, the point-slope equation of the line that goes through $(a, f(a))$ with slope m is given by $y = m(x - a) + f(a)$.

Therefore, since the tangent line goes through $(1, 13)$ and we just estimated its slope as $m = -4$, the point-slope equation of the tangent line to the graph of $f(x) = 15 - 2x^2$ at $x = 1$ is

$$y = -4(x - 1) + 13.$$

There are a couple of important points to note about our work above. First, we looked at points that were on both sides of $x = 1$. In this kind of process it is important to never assume that what is happening on one side of a point will also be happening on the other side as well. We should always look at what is happening on both sides of the point. In this example we could sketch a graph and from that guess that what is happening on one side will also be happening on the other, but we will often not have the graphs in front of us or be able to easily get them.

Look at what happens when we approach a point from the left and from the right!

Next, notice that when we say we're going to move in close to the point in question we do mean that we're going to move in very close and we also used more than just a couple of points. We should never try to determine a trend based on a couple of points that aren't really all that close to the point in question.

Test points that get *really* close to your point of interest.

The next thing to notice is really a warning more than anything. The values of m_{PQ} in this example were fairly "nice" and it was pretty clear what value they were approaching after a couple of computations. In most practical situations, this will not be the case, and you'll often need quite a few computations to be able to get an estimate. Two points is never sufficient to get a good estimate and three points will also often not be sufficient to get a good estimate. Generally, you keep picking points closer and closer to the point you are looking at until the change in the value between two successive points is getting very small.

Choose points that get successively closer to your point of interest to get the most accurate picture of what's happening.

Last, we were after something that was happening at $x = 1$ and we couldn't actually plug $x = 1$ into our formula for the slope (this would've caused a division by zero error). Despite this limitation we were able to determine some information about what was happening at $x = 1$ simply by looking at what was happening around $x = 1$. This is more important than you might at first realize and we will be discussing this point in detail in later sections.

We're more concerned about what's happening near our point of interest than what's happening at our point of interest.

Use the below practice to try this a few times before moving on to the next topic of this section.

Example 1.1.4

For the function $f(x) = 3(x + 2)^2$ and the point P given by $x = -3$, answer each of the following questions.

- a. For the points Q given by the following values of x , compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .

- | | |
|------------|-------------|
| i. -3.5 | vi. -2.5 |
| ii. -3.1 | vii. -2.9 |
| iii. -3.01 | viii. -2.99 |
| iv. -3.001 | ix. -2.999 |
| v. -3.0001 | x. -2.9999 |

- b. Use the information from (a) to estimate the slope of the tangent line to $f(x)$ at $x = -3$ and write down the equation of the tangent line.

Solution:

- a. The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$m_{PQ} = \frac{f(x) - f(-3)}{x - (-3)} = \frac{3(x + 2)^2 - 3}{x + 3}.$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places, but in this case the values terminated prior to 8 decimal places and so the “trailing” zeros are not shown.

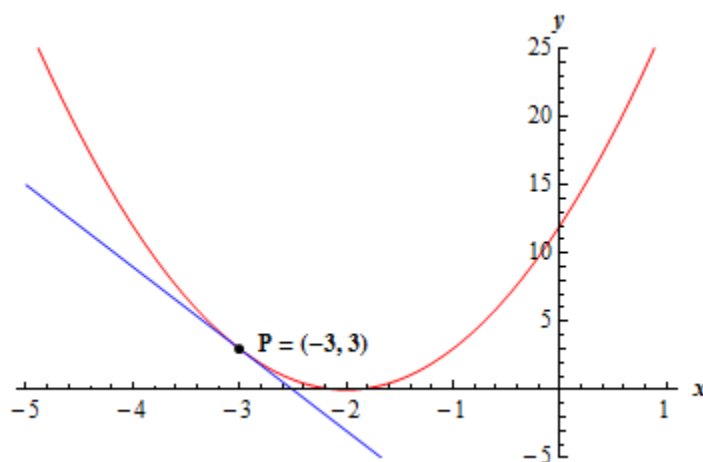
x	m_{PQ}	x	m_{PQ}
-3.5	-7.5	-2.5	-4.5
-3.1	-6.3	-2.9	-5.7
-3.01	-6.03	-2.99	-5.97
-3.001	-6.003	-2.999	-5.997
-3.0001	-6.0003	-2.9999	-5.9997

- b. From the table of values above we can see that the slope of the secant lines appears to be moving towards a value of -6 from both sides of $x = -3$ and so we can estimate that the slope of the tangent line is: $m = -6$.

The equation of the tangent line is then

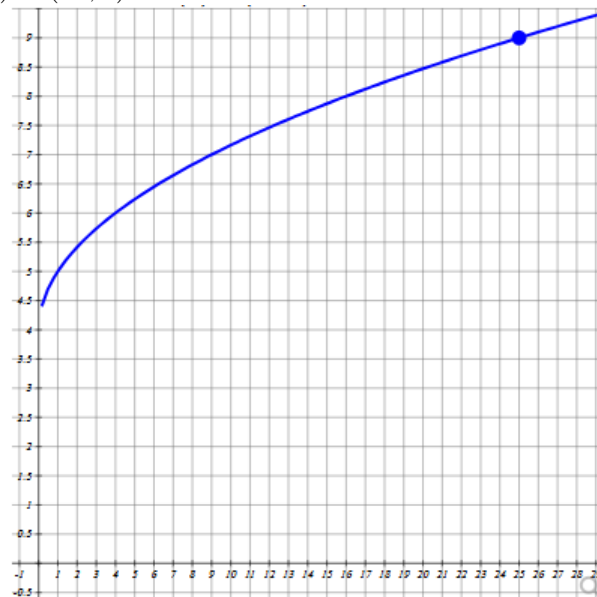
$$y = f(-3) + m(x - (-3)) = 3 - 6(x + 3) \Rightarrow y = -6x - 15.$$

Here is a graph of the function and the tangent line.



Check Point 1.1.1

The graph of $f(x) = \sqrt{x} + 4$ is below, and we are going to try and estimate the slope of the tangent line to the graph of $f(x)$ at $(25, 9)$.



Find the slope of the secant line between $(25, 9)$ and the point on the graph with the given x coordinate. Give your answers accurate to at least 4 decimal places.

- $x = 25.1$
- $x = 25.01$
- $x = 24.9$
- $x = 24.99$
- Use the results of a-d to estimate the slope of the tangent line of $f(x)$ at $x = 25$.

Rates of Change

The next problem that we need to look at is the rate of change problem. As mentioned earlier, this will turn out to be one of the most important concepts that we will look at throughout this course, and it is very closely related to the tangent line problem.

Let's consider a function $f(x)$ that represents some quantity that varies as x varies. For instance, maybe $f(x)$ represents the amount of water in a holding tank after x minutes. Or maybe $f(x)$ is the distance traveled by a car after x hours. In both of these examples we used x to represent time. Of course x doesn't have to represent time, but it makes for examples that are easy to visualize. We can also use input variables other than x .

To estimate an instantaneous rate of change, we'll calculate average rates of change over successively smaller intervals.

What we want to do here is determine just how fast $f(x)$ is changing at some point, say $x = a$. This is called the instantaneous rate of change or sometimes just rate of change of $f(x)$ at $x = a$.

This means we'll be calculating more slopes, but we'll be thinking about them now as rates of change.

As with the tangent line problem all that we're going to be able to do at this point is to estimate the rate of change. So, let's continue with the examples above and think of $f(x)$ as something that is changing in time and x being the time measurement. Again, x doesn't have to represent time but it will make the explanation a little easier. We don't quite have the tools to compute the instantaneous rate of change at this point, but we can find the average rate of change between two points with just what you know already.

To compute the average rate of change of $f(x)$ between $x = a$ and $x = b$, we use the formula

$$\text{average rate of change} = \frac{\text{change in } f(x)}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

This may remind you of the slope calculations we did earlier in this section (remember that slope simply measures change in output divided by change in input), since this is just the formula for slope between two points on the graph of $f(x)$.

To estimate the instantaneous rate of change at $x = a$, we can use the variable x for the second point instead of b and choose values for x closer and closer to $x = a$ (don't forget to choose them on both sides of $x = a$) and compute values of the average rate of change $\frac{f(x) - f(a)}{x - a}$. We can then estimate the instantaneous rate of change from these values.

Let's take a look at an example.

Example 1.1.5

Suppose that the amount of air in a balloon after t hours is given by $V(t) = t^3 - 6t^2 + 35$.

Estimate the instantaneous rate of change of the volume after 5 hours.

Solution:

First, let's identify the information we were given and how it fits into our formula for average rate of change. Our function is $V(t)$ (instead of $f(x)$), our input variable is t (instead of x), and we are looking for the instantaneous rate of change at $t = 5$. We can also go ahead and calculate $V(5) = 5^3 - 6(5^2) + 35 = 10$.

Then we can use this to get a formula for the average rate of change of the volume:

$$\text{average rate of change} = \frac{V(t) - V(5)}{t - 5} = \frac{t^3 - 6t^2 + 35 - 10}{t - 5} = \frac{t^3 - 6t^2 + 25}{t - 5}$$

Next, to estimate the instantaneous rate of change of the volume at $t = 5$, we just need to pick values of t that are getting closer and closer to $t = 5$. Here is a table of values of t (on both sides of $t = 5$) and the average rate of change for those values.

t	average rate of change	t	average rate of change
6	25	4	7
5.5	19.75	4.5	10.75
5.1	15.91	4.9	14.11
5.01	15.0901	4.99	14.9101
5.001	15.009001	4.999	14.991001
5.0001	15.00090001	4.9999	14.99910001

From this table it looks like the average rate of change is approaching 15 and so we can estimate that the instantaneous rate of change is 15 at this point.

What exactly does this estimate for the instantaneous rate of change tell us about the volume at $t = 5$? Let's put some units on the answer from above in the hopes that this context might help us to see what is happening to the volume at this point. Let's suppose that the units on the volume were in cm^3 . The units on the rate of change (both average and instantaneous) are then cm^3/hr .

Thus we have estimated that at $t = 5$ the volume is changing at a rate of $15\text{cm}^3/\text{hr}$. So at $t = 5$ the volume is changing in such a way that, if the rate were constant, then an hour later there would be 15cm^3 more air in the balloon than there was at $t = 5$.

In general, the units for the rate of change will be

$$\frac{\text{output (y) units}}{\text{input (x) units}}$$

We do need to be careful here however. In many practical situations, the rate at which the volume is changing will not remain constant, so there probably won't be exactly 15cm^3 more air in the balloon after an hour. For instance, suppose a person is blowing air into the balloon causing the volume to increase.

The instantaneous rate of change often does not stay the same over time. That is, the rate of change *changes* as time goes on!

In general, we are only getting information about the rate of change of volume of the balloon at $t = 5$ hours. We can't make any real determination as to what the volume will be in another hour using only this rate of change. What we can say is that the volume is increasing, since the instantaneous rate of change is positive, and if we had rates of change for other values of t we could compare the numbers and see if the rate of change is faster or slower at the other points.

For instance, at $t = 4$, it turns out that the instantaneous rate of change is $0\text{ cm}^3/\text{hr}$ for our specific volume function, and at $t = 3$ the instantaneous rate of change is $-9\text{ cm}^3/\text{hr}$. We'll leave it to you to check these rates of change. In fact, that would be a good exercise to see if you can build a table of values that will support our claims on these rates of change.

We can also interpret what these rates of change mean. At $t = 4$, the rate of change is zero, and so at this point in time the volume is not changing at all. That doesn't mean that it will not change in the future. It just means that exactly at $t = 4$ the volume isn't changing. Likewise, at $t = 3$ the volume is decreasing since the rate of change at that point is negative. We can also say that, regardless of the increasing/decreasing aspects of the rate of change, the volume of the balloon is changing faster at $t = 5$ than it is at $t = 3$ since 15 is larger than 9.

We will be talking a lot more about rates of change when we get into the next chapter.

Example 1.1.6

The population (in hundreds) of fish in a pond is given by $P(t) = 2t + \sin(2t - 10)$, where t is years after 2020. Answer each of the following questions.

- a. Compute (accurate to at least 8 decimal places) the average rate of change of the population of fish between $t = 5$ and the following values of t . Make sure your calculator is set to radians for the computations.

i. 5.5	vi. 4.5
ii. 5.1	vii. 4.9
iii. 5.01	viii. 4.99
iv. 5.001	viv. 4.999
v. 5.0001	vv. 4.999
- b. Use the information from (a) to estimate the instantaneous rate of change of the population of the fish at $t = 5$.

Solution

- a. The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$A.R.C. = \frac{P(t) - P(5)}{t - 5} = \frac{2t + \sin(2t - 10) - 10}{t - 5}.$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places.

x	$A.R.C.$	x	$A.R.C.$
5.5	3.68294197	4.5	3.68294197
5.1	3.98669331	4.9	3.98669331
5.01	3.99986667	4.99	3.99986667
5.001	3.99999867	4.999	3.99999867
5.0001	3.99999999	4.9999	3.99999999

- b. From the table of values above we can see that the average rate of change of the population of fish is moving towards a value of 4 from both sides of $t = 5$ and so we can estimate that the instantaneous rate of change of the population of the fish is 400 fish per year (remember the population is in hundreds and the time is in years since 2020).

Check Point 1.1.2

The volume of air in a balloon is given by $V(t) = \frac{8}{5t+1}$, where V is in cubic inches and t is in seconds.

- a. Compute (accurate to at least 8 decimal places) the average rate of change of the volume of air in the balloon between $t = 0.64$ and the following values of t :
- | | |
|--------------|---------------|
| i. 0.641 | iv. 0.639 |
| ii. 0.6401 | iiiv. 0.6399 |
| iii. 0.64001 | iiiv. 0.63999 |
- b. Use the rates of change above to estimate the instantaneous rate of change when $t = 0.64$.

Velocity Problem

One specific example of the rate of change problem that is often studied in calculus books is the velocity problem. In the velocity problem we are given a position function of an object, $f(t)$, that gives the position of an object at time t along an axis. For instance, think about a car driving down a highway and $f(t)$ measures how far it has traveled on that highway. Then to compute the instantaneous velocity of the object we just need to recall that the velocity is nothing more than the rate at which the position is changing. In other words, to estimate the instantaneous velocity of the object at time $t = a$ we would first compute the average velocity,

$$\text{average velocity} = \frac{\text{change in position}}{\text{time traveled}} = \frac{f(t) - f(a)}{t - a}$$

and then take values of t closer and closer to $t = a$ and use these values to estimate the instantaneous velocity.

Example 1.1.7

The position of an object is given by $s(t) = (8 - t)(t + 6)^{3/2}$. Note that a negative position here simply means that the position is to the left of the “zero position” and is perfectly acceptable. Answer each of the following questions.

- a. Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 10$ and the following values of t . Make sure your calculator is set to radians for the computations.

i. 10.5	vi. 9.5
ii. 10.1	vii. 9.9
iii. 10.01	viii. 9.99
iv. 10.001	viv. 9.999
v. 10.0001	vv. 9.999
- b. Use the information from (a) to estimate the instantaneous velocity of the object at $t = 10$ and determine if the object is moving to the right (i.e. the instantaneous velocity is positive), moving to the left (i.e. the instantaneous velocity is negative), or not moving (i.e. the instantaneous velocity is zero).

Solution:

- a. The first thing that we need to do is set up the formula for the slope of the secant lines. As discussed in this section this is given by,

$$A.V. = \frac{s(t) - s(10)}{t - 10} = \frac{(8 - t)(t + 6)^{3/2} + 128}{t - 10}.$$

Now, all we need to do is construct a table of the value of m_{PQ} for the given values of x . All of the values in the table below are accurate to 8 decimal places.

x	$A.V.$	x	$A.V.$
10.5	-79.11658419	9.5	-72.92931693
10.1	-76.61966704	9.9	-75.38216890
10.01	-76.06188418	9.99	-75.93813418
10.001	-76.00618759	9.999	-75.99381259
10.0001	-76.00061875	9.9999	-75.99938125

- b. From the table of values above we can see that the average velocity of the object is moving towards a value of -76 from both sides of $t = 10$ and so we can estimate that the instantaneous velocity is -76 and so the object will be moving to the left at $t = 10$.

Check Point 1.1.3

If a ball is thrown straight up into the air with an initial velocity of 90 ft/s, its height in feet after t seconds is given by $y = 90t - 16t^2$. Find the average velocity for the time period beginning when $t = 2$ and lasting

- 0.1 second
- 0.01 seconds
- 0.001 seconds

Based on the above results, guess what the instantaneous velocity of the ball is when $t = 2$.

Alternate Notation

There is one last bit of notation that we will cover in this section before we move on. The main point of this section was to introduce us to a couple of key concepts and ideas that we will see throughout the first portion of this course as well as get us started down the path towards limits and eventually derivatives.

For all three problems we studied this section, there was a common thread. Whether we wanted the tangent line, instantaneous rate of change, or instantaneous velocity each of these came down to using exactly the same formula, sometimes called a difference quotient:

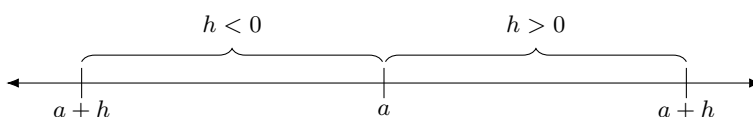
$$\frac{f(x) - f(a)}{x - a}$$

This should suggest that all three of these problems involve the same problem, and we'll make this connection more explicit in the next chapter. The main difference between these three problems is the interpretation of what the above difference quotient helps us approximate, whether that be the slope of a function's graph, the rate of change of a quantity, or the velocity of an object.

In all of these problems we wanted to determine what was happening at $x = a$. To do this we chose a second value of x (other than $x = a$) and plugged it into the above formula.

This provides an intuitive way of writing down what we are doing, but when we start looking at solving limits involving difference quotients, there is an alternate form for the difference quotient that will sometimes be easier to work with.

Instead of having two points with totally different inputs, $(a, f(a))$ and $(x, f(x))$, we are going to focus on the point $(a, f(a))$ and a second point defined relative to it. Suppose we want to move on the x -axis a distance of h from $x = a$. Then our new x value is $x = a + h$. This is shown below in a sketch.



For instance, if $a = 5$ and $h = 0.1$, then $a + h = 5 + 0.1 = 5.1$, an x -value a little bit to the right of $x = 5$. Likewise, if $h = -0.01$, then $a + h = 5 - 0.01 = 4.99$ for these example numbers, an x -value a little bit to the left of $x = 5$. Moreover, as h gets closer and closer to 0, $a + h$ is getting closer and closer to a .

As we noted earlier in this section, it is important to take values of x that are on both sides of $x = a$. We can do that with this new way of choosing $x = a + h$, since choosing $h > 0$ will cause $x = a + h$ to be an x -value right of $x = a$, while choosing $h < 0$ will cause $x = a + h$ to be an x -value left of $x = a$.

Now, with this new way of obtaining a second x -value, namely $x = a + h$, the difference quotient becomes

$$\frac{f(x) - f(a)}{x - a} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}.$$

This is for a specific value of x , i.e., $x = a$ (for example, $x = 5$ or $x = -2.1$). If we want to look at a difference quotient centered at the general value of x , we simply replace a with x to get

$$\frac{f(x + h) - f(x)}{h}.$$

Let's see how we can apply this definition to some of our previous examples.

Example 1.1.8

Find the equation of the tangent line to the graph of $f(x) = 15 - 2x^2$ at $x = 1$. Use the formula $\frac{f(a+h)-f(a)}{h}$ to estimate the slope.

Solution:

First, we want to come up with an estimate for the slope, but now we'll use

$$\frac{f(a+h) - f(a)}{h}$$

where $a = 1$, and where we let h get closer and closer to 0. From a formula perspective, that will look like this:

$$\begin{aligned} \frac{f(1+h) - f(1)}{h} &= \frac{15 - 2(1+h)^2 - (15 - 2(1)^2)}{h} = \frac{15 - 2(1+h)^2 - 13}{h} \\ &\Rightarrow \frac{2 - 2(1+h)^2}{h}. \end{aligned}$$

Now, we'll create tables of values where we choose smaller and smaller value of h , and use that to come up with our estimate of the slope.

h	m_{PQ}	x	m_{PQ}
0.5	-5	-0.5	-3
0.1	-4.2	-0.1	-3.8
0.01	-4.02	-0.01	-3.98
0.001	-4.002	-0.001	-3.998
0.0001	-4.0002	-0.0001	-3.9998

We reach the same conclusion as before: it looks like the slopes of the secant lines are approaching -4, but this time as h approaches 0. Therefore, we will estimate that the slope of the tangent line is -4, and our equation is still $y = -4(x - 1) + 13$.

Example 1.1.9

Suppose that the amount of air in a balloon after t hours is given by $V(t) = t^3 - 6t^2 + 35$.

Estimate the instantaneous rate of change of the volume after 5 hours, this time using the formula $\frac{f(a+h)-f(a)}{h}$.

Solution:

Once again, we're aiming to use $\frac{f(a+h)-f(a)}{h}$, with $a = 5$ and with increasingly small values of h . Let's tease out the formula first.

$$\frac{f(5+h) - f(5)}{h} = \frac{(5+h)^3 - 6(5+h)^2 + 35 - (5^3 - 6(5)^2 + 35)}{h}$$

$$= \frac{(5+h)^3 - 6(5+h)^2 + 35 - (10)}{h} = \frac{(5+h)^3 - 6(5+h)^2 + 25}{h}$$

Note that the only reason we're simplifying is to make our subsequent calculations a bit more efficient. Now, let's create some tables of values.

h	$A.R.C.$	x	$A.R.C.$
0.5	19.75	-0.5	10.75
0.1	15.91	-0.1	14.11
0.01	15.0901	-0.01	14.9101
0.001	15.009001	-0.001	14.991001
0.0001	15.00090001	-0.0001	14.99910001

Once again, we arrive at an estimated 15 for our instantaneous rate of change at 5 hours.

We won't dictate which method you should use to make these calculations, for now. Experiment with each version to get a better understanding of how they're similar and how they're different.

Practice Problems

Full solutions available in Appendix A

- For the function $g(x) = \sqrt{4x+8}$ and the point P given by $x = 2$, answer each of the following questions.
 - For the points Q given by the following values of x , compute (accurate to at least 8 decimal places) the slope m_{PQ} of the secant line through points P and Q .

i. 2.5	vi. 1.5
ii. 2.1	vii. 1.9
iii. 2.01	viii. 1.99
iv. 2.001	ix. 1.999
v. 2.0001	x. 1.9999
 - Use the information from (a) to estimate the slope of the tangent line to $g(x)$ at $x = 2$ and write down the equation of the tangent line.
- For the function $W(x) = \ln(1+x^4)$ and the point P given by $x = 1$, answer the following.
 - Compute slopes of secant lines.

- | | |
|-----------|------------|
| i. 1.5 | vi. 0.5 |
| ii. 1.1 | vii. 0.9 |
| iii. 1.01 | viii. 0.99 |
| iv. 1.001 | ix. 0.999 |
| v. 1.0001 | x. 0.9999 |

b. Estimate the tangent line at $x = 1$.

3. The position of an object is given by

$$s(t) = \cos^2\left(\frac{3t-6}{2}\right).$$

a. Compute average velocity between $t = 2$ and:

- | | |
|-----------|------------|
| i. 2.5 | vi. 1.5 |
| ii. 2.1 | vii. 1.9 |
| iii. 2.01 | viii. 1.99 |
| iv. 2.001 | ix. 1.999 |
| v. 2.0001 | x. 1.9999 |

b. Estimate instantaneous velocity at $t = 2$.

§1.1.1 Assignment Problems

1. For the function $f(x) = x^3 - 3x^2$ and the point P given by $x = 3$, answer each of the following questions.

a. For the points Q given by the following values of x , compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .

- | | |
|-----------|------------|
| i. 3.5 | vi. 2.5 |
| ii. 3.1 | vii. 2.9 |
| iii. 3.01 | viii. 2.99 |
| iv. 3.001 | viv. 2.999 |
| v. 3.0001 | vv. 2.999 |

b. Use the information from (a) to estimate the slope of the tangent line to $f(x)$ at $x = 3$ and write down the equation of the tangent line.

2. For the function $g(x) = \frac{x}{x^2+4}$ and the point P given by $x = 0$, answer each of the following questions.
- a. For the points Q given by the following values of x , compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .
- | | |
|----------|------------|
| i. 1 | vi. -1 |
| ii. 0.5 | vii. -0.5 |
| iii. 0.1 | viii. -0.1 |
| iv. 0.01 | viv. -0.01 |
| v. 0.001 | vv. -0.001 |
- b. Use the information from (a) to estimate the slope of the tangent line to $g(x)$ at $x = 0$ and write down the equation of the tangent line.
3. For the function $h(x) = 2 - (x + 2)^2$ and the point P given by $x = -2$, answer each of the following questions.
- a. For the points Q given by the following values of x , compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .
- | | |
|------------|------------|
| i. -2.5 | i. -1.5 |
| ii. -2.1 | ii. -1.9 |
| iii. -2.01 | iii. -1.99 |
| iv. -2.001 | iv. -1.999 |
| v. -2.0001 | v. -1.9999 |
- b. Use the information from (a) to estimate the slope of the tangent line to $h(x)$ at $x = -2$ and write down the equation of the tangent line.
4. For the function $P(x) = e^{2-8x^2}$ and the point P given by $x = 0.5$, answer each of the following questions.
- a. For the points Q given by the following values of x , compute (accurate to at least 8 decimal places) the slope, m_{PQ} , of the secant line through points P and Q .
- | | |
|------------|-------------|
| i. 1 | vi. 0 |
| ii. 0.51 | vii. 0.49 |
| iii. 0.501 | viii. 0.499 |
| iv. 0.5001 | viv. 0.4999 |
| v. 0.50001 | vv. 0.49999 |
- b. Use the information from (a) to estimate the slope of the tangent line to $h(x)$ at $x = 0.5$ and write down the equation of the tangent line.
5. The amount of grain in a bin is given by $V(t) = \frac{11t+4}{t+4}$. Answer each of the following questions.
- a. Compute (accurate to at least 8 decimal places) the average rate of change of the amount of grain in the bin between $t = 6$ and the following values of t .

- | | |
|-----------|------------|
| i. 6.5 | vi. 5.5 |
| ii. 6.1 | vii. 5.9 |
| iii. 6.01 | viii. 5.99 |
| iv. 6.001 | viv. 5.999 |
| v. 6.0001 | vv. 5.9999 |
- b. Use the information from (a) to estimate the instantaneous rate of change of the volume of grain in the bin at $t = 6$.
6. The population (in thousands) of insects is given by $P(t) = 2 - \frac{1}{\pi} \cos(3\pi t) \sin\left(\frac{\pi t}{2}\right)$. Answer each of the following questions.
- a. Compute (accurate to at least 8 decimal places) the average rate of change of insects between $t = 4$ and the following values of t . Make sure your calculator is set to radians for the computations.
- | | |
|-----------|------------|
| i. 4.5 | vi. 3.5 |
| ii. 4.1 | vii. 3.9 |
| iii. 4.01 | viii. 3.99 |
| iv. 4.001 | viv. 3.999 |
| v. 4.0001 | vv. 3.9999 |
- b. Use the information from (a) to estimate the instantaneous rate of change of the population of insects at $t = 4$.
7. The amount of water in a holding tank is given by $V(t) = 8t^4 - t^2 + 7$. Answer each of the following questions.
- a. Compute (accurate to at least 8 decimal places) the average rate of change of the amount of water in the tank between $t = 0.25$ and the following values of t .
- | | |
|------------|------------|
| i. 1 | vi. 0 |
| ii. 0.5 | vii. 0.1 |
| iii. 0.251 | viii. 2.49 |
| iv. 0.2501 | viv. 2.499 |
| v. 0.25001 | vv. 2.4999 |
- b. Use the information from (a) to estimate the instantaneous rate of change of the volume of water in the tank at $t = 0.25$.
8. The position of an object is given by $s(t) = t^2 + \frac{72}{t+1}$. Answer each of the following questions.
- a. Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 5$ and the following values of t .
- | | |
|-----------|------------|
| i. 5.5 | vi. 4.5 |
| ii. 5.1 | vii. 4.9 |
| iii. 5.01 | viii. 4.99 |
| iv. 5.001 | viv. 4.999 |
| v. 5.0001 | vv. 4.9999 |

- b. Use the information from (a) to estimate the instantaneous velocity of the object at $t = 5$ and determine if the object is moving to the right (i.e. the instantaneous velocity is positive), moving to the left (i.e. the instantaneous velocity is negative), or not moving (i.e. the instantaneous velocity is zero).
9. The position of an object is given by $s(t) = 2 \cos(4t - 8) - 7 \sin(t - 2)$. Note that a negative position here simply means that the position is to the left of the “zero position” and is perfectly acceptable. Answer each of the following questions.
- a. Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 2$ and the following values of t . Make sure your calculator is set to radians for the computations.
- | | |
|-----------|------------|
| i. 2.5 | vi. 1.5 |
| ii. 2.1 | vii. 1.9 |
| iii. 2.01 | viii. 1.99 |
| iv. 2.001 | viv. 1.999 |
| v. 2.0001 | vv. 1.9999 |
- b. Use the information from (a) to estimate the instantaneous velocity of the object at $t = 2$ and determine if the object is moving to the right (i.e. the instantaneous velocity is positive), moving to the left (i.e. the instantaneous velocity is negative), or not moving (i.e. the instantaneous velocity is zero).
10. The position of an object is given by $s(t) = t^2 + \frac{72}{t+1}$. Answer each of the following questions.
- a. Determine the time(s) in which the position of the object is at $s = -5$.
- b. Estimate the instantaneous velocity of the object at each of the time(s) found in part (a) using the method discussed in this section.