College Algebra

Edition 1.0

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Content Adapted from
Precalculus: An Investigation of Functions (2nd Ed)
by Lippman and Rassumsen

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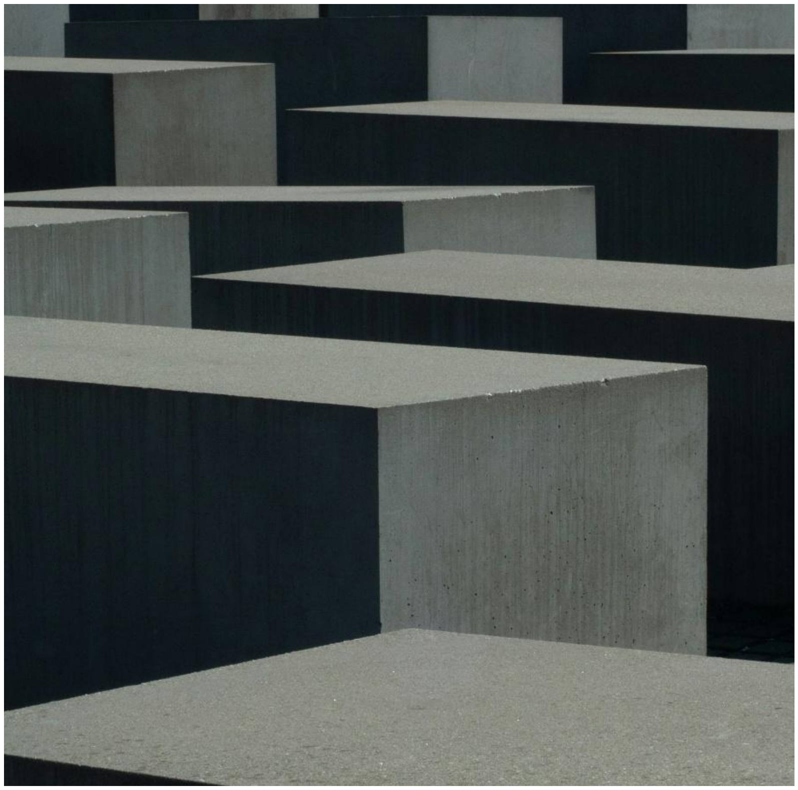
Chapter 0: Prerequisites



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In this chapter, we're laying foundations for the content to come. You may be comfortable with some of the topics, or you may need a little support in dusting off your skills. Read the examples, try the practice problems, and always focus on the reasoning, not just the answer. If nothing else, we recommend you check out Section 0.4 on solving equations. These foundational skills will be essential for the rest of the course.

Section 0.1: Expressions and Equations

Expressions vs. Equations: Why should we care?

Mathematicians like to spend a lot of time making definitions, and math teachers like to spend a lot of time teaching those definitions to students. This can feel like a right of passage or hazing, depending on your point of view. With a lot of the definitions that we offer in before-calculus mathematics, students might justifiably ask: “Why should we care?”

Here, we’d like to make the argument that:

Once you know what something IS, then you know what you can and can’t DO with it.

You might be familiar with the old adage “Once you have a hammer, every problem looks like a nail.” We’d like you to consider the reverse; once you recognize that you’re trying to use a nail, you know you should reach for your hammer!

Two of our most fundamental structures to begin with in algebra will be **expressions** and **equations**. We’ll define these structures and take a look at some of the things we can do with them.

Expressions

Algebraic Expression
An **algebraic expression** consists of numbers and variables joined together with mathematical operations (such as addition, subtraction, multiplication, division, exponentiation). An algebraic expression DOES NOT have an equal sign in it.

A great way to internalize a definition is to look at examples and non-examples. Let’s practice identifying algebraic expressions!

Example 1

Is each of the following an algebraic expression? Explain why or why not.

a. $5x^2 - 2$
b. $3x = 9$
c. $5 + 2 = 7$
d. $\frac{5}{x} - 1$

Solutions:
Remember, what we're looking for are "numbers and variables joined together with math", and we want to make sure we *don't* have any equal signs. That makes options a and d examples of algebraic expressions. Examples b and c *aren't* algebraic expressions, because they each include an equal sign!

Try it now
Write two examples of algebraic expressions, and two examples of things that *aren't* algebraic expressions.

Now we can identify algebraic expressions: so what? Well, if we know we have an algebraic expression, then we know there are a few things we can do with it.

What can we do with an algebraic expression?

Once we know we have an expression, what can we do with it? Our guiding principle will be this:

When working with an algebraic expression, we never want to change its value.

What does this mean? Well, let's work with the super sophisticated expression of 2 to investigate.

Example 2: Multiplying by 1 or Adding 0
When working with the number 2, we **can** do things like:
- Multiply by 1, because $2 \cdot 1$ is still just 2
- Add 0, because $2 + 0$ is 2

However, we **can't** do things like:
- Multiply by 2, because $2 \cdot 2$ is 4; we've changed the value of our expression from the original 2 into a 4.
- Add 3, because $2 + 3$ is 5; we've changed the value of our expression from the original 2 into a 5.

Example 3: Fancy forms of 1

Rewrite $\frac{4x}{5y}$ by multiplying by a fancy form of 1.

What do we mean by a fancy form of 1? Well, we use the idea that "a number divided by itself equals 1" to conclude that things like $\frac{2}{2}$ or $\frac{17}{17}$ are actually 1, they're just a little fancier looking. So, we could rewrite our expression as

$$\frac{4x}{5y} \cdot \frac{2}{2}$$
$$= \frac{8x}{10y}$$

Or

$$\frac{4x}{5y} \cdot \frac{17}{17}$$
$$= \frac{68x}{85y}$$

In the end, then, we're saying that $\frac{4x}{5y}$ is the same as $\frac{8x}{10y}$, and the same as $\frac{68x}{85y}$. The new versions are just a bit fancier than the original!

Example 4: Fancy forms of 0

Rewrite $6x - 2$ by adding a fancy form of 0

What does a fancy form of 0 look like? We'll use the idea that a number minus itself makes 0, so things like $7 - 7$ or $13x - 13x$ are forms of 0. That means that

$$6x - 2 + 7 - 7$$

and

$$6x - 2 + 13x - 13x$$

are both equivalent to the original $6x - 2$.

To see what else we can do, let's look at a slightly more complicated expression, like $3x + 5y$.

Example 5: Rearranging Addition or Multiplication

In the expression $3x + 5y$, we can rearrange the order of addition:

$$3x + 5y$$
$$= 5y + 3x$$

If we wanted, we could also reorder the multiplication represented by $3x$:

$$3x + 5y$$
$$= x3 + 5y$$

We can do this because **addition is commutative** and **multiplication is commutative**. That means that if we change the order of addition (or the order of multiplication), it doesn't affect the final value of the expression.

We've focused a lot on addition and multiplication; what about subtraction and division? If we are careful, we can think about subtraction and division as being special forms of addition and multiplication, respectively.

Example 6: Changing Subtraction to Addition

Rewrite the expression $7x - 4y$ in terms of addition. Then, rearrange.

Subtraction is equivalent to **adding the opposite**, so we can rewrite as follows:

$$7x + (-4y)$$

The parentheses are mostly here to help us visually sort out the addition followed by a negative symbol. Now, we can rearrange.

$$7x + (-4y)$$
$$= -4y + 7x$$

Notice: the original subtraction is **not** commutative! $7x - 4y$ would have a different value than $4y - 7x$. If we want to rearrange, we should first think in terms of addition!

Example 7: Changing division to Multiplication

Rewrite the expression $\frac{7x}{4+y}$ in terms of multiplication. Then, rearrange.

Here, the fraction bar represents division, so our expression could be thought of as

$$(7x) \div (4 + y)$$

Division is equivalent to **multiplying by the reciprocal**, or what you may know as **copy dot flip** or **keep change flip**. The thing to keep in mind here is that you must flip the entire denominator together (not each piece of it individually). So,

$$\frac{7x}{4 + y}$$
$$= 7x \div (4 + y)$$
$$= 7x \cdot \frac{1}{4 + y}$$

Then, if we want to rearrange, we could do

$$= \frac{1}{4 + y} \cdot 7x$$

Ways to rearrange algebraic expressions

To rearrange an algebraic expression without affecting its value, we may:

- Multiply by 1
- Add zero
- Reorder addition (or change subtraction to addition, then reorder)
- Reorder multiplication (or change multiplication to division, then reorder)

Example 8: Rearrange an algebraic expression

Which of the following expressions are equivalent to $\frac{6+x}{2} - 3y$?

a. $6 + x \cdot \frac{1}{2} - 3y$
b. $\frac{x+6}{2} - 3y$
c. $3y - \frac{x+6}{2}$
d. $-3y + \frac{x+6}{2}$
e. $\frac{4(6+x)}{4 \cdot 2} - 3y$
f. $\frac{6+x}{2} + 5 - 3y$
g. $\frac{6+x}{2} + 5 - 3y - 5$

Solutions:
We'll take this an expression at a time.

a. When we look at $6 + x \cdot \frac{1}{2} - 3y$, it's almost a rewrite of division, but it's missing a key feature: parentheses. We want to make sure we're multiplying the **entire** numerator by the reciprocal of the **entire** denominator. When we write $6 + x \cdot \frac{1}{2}$, only the x is being multiplied by the $\frac{1}{2}$, not the entire $6 + x$. To fix this, then, our expression should be $(6 + x) \cdot \frac{1}{2} - 3y$. Now it's equivalent to the original.
b. Next we consider $\frac{x+6}{2} - 3y$. The only thing that's happened here is that the addition in the numerator has been rearranged, a perfectly valid move. Therefore, this expression is equivalent to the original.
c. In $3y - \frac{x+6}{2}$, we swapped around the subtraction. The problem is, subtraction isn't commutative, so this is not equivalent to the original expression.
d. Now this is more like it. If we first change $\frac{6+x}{2} - 3y$ into $\frac{6+x}{2} + (-3y)$, then we have some addition that we can rearrange. So, we rewrite as $-3y + \frac{6+x}{2}$ for an equivalent expression.
e. Check out the extra 4's in $\frac{4(6+x)}{4 \cdot 2} - 3y$. This comes from multiplying that first fraction by $\frac{4}{4}$, a fancy form of 1, so this expression is still equivalent to the original.

f. In $\frac{6+x}{2}+5-3y$, we simply added a 5 to the expression. That definitely changes the value of the expression, so it is not equivalent to the original.
g. This last expression can be created if we add $5-5$ to the original expression. Is that allowed? Sure, because $5-5$ is a fancy form of 0. So we have $\frac{6+x}{2}-3y+5-5$, which then allows us to (carefully) rearrange our addition to get $\frac{6+x}{2}+5-3y-5$.

Try it Now

Which of the following expressions are equivalent to $7y+3-\frac{5}{2-x}$?

a. $3+y7-\frac{5}{2-x}$
b. $7y+\frac{5}{2-x}-3$
c. $7y-\frac{5}{2-x}+3$
d. $7y+3-\frac{5}{x-2}$
e. $7y+3-5\cdot\frac{1}{2-x}$
f. $7y+3-5\cdot\frac{1}{2}-x$

When faced with an algebraic expression, we can also do a whole host of things that usually fall under the category of "simplification". Depending on our goals, simplification might mean **distribution, combining like terms, factoring, and more.**

Example 9: Distribution

Simplify $2(x+5)$.

Solution:
Distribution is what we do when multiplication meets addition. Here we have 2 multiplied by an addition expression, $x+5$.

When we first learned about multiplication, it was likely presented as repeated addition. For example, 2 times 3 is 3 added to itself two times:

$$2\cdot 3$$
$$=3+3$$
$$=6$$

The same principle applies here. When we have 2 times $x+5$, that means we want to add $x+5$ to itself two times:

$$2\cdot(x+5)$$
$$=x+5+x+5$$

which we can then reorder because of the commutativity of addition:

$$=x+x+5+5$$

Now, we notice that $x + x$ is 2 times x, and $5 + 5$ is 2 times 5, so we have

$$= 2x + 2 \cdot 5$$
$$= 2x + 10$$

In short, the 2 is "distributed" to both the x and the 5:

$$2(x + 5)$$
$$= 2 \cdot x + 2 \cdot 5$$

Distributive Property

If a, b, and c are real numbers, or variables representing real numbers, then

$$a(b + c)$$
$$= a \cdot b + a \cdot c$$

Because subtraction can be rewritten as adding the opposite, the distributive property also applies to

$$a(b - c)$$
$$= a \cdot b - a \cdot c.$$

Notice that the distributive property only applies in this specific case: when we want to multiply by an addition (or subtraction) expression. Let's take a detour to make sure we understand when distribution does and doesn't apply.

Example 10: To Distribute, or Not to Distribute

In each expression, determine whether distribution can be applied. Then, simplify.

a. $5(3x)$
b. $5(3 + x)$
c. $5(3 - x)$
d. $(3 + x)^5$
e. $(x + 5)(x + 3)$

Solutions:

a. Check out the operations in $5(3x)$. That's 5 times 3 times x; there's no addition or subtraction in sight, so distribution doesn't apply here. To simplify, we can think of this instead as $5 \cdot 3 \cdot x$, so $15x$.

b. Now we have 5 times $3 + x$, so we have the required multiplication by an addition expression. So, we distribute the 5:

$$5 \cdot 3 + 5 \cdot x$$
$$= 15 + 5x$$

c. Distribution still applies when we have a number (5) times a subtraction expression $(3 - x)$, so to simplify we can distribute the 5:

$$5 \cdot 3 - 5 \cdot x$$
$$= 15 - 5x$$

d. Now we have an addition expression, $3 + x$, **raised to a power of 5.** That's not multiplication, so distribution doesn't apply here. It would be incorrect to write $3^5 + x^5$. In fact, if we wanted to simplify, we'd need to think about $(3 + x)$ multiplied by itself 5 times, like $(3 + x)(3 + x)(3 + x)(3 + x)(3 + x)$.

e. We in fact *do* have multiplication meeting with addition here, so we can distribute…carefully. We can think of this as $x + 5$ times an addition statement, $x + 3$. So we'll distribute the $x + 5$ itself:

$$(x + 5) \cdot x + (x + 5) \cdot 3$$

We can rearrange the multiplication a little, then distribute again:

$$x(x + 5) + 3(x + 5)$$
$$= x \cdot x + x \cdot 5 + 3 \cdot x + 3 \cdot 5$$
$$= x^2 + 5x + 3x + 15$$

This expression can be simplified even further, but first we must discuss some more simplification methods!

Try it Now

Decide whether the distributive property can be used in each expression. Then simplify.

a. $2 + (1 + x)$
b. $2(1 + x)$
c. $(x + 2)(x - 1)$
d. $(1 - x)2$
e. $(1 + x)^2$
f. $2(1x)$

Another major move we can make while simplifying expressions is **combining like terms.** The "combining" part usually refers to adding or subtracting. To think about what we're adding or subtracting, let's define **terms** and **like terms**.

Algebraic Terms

An algebraic term consists of a constant (a number) multiplied by a variable (or variables) raised to a power.

The constant part of the algebraic term is called a **coefficient.**

Like Terms

Two algebraic terms are **like terms** if they have the exact same combination of variables raised to the exact same powers.

Example 11: Terms and Like Terms

Determine which of the following expressions represent terms. Then, determine which are like terms.

a. $3x$
b. $3 + x$
c. $3x^2$
d. $3x^2y$
e. $5x^2$
f. $7x^2y$
g. $2xy^2$

Solutions:
Only item b is not a term. The 3 and x are joined with addition, not multiplication. In fact, the 3 and x are each an individual term!

The like terms are $3x^2$ and $5x^2$, because they both have the same variable (x) raised to the same power (2). In the $3x^2$ term, 3 is the coefficient. Similarly in the $5x^2$ term, the 5 is the coefficient. Notice that $3x$ is **not** a like term in this group, because though it has an x, that x is raised to the (invisible) first power.

The $3x^2y$ and $7x^2y$ are also like terms, because they both have the same combination of variables (x and y) each raised to the same power (2 and 1, respectively). Notice that $2xy^2$ is **not** a like term in this group, because its x is raised to the 1st power and its y is raised to the 2nd power; it has the same variables, but the wrong powers on those variables.

Now we can work on combining like terms; that is, we'll add or subtract them.

Example 12: Combining Like Terms

Simplify the algebraic expression by combining like terms.

$$3xy^2 + 7x^2y + 2xy^2 - x^2y$$

Let's start by identifying our like terms. We have $3xy^2$ and $2xy^2$ for one group, and $7x^2y$ and $-x^2y$ for our other group. If we change that subtraction to adding the opposite, we can rearrange all the addition to put the like terms next to each other.

$$3xy^2 + 7x^2y + 2xy^2 + (-x^2y)$$
$$= 3xy^2 + 2xy^2 + 7x^2y + (-x^2y)$$

Now for the combining part. We're basically counting up how many of each like term we have. So there are 3 xy^2's plus another 2 of the xy^2's , making a total of $5xy^2$. Similarly, we have 7 of the x^2y's, plus a negative 1 x^2y, so there are 6 total $x^2y's$. This means, we have

$$5xy^2 + 6x^2y$$

for our final simplified answer.

Notice, we added the coefficients of the like terms, but did nothing to change the variables.

Combining Like Terms

To combine like terms, add or subtract the coefficients, but do not change the variables in any way.

Try it Now

Simplify by combining like terms:

$$4x^2 - 3x - 2 + 8x^2 - 2x + 1$$

Example 13

Simplify completely:

$$5(x - 6) - 3(x + 4)$$

Now we have some distribution and eventually some like terms to combine. To ensure that the negatives don't mess us up, though, let's change that -3 into adding the opposite:

$$5(x - 6) + (-3)(x + 4)$$

Now we distribute, rearrange, and combine like terms.

$$\begin{gathered} 5x - 5 \cdot 6 + \ -3x + \ -3 \cdot 4 \\ = 5x + -30 + \ -3x + \ -12 \\ = 5x + \ -3x + \ -30 + \ -12 \\ = 2x - 42 \end{gathered}$$

Try it Now

Simplify completely:

$$7(xy + 3x^2) - 2(x^2 + 5xy)$$

Simplifying Algebraic Expressions

To simplify an algebraic expression, we might want to:

- Distribute
- Combine like terms

Now we have a lot of tools for rewriting and simplifying algebraic expressions **without** changing the value of the expression. We can also **evaluate** an expression to find its value in particular circumstances. To do this kind of evaluation, we should make sure we're aware of the order of operations.

Order of Operations

When finding the value of an expression, we should do operations in the following order:

1. Parentheses
2. Exponents
3. Multiplication/Division (from left to right)
4. Addition/Subtraction (from left to right)

A common mnemonic for remembering this order is **PEMDAS**.

Some things in mathematics reflect natural truths about the world we live in, while others reflect agreements made by the mathematical community. The order of operations is an agreement to do operations in a particular order so that, given a string of math, everyone ends up with the same answer. Without this sort of agreement, it would be very difficult to do everything from splitting a check and figuring tip at a restaurant, to sending people to Mars.

Example 14

Evaluate each of the following expressions.

a. $3 \cdot 4 - (5 - 3)^2 + 2$

b. $\frac{4+2}{4-1}$

c. $\frac{1}{2}\left(\frac{3}{4} + \frac{1}{2}\right)$

Solutions:

For each of these, our primary job is to attend to the order of operations.

a.

$3 \cdot 4 - (5 - 3)^2 + 2$	
$= 3 \cdot 4 - (\mathbf{2})^2 + 2$	parentheses
$= 3 \cdot 4 - \mathbf{4} + 2$	exponents
$= \mathbf{12} - 4 + 2$	multiplication and division
$= \mathbf{8} + 2$	addition and subtraction
$= 10$	addition and subtraction

Notice that once this expression simplified down to all addition and subtraction, we *didn't* necessarily do addition first, then subtractions. The "AS" in our "PEMDAS" mnemonic tells us to do addition and subtraction **from left to right**.

b. When we have a big fraction like this, we want to think of the entire numerator and the entire denominator as each being in their own set of parentheses (even if the expression isn't explicitly written like that).

$$\begin{aligned} &\frac{(4+2)}{(4-1)} \\ &= \frac{6}{3} && \text{parentheses} \\ &= 2 && \text{division} \end{aligned}$$

c. Our friends, fractions! We'll review some fraction basics *while* using the order of operations.

First, recall that to add fractions, we need to make sure they have a common denominator. How can we make a common denominator? Multiply our fractions by a fancy form of 1 (one of the things we're always allowed to do to an expression!).

$$\begin{aligned} &\frac{1}{2}\left(\frac{3}{4}+\frac{1}{2}\right) \\ &= \frac{1}{2}\left(\frac{3}{4}+\frac{1}{2}\cdot\frac{2}{2}\right) && \text{fancy form of 1} \\ &= \frac{1}{2}\left(\frac{3}{4}+\frac{2}{4}\right) && \text{multiplication} \\ &= \frac{1}{2}\left(\frac{5}{4}\right) && \text{addition} \\ &= \frac{5}{8} && \text{multiplication} \end{aligned}$$

If you find yourself in need of a fraction review, check out this resource: Fractions (Review) – Intermediate Algebra.

Try it Now

Evaluate each of the following expressions:

a. $\frac{3^2+2(-4)}{(4-3)^3}$

b. $\frac{2}{25}-\left(1+\frac{1}{5}\right)^2$

Now we return to algebraic expressions. Oftentimes, we want to find the value of an algebraic expression under particular conditions. This requires careful attention to parentheses and the order of operations.

Example 15

Consider the expression x^2-3xy. Evaluate for the following values of x and y.

a. $x=2,\ y=1$

b. $x=-2,\ y=3$

c. $x=z,\ y=z+2$

Solutions:
We want to replace the x and y in our expression with the indicated values. As we do so, it's a good idea to put our values into parentheses, so we can keep proper track of the operations we're doing.

a. $x^2 - 3xy$ where $\boldsymbol{x = 2}, \boldsymbol{y = 1}$
We'll replace each $\boldsymbol{x}$ with $(\mathbf{2})$ and each $\boldsymbol{y}$ with $(\mathbf{1})$.

$$\begin{aligned} & \boldsymbol{x}^2 - 3\boldsymbol{xy} & \\ &= (\mathbf{2})^2 - 3(\mathbf{2})(\mathbf{1}) && \text{substitute} \\ &= \mathbf{4} - 3(2)(1) && \text{exponents} \\ &= 4 - \mathbf{6}(1) && \text{multiplication} \\ &= 4 - \mathbf{6} && \text{multiplication} \\ &= -2 && \text{subtraction} \end{aligned}$$

b. $x^2 - 3xy$ where $x = -2$, $y = 3$
This time, we replace each $\boldsymbol{x}$ with $(\mathbf{-2})$ and each $\boldsymbol{y}$ with $(\mathbf{3})$.

$$\begin{aligned} & \boldsymbol{x}^2 - 3\boldsymbol{xy} & \\ &= (\mathbf{-2})^2 - 3(\mathbf{-2})(\mathbf{3}) && \text{substitute} \\ &= \mathbf{4} - 3(-2)(3) && \text{exponents}^* \\ &= 4 + (\mathbf{-3})(-2)(3) && \text{change subtraction}^{**} \\ &= 4 + \mathbf{6}(3) && \text{multiplication} \\ &= 4 + \mathbf{18} && \text{multiplication} \\ &= 22 && \text{addition} \end{aligned}$$

*Notice that $(-2)^2$ is $-2 \cdot -2$, hence **positive** 4.
**Changing the subtraction into adding the opposite helps us to keep proper track of the signs as we work on the multiplying steps.

d. $x^2 - 3xy$ where $x = z$, $y = z + 2$
Now we have an extra variable floating around, but our process is still the same! We'll replace each $\boldsymbol{x}$ with $\boldsymbol{z}$ and replace each $\boldsymbol{y}$ with $(\boldsymbol{z} + \mathbf{2})$.

$$\begin{aligned} & \boldsymbol{x}^2 - 3\boldsymbol{xy} & \\ &= \boldsymbol{z}^2 - 3\boldsymbol{z}(\boldsymbol{z} + \mathbf{2}) && \text{substitute} \\ &= z^2 + (\mathbf{-3}\boldsymbol{z})(z + 2) && \text{change subtraction}^* \\ &= z^2 + \mathbf{-3}\boldsymbol{z} \cdot \boldsymbol{z} + \mathbf{-3}\boldsymbol{z} \cdot \mathbf{2} && \text{parentheses (distribute)}^{**} \\ &= z^2 + -3z \cdot z + \mathbf{-3} \cdot \mathbf{2} \cdot \boldsymbol{z} && \text{reorder multiplication} \\ &= z^2 + \mathbf{-3}\boldsymbol{z}^{\mathbf{2}} + \mathbf{-6}\boldsymbol{z} && \text{multiply} \\ &= \mathbf{-2}\boldsymbol{z}^{\mathbf{2}} - 6z && \text{combine like terms} \end{aligned}$$

$*$ Again, we change the subtraction to help keep track of the signs.
**Here, the $z + 2$ in the parentheses can't be added more, because z and 2 aren't like terms. So, instead, we use the distributive property to evaluate the parentheses.

We'll take a look at a few more examples before moving on.

Example 16

Evaluate each expression for the given values.

a. $\frac{x}{2x-1}$ where $x = 3$

b. $x - (4 - x)$ where $x = -1$

c. $(xy^2)^3$ where $x = \frac{1}{2}$ and $y = 2$

Solutions:

a. $\frac{x}{2x-1}$ where $x = 3$

$$\frac{\mathbf{x}}{2\mathbf{x} - 1}$$

$$= \frac{\mathbf{3}}{(2(\mathbf{3}) - 1)} \quad \text{Substitute}^*$$

$$= \frac{3}{(\mathbf{6} - 1)} \quad \text{Parentheses: multiplication}$$

$$= \frac{3}{\mathbf{5}} \quad \text{Parentheses: subtraction}$$

When we substituted, we also put the denominator in a set of parentheses. That helps us to keep track of tl

b. $x - (4 - x)$ where $x = -1$

$$\mathbf{x} - (4 - \mathbf{x})$$
$$= (\mathbf{-1}) - \left(4 - (\mathbf{-1})\right) \quad \text{Substitute}$$
$$= -1 - (4 + \mathbf{1}) \quad \text{Change Subtraction}$$
$$= -1 - (\mathbf{5}) \quad \text{Parentheses}$$
$$= -6 \quad \text{Subtraction}$$

Alternatively, we could have chosen to simplify this expression, then substitute.

$$x - (4 - x)$$
$$= x + (\mathbf{-1})(4 + -x) \quad \text{Change Subtraction}^*$$
$$= x + \mathbf{-1 \cdot 4 + (-1)(-x)} \quad \text{Distribute}$$
$$= x \mathbf{- 4} + \mathbf{x} \quad \text{Multiply}$$
$$= \mathbf{2x} - 4 \quad \text{Combine like terms}$$
$$= 2(\mathbf{-1}) - 4 \quad \text{Substitute } \mathbf{x = -1}$$
$$= \mathbf{-2} - 4 \quad \text{Multiplication}$$
$$= -6 \quad \text{Subtraction}$$

*When we changed the subtraction in front of the parentheses, we also wrote down the invisible 1 that is implied there. So we have $x - 1(4 - x)$, which can be rewritten as $x + (-1)(4 - x)$.

c. $(xy^2)^3$ where $x = \frac{1}{2}$ and $y = 2$

$$
\begin{aligned}
&(xy^2)^3 \\
&= \left(\left(\frac{1}{2}\right)(2)^2\right)^3 && \text{Substitute} \\
&= \left(\left(\frac{1}{2}\right)(4)\right)^3 && \text{Parentheses (exponents)} \\
&= \left(\frac{4}{2}\right)^3 && \text{Parentheses (multiplication)} \\
&= 2^3 && \text{Parentheses (division)} \\
&= 8 && \text{Exponents}
\end{aligned}
$$

Try it Now

Evaluate each expression for the given values.

a. $a(b+2)$ for $a = 3, b = -4$
b. $a(b+2)^2$ for $a = 3$, $b = -1$
c. $4a - (b-2)$ for $a = 2, b = 3$
d. $4a - (b-2)$ for $a = \frac{1}{4}, b = \frac{1}{3}$
e. $\frac{a^2-2a}{a+4}$ for $a = 1$

Equations

Now we turn our sights to equations. Equations are a lot like expressions, with one key difference: they have an equal sign!

Equations
A mathematical statement in which we write that one algebraic expression **equals** another. Equations are written in the structure: $\text{algebraic expression } 1 = \text{algebraic expression } 2$ The key feature we're looking for is an **equal** sign in the given statement!

One of our main tasks when faced with an equation is solving for a value of the variable(s) which makes the equation a true statement.

Notice that we were using equals signs when we simplified or evaluated algebraic expressions, but we did not write our work all in one line. That is, when we simplify something like $2(x-3)$, we write

$$
\begin{aligned}
&2(x-3) \\
&= 2 \cdot x - 2 \cdot 3
\end{aligned}
$$

instead of

$$2(x-3) = 2x - 2 \cdot 3.$$

When we simplify algebraic expressions, we'll write each step on a new line, with the equal sign at the beginning, to distinguish from when we're simplifying or evaluating versus when we're solving an equation.

Example 17: Verifying solutions

For the equation $x^2 - 4 = 3x$, determine whether each value of x is a valid solution.

a. $x = 0$
b. $x = 1$
c. $x = -1$

Solutions:

When we are determining whether a value of a variable is a valid solution to an equation, we want to evaluate each side of the equation for the given value of the variable and see if they are equal.

a.

$$\begin{array}{ccccl} x^2 - 4 & = & 3x & & \\ (0)^2 - 4 & ?= & 3(0) & & \text{Substitute } x = 0 \\ 0 - 4 & ?= & 0 & & \text{Evaluate each side}^* \\ -4 & \neq & 0 & & \end{array}$$

Let's dig into what we're doing here. First of all, while we're checking our potential solution, we use a $?=$ instead of an $=$. We're not claiming that the two sides are equal, we're checking whether they are indeed equal.

Notice also that we're working with each side independently, simplifying each expression using the order of operations.

Now, at the end, we have -4 on the left and 0 on the right, which are not equal. This means that the value we substituted in, $x = 0$, is *not* a solution for the equation, because it doesn't cause the expression on the left to be equal to the expression on the right.

b.

$$\begin{array}{ccccl} x^2 - 4 & = & 3x & & \\ (1)^2 - 4 & ?= & 3(1) & & \text{Substitute } x = 1 \\ 1 - 4 & ?= & 3 & & \text{Evaluate each side} \\ -3 & \neq & 3 & & \end{array}$$

When we substitute $x = 1$ into the equation, the left side evaluates to -3, but the right side evaluates to 3. Since $-3 \neq 3$, $x = 1$ is not a solution to the equation.

c.

$$\begin{array}{ccc} x^2-4 & = & 3x & \\ (-1)^2-4 & ?= & 3(-1) & \text{Substitute } x=-1 \\ 1-4 & ?= & -3 & \text{Evaluate each side} \\ -3 & = & -3 & \end{array}$$

This time, finally, when we substitute in -1 for x, we get a true statement at the end: $-3=-3$. This means $x=-1$ **is** a solution to the equation!

As we see from the example above, one thing we might do with equations is evaluate them for given values of the variable, just like we did with expressions. When we do this, it's a good idea to use a ?= instead of an = while we're deciding whether the two sides are indeed equal.

How can we find potential solutions to test, though? That's a really big topic, which we'll revisit time and again throughout this text. For now, we'll review a few of the main concepts for solving equations.

Solving equations: balancing

When our goal is solving an equation, we're now allowed to do things that alter the value of the algebraic expressions, **as long as we alter the expressions on both sides in the same way.** This is often called "balancing" the equation.

Example 18: Balancing an equation

Consider the equation $x+4=2x-2$. There are many things we could do to the equation to keep it balanced.

$$x+4\mathbf{+1}=2x-2\mathbf{+1}$$

Here we added 1 to both sides of the equation. If we had just the expression $x+4$, we couldn't add 1 to that; it changes the value! But, if $x+4$ is the same as $2x-2$, then they'll still be the same if we add 1 to both of them.

$$\mathbf{2}(x+4)=\mathbf{2}(2x-2)$$

This time we multiplied both sides by 2. This is allowed as long as that multiplication affects the whole expression on each side; that's why we put the $x+4$ and the $2x-2$ into parentheses.

$$(x+4)^{\mathbf{3}}=(2x-2)^{\mathbf{3}}$$

We could also raise both sides to the same power.

$$\frac{x+4}{-1} = \frac{2x-2}{-1}$$

Or divide both sides by the same number.

$$\sqrt{x+4} = \sqrt{2x-2}$$

Or square root both sides.

The point of the example above is that we can do a whole bunch of things to an equation that we weren't able to do to an individual expression, as long as we do the exact same thing to both sides of the equation.

The things we were doing to that equation weren't very helpful, though, for one of our most common goals: we'd like to solve for the value(s) of x which make the equation true. So, we'll do the same operations to both sides of the equation with the goal of isolating x.

Example 19: Solving an equation

Solve $x + 4 = 3x - 2$

Solution:
Goal 1: move both x terms to the same side.
Currently we have an x on the left side and a $3x$ on the right side. We'd like to combine those, but we can't when they're on opposite sides. So, we can move one of these x terms to the other side by *undoing* an operation.

On the left, we have x plus 4. If we want to move that x, we need to zero it out; so we'll subtract an x. To keep the equation balanced, we'll subtract x from both sides.

$$x + 4 \mathbf{- x} = 3x - 2 \mathbf{- x}$$

Now we'll combine the like terms in the expressions on each side of the equation.

$$\begin{array}{rcl} x - x + 4 & = & 3x - x - 2 \\ 0x + 4 & = & 2x - 2 \\ 4 & = & 2x - 2 \end{array}$$

Now that we only have one variable term, we want to work on isolating the variable. To work on this goal, we'll undo anything else happening to the variable, in **reverse** of the order of operations.

Our x is being multiplied by 2, then subtracted by 2. If we were evaluating, we'd do the multiplication first, then the subtraction. We're solving, so we'll go in reverse. We want to

undo the -2 first by adding 2. But, because that changes the value of the expression, we have to do it to both sides!

$$\begin{array}{ccc} 4+\mathbf{2} & = & 2x-2+\mathbf{2} \\ 6 & = & 2x+0 \\ 6 & = & 2x \end{array}$$

Now our x is only being multiplied by 2. We'll undo that by dividing by 2, on both sides.

$$\begin{array}{ccc} \frac{6}{2} & = & \frac{2x}{2} \\ 3 & = & x \end{array}$$

Our claim now is that $x = 3$ is a solution to the equation $x + 4 = 3x - 2$. We can check that claim by substituting and evaluating.

$$\begin{array}{cccc} x+4 & = & 3x-2 & \\ 3+4 & ?= & 3(3)-2 & \text{substitute} \\ 7 & ?= & 9-2 & \text{simplify} \\ 7 & = & 7 & \end{array}$$

When we substitute in $x = 3$ and evaluate each side independently, we end up with a true statement, so $x = 3$ is a solution to the equation $x + 4 = 3x - 2$.

Let's check out an example which requires a bit more work.

Example 20

Solve $4(x-3) = 1-(x+2)$.

Solution:
This time, before we worry about moving x terms or undoing operations, it might be in our best interests to simplify first.

$$\begin{array}{cccr} 4(x-3) & = & 1-\mathbf{1}(x+2) & \text{Write invisible 1} \\ 4(x-3) & = & 1+(-\mathbf{1})(x+2) & \text{Change subtraction} \\ 4\cdot x-4\cdot 3 & = & 1+\ -\mathbf{1}\cdot \boldsymbol{x}+\ -\mathbf{1}\cdot\mathbf{2} & \text{Distribute} \\ 4x-\mathbf{12} & = & 1-\boldsymbol{x}-\mathbf{2} & \text{Simplify} \\ 4x-12 & = & -x-\mathbf{1} & \text{Combine like terms} \end{array}$$

In this first part of the process, we worked with each side individually, so we had to stick to do's and don'ts of working with expressions (we were careful not to change the value of any expression). Now, we want to start moving things to the other side of the equal sign, so we'll change both sides of the equation in the same way.

$$\begin{array}{rcll}
4x-12 &=& -x-1 & \\
4x-12+\boldsymbol{x} &=& -x-1+\boldsymbol{x} & \text{Add } x \text{ to both sides} \\
4x+x-12 &=& -x+x-1 & \text{Rearrange} \\
5x-12 &=& 0x-1 & \text{Combine like terms} \\
5x-12 &=& -1 & \text{Simplify} \\
5x-12+\mathbf{12} &=& -1+\mathbf{12} & \text{Add 12 to both sides} \\
5x+0 &=& 11 & \text{Simplify} \\
5x &=& 11 & \\
\dfrac{5x}{\mathbf{5}} &=& \dfrac{11}{\mathbf{5}} & \text{Divide by 5 on both sides} \\
x &=& \dfrac{11}{5} & \text{Simplify}
\end{array}$$

Let's check that work by substituting $x = \frac{11}{5}$ into the original equation, $4(x-3) = 1-(x+2)$.

$$\begin{array}{rcll}
4\left(\dfrac{11}{5}-3\right) &?=& 1-\left(\dfrac{11}{5}+2\right) & \\
4\left(\dfrac{11}{5}-\dfrac{3}{1}\cdot\dfrac{5}{5}\right) &?=& 1\cdot\dfrac{5}{5}-\left(\dfrac{11}{5}+\dfrac{2}{1}\cdot\dfrac{5}{5}\right) & \text{Make common denominator} \\
4\left(\dfrac{11}{5}-\dfrac{15}{5}\right) &?=& \dfrac{5}{5}-\left(\dfrac{11}{5}+\dfrac{10}{5}\right) & \text{Simplify} \\
4\left(-\dfrac{4}{5}\right) &?=& \dfrac{5}{5}-\left(\dfrac{21}{5}\right) & \text{Subtract} \\
\dfrac{4}{1}\cdot-\dfrac{4}{5} &?=& \dfrac{5}{5}-\dfrac{21}{5} & \\
-\dfrac{16}{5} &=& -\dfrac{16}{5} &
\end{array}$$

Looks like $x = \frac{11}{5}$ is a good solution to the equation!

Solving equations: The Zero Product Property

Besides undoing operations to isolate x, our other really important tool for solving equations will be the Zero Product Property.

Zero Product Property (ZPP)
Given two real numbers or algebraic expressions, a and b: If $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Example 21

Use the Zero Product Property to solve each equation.

a. $3x = 0$
b. $3(x-2) = 0$
c. $3x(x-2) = 0$
d. $(x+3)(x-2) = 0$

Solutions:

a. Here we have $3 \cdot x = 0$. According to the ZPP, that's only possible if either 3 equals 0 or x equals 0. Since $3 \neq 0$, the only solution is $x = 0$.

b. This time we have $3 \cdot (x-2) = 0$. Again, $3 \neq 0$, so if we use the ZPP, we'll only have $x - 2 = 0$. Then, we can finish isolating x in that little equation.

$$\begin{aligned} x-2 &= 0 \\ x-2+2 &= 0+2 \\ x &= 2 \end{aligned}$$

So our solution to the equation is $x = 2$. Let's double check that.

$$\begin{array}{rcll} 3(x-2) & = & 0 & \\ 3(2-2) & ?= & 0 & \text{Substitute 2 for } x. \\ 3(0) & ?= & 0 & \text{Simplify} \\ 0 & = & 0 & \end{array}$$

We ended with a true statement ($0 = 0$), so it looks like $x = 2$ is a good solution.

c. This time we have a full $3x$ times our $x - 2$, so according to the ZPP, either $3x = 0$ or $x - 2 = 0$. That gives us two equations to finish solving for x.

We already found that when $3x = 0$, $x = 0$, and when $x - 2 = 0$, the solution is $x = 2$. That means our equation has two solutions, $x = 0$ and $x = 2$.

d. Now the things being multiplied are $(x+3)$ and $(x-2)$. According to the ZPP then, either $x + 3 = 0$ or $x - 2 = 0$. Isolate x in each of these equations to get solutions of $x = -3$ or $x = 2$.

Try it Now

Solve each equation.

a. $6x = 0$
b. $6(x+4) = 0$
c. $6x(x+4) = 0$
d. $(x+6)(x+4) = 0$

Section 0.1 Exercises

Conceptual Questions

1. What is the difference between an algebraic expression and an equation?
2. What is an algebraic term?
3. What are like terms?
4. What are things we can do to an algebraic expression which don't change its value?
5. How can we check a solution to an equation?
6. How can we solve an equation?
7. What is the Zero Product Property?

Practice Problems

1. Which of the following expressions are equivalent to $3x - 4(x + 5)$?
   a. $-4(x + 5) + 3x$
   b. $4(x + 5) - 3x$
   c. $3x - (x + 5)4$
   d. $3x - 4x + 5$
   e. $3x - 4x - 20$
   f. $3x - 4x + 20$
   g. $-x^2 + 20$
   h. $-x + 20$
   i. $20 - x$
   j. $x = 20$

2. Which of the following expressions are equivalent to $\frac{4}{x} + \frac{3}{y}$?
   a. $\frac{4+3}{x+y}$
   b. $\frac{4}{x} \cdot \frac{y}{y} + \frac{3}{y}$
   c. $\frac{4}{x} + \frac{3}{y} \cdot \frac{x}{x}$
   d. $\frac{4y}{xy} + \frac{3x}{xy}$
   e. $\frac{4y+3x}{xy}$
   f. $4 \cdot \frac{1}{x} + 3 \cdot \frac{1}{y}$
   g. $\frac{1}{4} \cdot x + \frac{1}{3} \cdot y$
   h. $\frac{4}{x} + \frac{3}{y} + 3 - 3$
   i. $\frac{4}{x} + \frac{3}{y} + 3$

Write at least four algebraic expressions which are equivalent to the given expressions.

3. $5a + 3b - c$
4. $\frac{6}{6+x} + 8y - 10$
5. $3(x - 4) - (x + 2)$

Simplify each algebraic expression completely.

6. $8x^2 - 7xy + 3y - 2x + 9xy + 2x^2$
7. $10a^2b + 3ab^2 + 4a^2b - 8ab^2 + 3a - 2b$
8. $3(x-4) - (x+2)$
9. $3(4x)$
10. $2(x+3)^2$
11. $7 - 2(x+3)^2$
12. $3y - (5)^2y - 10$
13. $9x - 3x(2) + 7$
14. $\left(\frac{1}{2}x - 1\right)3$
15. $3^2 - 2(4x)$

Evaluate each expression using the given value of the variable.[1]

16. $8(x+3) - 64$ for $x = 2$
17. $4y + 8 - 2y$ for $y = 3$
18. $(11a + 3) - 18a + 4$ for $a = -2$
19. $4z - 2z(1+4) - 36$ for $z = 5$
20. $4y(7-2)^2 + 200$ for $y = -2$
21. $-(2x)^2 + 1 + 3$ for $x = 2$
22. $8(2+4) - 15b + b$ for $b = -3$
23. $2(11c - 4) - 36$ for $c = 0$
24. $4(3-1)x - 4$ for $x = 10$
25. $\frac{1}{4}(8w - 4^2)$ for $w = 1$

For each equation, check whether the given values of the variable are valid solutions.

26. $8x - 5x = 6$
    a. $x = 2$
    b. $x = 0$
    c. $x = -2$

[1] The problems in this section are taken from 1.1 Real Numbers: Algebra Essentials - College Algebra with Corequisite Support 2e | OpenStax and shared under a Creative Commons Attribution 4.0 International License.

For each equation, check whether the given values of the variable are valid solutions.

27. $-\frac{5}{4}x + \frac{1}{2}x = -\frac{3}{2}$
    a. $x = -1$ b. $x = 1$ c. $x = 2$

28. $17 = -3 + 8a + 2a$
    a. $a = 2$ b. $a = 1$ c. $a = 0$

29. $35 + 3b = -5(6b - 7)$
    a. $b = -1$ b. $b = 0$ c. $b = 1$

30. $2m - \frac{25}{12} = -\frac{3}{2}\left(-\frac{3}{2}m + 1\right)$
    a. $m = -1$ b. $m = -\frac{1}{3}$ c. $m = -\frac{7}{3}$

Solve each equation.

31. $2x = -15 - 3x$

32. $2y + \frac{11}{4} - \frac{9}{4}y = \frac{39}{16}$

33. $-11 + 5x = 1 - x$

34. $6(3a - 1) + 4 = 106$

35. $32 - 7b = -4(1 + 4b)$

36. $\frac{11}{3} - \frac{7}{2}x = -2\left(\frac{5}{4}x - \frac{1}{2}\right)$

37. $4x = 0$

38. $x(x - 5) = 0$

39. $x(5 - x) = 0$

40. $5(x - 3) = 0$

41. $(x + 2)(x - 1) = 0$

42. $x(x + 3)(x - 6) = 0$

43. $4x(x - 5)(x + 7) = 0$

Section 0.2: Exponent Properties

A foundational skill for most math classes is simplifying expressions using exponent properties. This is a skill that can get confusing if we're not paying very close attention to what we're doing. We'll build up each property in an intuitive way, then practice using all the properties together.

Exponents

Let's start from the very beginning with what an exponent means. An exponent is a way of indicating repeated multiplication, so

$$2^3$$
$$= 2 \cdot 2 \cdot 2$$

or

$$x^5$$
$$= x \cdot x \cdot x \cdot x \cdot x.$$

Exponents

In the expression

$$m^n$$

m is called the **base** and n is called the **exponent**, or the power.

An exponent indicates repeated multiplication. We multiply m by itself n times.

An exponent of 1

In the expression

$$m$$

we assume that the base has an exponent of 1. In other words, m is the same as m^1.

Product Property of Exponents

Now let's look at what happens when we multiply two expressions with the same base together.

Example 1:

Simplify

$$x^3 \cdot x^2$$

Solution:

Let's write out what each of the exponents means. We'll multiply x by itself three times, then multiply x by itself another 2 times.

$$x^3 \cdot x^2$$
$$= x \cdot x \cdot x \quad \cdot \quad x \cdot x$$

So, how many times are we multiplying x by itself? $3 + 2 = 5$ times.

$$x^3 \cdot x^2$$
$$= x^{(3+2)}$$
$$= x^5$$

In fact, this strategy of adding the exponents works in general!

Product Property of Exponents

When multiplying two exponential expressions with the same base, we can simplify by adding their exponents:

$$m^n \cdot m^p$$
$$= m^{(n+p)}$$

One way of thinking about this property is that it is asking: "How many bases are we multiplying together?" To count them up, we must add!

Example 2:

Simplify each expression:

a. $x^5 \cdot x^7$
b. $3^2 \cdot 3^x$
c. $y \cdot y^4$
d. $a^2 \cdot b^3$

Solutions:

a. We have x^5 times x^7 (a product). The two exponential expressions have the same base (x), so we can use the Product Property of Exponents to simplify:

$$x^5 \cdot x^7$$
$$= x^{5+7}$$
$$= x^{12}$$

In other words, we have five x's being multiplied, times another seven x's being multiplied, for a total string of twelve x's being multiplied together, so x^{12}.

b. Again, we have a product, 3^2 times 3^x, and our two exponential expressions have the same base. So, to simplify we can add the exponents.

$$3^2 \cdot 3^x$$
$$= 3^{2+x}$$

c. This time we have y times y^4, so we should be able to use the Product Property of Exponents. But what exponents, exactly, are we adding? The first y has an invisible exponent of 1, so

$$\begin{aligned} &y \cdot y^4 \\ &= y^1 \cdot y^4 \\ &= y^{(1+4)} \\ &= y^5 \end{aligned}$$

d. Hold on a second. a^2 times b^3 is a product, but the bases (the a and b) don't match! We can't simplify this expression any further.

Try it Now

Simplify each expression.

a. $a \cdot a^5$
b. x^2y^2
c. $b^6 \cdot b^8$

Before our next example, a reminder: when working with numbers or algebraic expressions, we are allowed to reorder multiplication. We can intuitively check this idea by thinking about numbers: $3 \cdot 5$ is the same value as $5 \cdot 3$, for example. This is called the **commutative property** of multiplication.

Example 3:

Simplify each expression:

a. $x^5y^7 \cdot xy^2$
b. $(4x^5)(3x)$
c. $x^6 + x^7$

Solutions:

a. Let's rearrange this expression so that our $x's$ are near each other and our $y's$ are near each other. This is a long string of multiplication, so we're allowed to rearrange it however we please.

$$\begin{aligned} &x^5y^7 \cdot xy^2 \\ &= x^5 \cdot x \cdot y^7 \cdot y^2 \\ &= x^5 \cdot x^1 \cdot y^7 \cdot y^2 \\ &= x^{5+1} \cdot y^{7+2} \\ &= x^6y^9 \end{aligned}$$

We see, then, that we add the exponents only of the pieces of our expression with matching bases.

b. This is expression is still a string of multiplication, it's just written in a different way. Let's rewrite for clarity, then apply our exponent properties.

$$\begin{aligned}&(4x^5)(3x)\\&= 4x^5 \cdot 3x\\&= 4 \cdot 3 \cdot x^5 \cdot x^1\\&= 12x^{5+1}\\&= 12x^6\end{aligned}$$

An expression like this might be confusing if we lose track of what we're doing. This is a bunch of multiplication, so when it comes to the 4 and 3, we can just multiply them together. When we look at the $x^5 \cdot x^1$, we want to stay grounded in the fact that this is five multiplied x's times another x, so six total multiplied x's.

c. Wait a second, this is addition, not multiplication! We can't simplify $x^6 + x^7$ with our exponent properties.

With parts b and c of the last example, we're trying to make the argument that **it's important to understand why you're doing what you're doing**. Math students can easily fall victim to overgeneralization: taking rules they have learned and applying them in the wrong situations. In part b, we notice that we're not just adding everything willy-nilly; we're actually multiplying, so the $4 \cdot 3$ makes a 12. In part c, we see that our adding the exponent rule doesn't work in situations outside of multiplication.

Try it Now

Simplify each expression, if possible.

a. $(7x)(3x^4)$

b. $x + x^2$

c. $10x^6y \cdot 2x^2y^3$

Quotient Property of Exponents

Next we turn our attention to division (quotients). First, though, let's review a couple of fraction rules

Fraction Multiplication

To multiply fractions, we multiply the numerators, and multiply the denominators:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

This idea works in reverse as well:

$$\frac{ac}{bd} = \frac{a}{b} \cdot \frac{c}{d}$$

Similarly,

$$\frac{ac}{b} = \frac{ac}{b \cdot 1} = \frac{a}{b} \cdot \frac{c}{1}$$

And

$$\frac{a}{bd} = \frac{a \cdot 1}{bd} = \frac{a}{b} \cdot \frac{1}{d}$$

Example 4

Simplify $\frac{x^5}{x^3}$.

Solution:

Once again, let's write out what each of these exponential expressions means:

$$\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x}$$

Now, how do we simplify this? Let's break up that fraction multiplication a bit:

$$= \frac{x}{x} \cdot \frac{x}{x} \cdot \frac{x}{x} \cdot x \cdot x$$

Notice that for each of the first three fractions, $\frac{x}{x} = 1$ (because a nonzero number divided by itself is 1). Now we have

$$= 1 \cdot 1 \cdot 1 \cdot x \cdot x$$
$$= x^2$$

Okay, so it seems $\frac{x^5}{x^3}$ simplifies down to $x^{5-3} = x^2$.

Once again, this idea turns out to work in general!

The Quotient Property of Exponents

When dividing two exponential expressions with the same base, we can simplify by subtracting the exponents:

$$\frac{m^n}{m^p}$$
$$= m^{n-p}$$

The Quotient Property of Exponents asks the questions: "Where are there more copies of the base? How many more?" We're looking for the difference in how many bases are in the numerator versus the denominator, so we subtract the exponents.

Example 5:

Simplify each expression

a. $\frac{7^6}{7^2}$
b. $\frac{x^6}{x^5}$
c. $\frac{x^4}{1+x^3}$

Solutions:

a. Same bases? Yes, each base is 7. We have a quotient (division), so to simplify we subtract the exponents:

$$\frac{7^6}{7^2}$$
$$= 7^{6-2}$$
$$= 7^4$$

b. Once again, we have matching bases being divided, so we can subtract the exponents to simplify:

$$\frac{x^6}{x^5}$$
$$= x^{6-5}$$
$$= x^1$$
$$= x$$

c. Initially, it looks like we have matching bases in the numerator and the denominator, but in fact, we have a problem. The denominator isn't just x to a power, it's 1 *plus* x to a power. So really, the denominator is $(1 + x^3)^1$. That means we actually don't have matching bases, and this expression cannot be simplified using the Quotient Property of Exponents.

Let's look at some more fraction properties before we try some more complicated Quotient Property examples.

Fraction addition

To add two fractions with a common denominator, add the numerators and keep the common denominator:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

This property works in reverse as well:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

However, we *cannot* similarly split up addition in the denominator:

$$\frac{a}{c+d} \neq \frac{a}{c} + \frac{a}{d}$$

The last fact is sometimes hard to accept. Experiment with numbers a little to convince yourself. Also, remember the principle that fractions *must* have common denominators to be added, so if we're splitting up fraction addition, we must end up (at least initially) with common denominators!

Example 6:

Simplify each expression, if possible.

a. $\frac{10x^4}{5x^2}$

b. $\frac{5x^4}{10x^2}$

c. $\frac{10+x^4}{5x^2}$

d. $\frac{10x^4}{5+x^2}$

Solution:

a. It will help to split up the fraction a bit first:

$$\frac{10x^4}{5x^2}$$
$$= \frac{10}{5} \cdot \frac{x^4}{x^2}$$

Now we can reduce the $\frac{10}{5}$ and simplify the $\frac{x^4}{x^2}$ using the Quotient Property.

$$= 2 \cdot x^{4-2}$$
$$= 2x^2$$

b. Once again, let's split then simplify.

$$\frac{5x^4}{10x^2}$$
$$= \frac{5}{10} \cdot \frac{x^4}{x^2}$$
$$= \frac{1}{2} \cdot x^{4-2}$$
$$= \frac{1}{2}x^2$$

c. This time, we must split up some fraction addition to help with the simplification.

$$\frac{10 + x^4}{5x^2}$$
$$= \frac{10}{5x^2} + \frac{x^4}{5x^2}$$

Let's split up the multiplication in each fraction to more clearly see what we can do next:

$$= \frac{10}{5} \cdot \frac{1}{x^2} + \frac{1}{5} \cdot \frac{x^4}{x^2}$$

Now we simplify the $\frac{10}{5}$ to a 2, and use the Quotient Property to simplify the $\frac{x^4}{x^2}$:

$$= 2 \cdot \frac{1}{x^2} + \frac{1}{5}x^2$$
$$= \frac{2}{x^2} + \frac{x^2}{5}$$

d. It's tempting to approach $\frac{10x^4}{5+x^2}$ in the same way as part c, but we have a major difference in structure here! If we wanted to split the fraction, the best we could do is break up the multiplication in the numerator; we CANNOT split up the addition in the denominator, because when we add fractions we need to keep common denominators!

$$\frac{10x^4}{5 + x^2}$$

$$= 10 \cdot \frac{x^4}{5+x^2} \quad or \quad \frac{10}{5+x^2} \cdot x^4$$

From there, it looks like there's no way to make use of the Quotient Property of Exponents; the addition in the denominator is in our way!

With these last few examples, a theme emerges.

Addition (and subtraction) ruin everything!

This will be a theme throughout our algebra journey. When addition or subtraction are present, especially inside a structure like a fraction or a root, we cannot simplify in the ways that may be instinctive to us. Before you cross out an x*, or simplify in some way, ask yourself: is addition in the way?*

Example 7

Simplify $\frac{a^2}{a^5}$.

Solution:

We'll do this in two ways to make a point. First, let's use the Quotient Property:

$$\frac{a^2}{a^5}$$
$$= a^{2-5}$$
$$= a^{-3}.$$

What does that negative exponent really mean, though? Let's backtrack and write out the exponents to see what's happening.

$$\frac{a^2}{a^5}$$
$$= \frac{a \cdot a}{a \cdot a \cdot a \cdot a \cdot a}$$
$$= \frac{a}{a} \cdot \frac{a}{a} \cdot \frac{1}{a \cdot a \cdot a}$$
$$= 1 \cdot 1 \cdot \frac{1}{a^3}$$
$$= \frac{1}{a^3}$$

In the previous example, we see that $\frac{a^2}{a^5}$ can be simplified to either a^{-3} or $\frac{1}{a^3}$. That brings us to a big idea about negative exponents.

Negative Exponents

In general,

$$m^{-n}$$

is the same as

$$\frac{1}{m^n}$$

Similarly,

$$\frac{1}{m^{-n}}$$
$$= \frac{1}{\left(\frac{1}{m^n}\right)}$$
$$= 1 \cdot \frac{m^n}{1}$$
$$= m^n$$

To make a negative power positive, move it across the fraction bar.

Example 8

Simplify each expression. Write your final answer in terms of only positive exponents.

a. x^{-4}

b. $\frac{a^{-1}}{b^{-4}}$

c. $\frac{y^2}{y^7}$

Solutions:

a. We'll use our negative exponents property to rewrite.

$$x^{-4}$$
$$= \frac{1}{x^4}$$

b. To clearly see what's happening, we can first split this into two fractions. Then we'll use the negative exponents property.

$$\frac{a^{-1}}{b^{-4}}$$
$$= a^{-1} \cdot \frac{1}{b^{-4}}$$
$$= \frac{1}{a} \cdot b^4$$
$$= \frac{b^4}{a}$$

So, the b moves into the numerator, turning its exponent positive; likewise, the a moves into the denominator, turning its exponent positive.

c. For this one, we can start with the Quotient Property of Exponents:

$$\frac{y^2}{y^7}$$
$$= y^{2-7}$$
$$= y^{-5}$$

We want to write our answer in terms of positive exponents, though, so we'll rewrite for a final answer of $\frac{1}{y^5}$.

Example 9

Simplify $\frac{w^3}{w^3}$

Solution:
Once more, we'll use the Quotient Property of Exponents.

$$\frac{w^3}{w^3}$$
$$= w^{3-3}$$
$$= w^0$$

What does that zero power mean, though? We can return to the original exponentials to investigate.

$$\frac{w^3}{w^3}$$
$$= \frac{w \cdot w \cdot w}{w \cdot w \cdot w}$$
$$= \frac{w}{w} \cdot \frac{w}{w} \cdot \frac{w}{w}$$
$$= 1 \cdot 1 \cdot 1$$
$$= 1$$

Therefore, we see that $\frac{w^3}{w^3}$ is w^0, which is also simply 1.

We'll take what we learned in that last example and write it as another exponent property.

An Exponent of Zero

A number or expression raised to the zero power is 1. That is,

$$m^0$$
$$= 1$$

That's a whole lot of properties so far. Let's do some examples where we mix them all together a bit.

Example 10

Simplify each expression, if possible.

a. $\frac{x^2x^5}{x^9}$

b. $\frac{a^5b^4}{ab^7}$

c. $\frac{2w^3}{4w^5}$

d. $\frac{x+y}{x^2-y}$

Solutions:

a. There are a couple of ways that we could proceed. The important thing is to approach the simplification systematically, taking one step at a time until we're comfortable with the properties.

$$\begin{aligned}
&\frac{x^2x^5}{x^9} \\
&= \frac{x^{2+5}}{x^9} && \text{Product Property} \\
&= \frac{x^7}{x^9} \\
&= x^{7-9} && \text{Quotient Property} \\
&= x^{-2} \\
&= \frac{1}{x^2} && \text{Negative Exponents}
\end{aligned}$$

Let's try another approach, too, for fun!

$$\begin{aligned}
&\frac{x^2x^5}{x^9} \\
&= \frac{x^2}{x^9} \cdot x^5 && \text{Fraction multiplication} \\
&= x^{2-9} \cdot x^5 && \text{Quotient Property} \\
&= x^{-7} \cdot x^5 \\
&= x^{-7+5} && \text{Product Property} \\
&= x^{-2} \\
&= \frac{1}{x^2} && \text{Negative Exponents}
\end{aligned}$$

That's comforting: both approaches get us to the same conclusion! Math works!

b. Here we have two different bases, a and b, so it's a good idea to group those bases together and then use our exponent properties.

$$\begin{aligned}
&\frac{a^5b^4}{ab^7} \\
&= \frac{a^5}{a^1}\cdot\frac{b^4}{b^7} && \text{Fraction multiplication} \\
&= a^{5-1}\cdot b^{4-7} && \text{Quotient Property} \\
&= a^4b^{-3} \\
&= a^4\cdot\frac{1}{b^3} && \text{Negative Exponents} \\
&= \frac{a^4}{b^3}
\end{aligned}$$

If we take the view of asking "Where are there more? How many more?" we see that there are 4 more a's in the numerator, and 3 more b's in the denominator.

c. This is another great case for breaking up the fraction before proceeding.

$$\begin{aligned}
&\frac{2w^3}{4w^5} \\
&= \frac{2}{4}\cdot\frac{w^3}{w^5} && \text{Fraction multiplication} \\
&= \frac{2}{4}\cdot w^{3-5} && \text{Quotient Property} \\
&= \frac{1}{2}w^{-2} \\
&= \frac{1}{2}\cdot\frac{1}{w^2} && \text{Negative exponents} \\
&= \frac{1}{2w^2}
\end{aligned}$$

d. Careful! If we try to split up $\frac{\mathrm{x}+\mathrm{y}}{\mathrm{x}^2-\mathrm{y}}$, the best we can do is write $\frac{\mathrm{x}}{\mathrm{x}^2-\mathrm{y}}+\frac{\mathrm{y}}{\mathrm{x}^2-\mathrm{y}}$, because when we split up the addition we have to keep the common denominator. That means we can't use our Quotient Property at all, because we have a whole $\mathrm{x}^2-\mathrm{y}$ in the denominator, not just an x to a power. We're stuck, and we can't simplify further.

Try it Now

Simplify each expression. Write your final answers with only positive exponents.

a. $yx^3(2x^2)$

b. $\frac{4xy}{16yx^2}$

c. $\frac{6x^3y^4}{yx^3\cdot 2y^3x^4}$

Example 11

Simplify each expression. Write your final answers with only positive exponents.

a. $3x^2 \cdot 5y^{-3}x^{-2}$

b. $\frac{x^{-4}y^3}{x^{-1}y^3}$

c. $\frac{3n^{-1}}{3m^4 \cdot 4m^4n^3}$

Solutions:

a. Here, we can reorder to get our like bases next to one another, then use our exponent properties little by little.

$$\begin{aligned}
&3x^2 \cdot 5y^{-3}x^{-2} && \\
&= 3 \cdot 5 \cdot x^2 \cdot x^{-2} \cdot y^{-3} && \text{Reorder multiplication} \\
&= 15x^{2+(-2)}y^{-3} && \text{Product Property} \\
&= 15x^0y^{-3} && \\
&= 15 \cdot 1\, y^{-3} && \text{Zero exponent} \\
&= \frac{15}{y^3} && \text{Negative exponent}
\end{aligned}$$

b. There are a couple of ways to approach this one. We could split the fractions and apply the Quotient Property, or we could rewrite the negative exponents first. We'll look at both approaches.

$$\begin{aligned}
&\frac{x^{-4}y^3}{x^{-1}y^3} && \\
&= \frac{x^{-4}}{x^{-1}} \cdot \frac{y^3}{y^3} && \text{Fraction multiplication} \\
&= x^{-4-(-1)} \cdot y^{3-3} && \text{Quotient Property} \\
&= x^{-3} \cdot y^0 && \\
&= x^{-3} \cdot 1 && \text{Zero exponent} \\
&= \frac{1}{x^3} && \text{Negative exponent}
\end{aligned}$$

Now, let's see how that compares to what happens if we rewrite the negative exponents first.

$$\frac{x^{-4}y^3}{x^{-1}y^3}$$
$$= \frac{x^1y^3}{x^4y^3} \qquad \text{Negative exponent}$$
$$= \frac{x^1}{x^4} \cdot \frac{y^3}{y^3} \qquad \text{Fraction multiplication}$$
$$= x^{1-4} \cdot y^{3-3} \qquad \text{Quotient Property}$$
$$= x^{-3} \cdot y^0$$
$$= \frac{1}{x^3}$$

So it seems both approaches give us the same final answer. As long as we are careful, we have a little bit of flexibility in how we simplify some of these expressions!

d. Again, we have some choices in how we want to simplify $\frac{3n^{-1}}{3m^4 \cdot 4m^4n^3}$. We'll work on one method; can you find some other ways?

$$\frac{3n^{-1}}{3m^4 \cdot 4m^4n^3}$$
$$= \frac{3}{3} \cdot \frac{n^{-1}}{n^3} \cdot \frac{1}{4} \cdot \frac{1}{m^4 \cdot m^4} \qquad \text{Multiplication}$$
$$= 1 \cdot n^{-1-3} \cdot \frac{1}{4} \cdot \frac{1}{m^{4+4}} \qquad \text{Quotient and Product Rules}$$
$$= \frac{1}{4}n^{-4} \cdot \frac{1}{m^8}$$
$$= \frac{1}{4} \cdot \frac{1}{n^4} \cdot \frac{1}{m^8} \qquad \text{Negative exponent}$$
$$= \frac{1}{4n^4m^8}$$

Try it Now

Simplify each expression. Write your final answers in terms of positive exponents.

a. $\frac{3yx^4}{5x^{-2}y^2}$

b. $4x^2y^{-4} \cdot 5x^3y^2$

c. $\frac{y^3}{3y(2x^4y^2)}$

Power Property of Exponents

We only have a couple more properties of exponents to learn. Next we'll investigate what happens when we raise an exponential expression to another power.

Example 12

Simplify $(x^2)^3$

Solution:
Let's work our way from the outside in. That is, the power of 3 tells us to multiply the base by itself 3 times. That base is x^2, so

$$(x^2)^3$$
$$= x^2 \cdot x^2 \cdot x^2.$$

We know from the Product Property that this will simplify to

$$= x^{2+2+2}$$
$$= x^{2\cdot 3}$$
$$= x^6.$$

We made use of a bit of arithmetic there; 2 added to itself 3 times is the very definition of $2 \cdot 3$.

Once again, this is a principle that works in general!

Power Property of Exponents

To simplify an exponential expression raised to a power, we can multiply the exponents:

$$(m^n)^p$$
$$= m^{n\cdot p}$$

Example 13

Simplify each expression.

a. $(2^2)^4$
b. $(\mathrm{a}^3)(\mathrm{a}^4)$
c. $(\mathrm{a}^3)^4$

Solutions

a. Here we have 2^2 all raised to the 4^{th} power, so we can use the Power Property of Exponents to simplify.

$$(2^2)^4$$
$$= 2^{2\cdot 4}$$
$$= 2^8$$

b. Wait a second, this isn't an exponential raised to a power. This is a^3 times a^4; we should be using the Product Property and adding the exponents to simplify!

$$\begin{aligned}&(a^3)(a^4)\\&= a^3 \cdot a^4\\&= a^{3+4}\\&= a^7\end{aligned}$$

c. Now, this one really does need the Power Property, since we have a^3 raised to the 4th power:

$$\begin{aligned}&(a^3)^4\\&= a^{3\cdot 4}\\&= a^{12}\end{aligned}$$

The previous example is another caution against overgeneralization: make sure you're not just using the Power Property every time you see parentheses! It's not the parentheses on their own that indicate we need the Power Property; it's that the stuff inside the parentheses is all being raised to a power outside the parentheses!

Only two more properties to go, and they're both just extensions of the Power Property. Before we get to the properties, we'll motivate them with some examples.

Example 14

Simplify the expression $(x^5y^3)^2$.

Solution:

Let's focus on writing out what that exponent of 2 means first. That 2 tells us to multiply the stuff inside the parentheses by itself twice, so

$$\begin{aligned}&(x^5y^3)^2\\&= x^5y^3 \cdot x^5y^3.\end{aligned}$$

Now, we can reorder the multiplication and use our Product Property:

$$\begin{aligned}&= x^5 \cdot x^5 \cdot y^3 \cdot y^3\\&= x^{5\cdot 2}y^{3\cdot 2}\\&= x^{10}y^6\end{aligned}$$

This is a terrific example of how to handle a power of a product!

Power of a Product
If we raise a product to a power, we multiply each exponent in the product by the outside power: $$(m^pn^q)^r$$ $$= m^{p\cdot r}n^{q\cdot r}$$

Let's also look at what happens when we raise a quotient to a power.

Example 15

Simplify $\left(\frac{x^5}{y^3}\right)^2$.

Again, let's focus on writing out what that exponent of 2 means first. That 2 tells us to multiply the stuff inside the parentheses by itself twice, so

$$\left(\frac{x^5}{y^3}\right)^2$$
$$= \frac{x^5}{y^3} \cdot \frac{x^5}{y^3}$$

To multiply fractions, we multiply the numerators together and multiply the denominators together. Then we'll be able to use the Product Property.

$$= \frac{x^5 \cdot x^5}{y^3 \cdot y^3}$$
$$= \frac{x^{5+5}}{y^{3+3}}$$
$$= \frac{x^{5\cdot 2}}{y^{3\cdot 2}}$$
$$= \frac{x^{10}}{y^6}$$

So, similar to what we saw with a power of a product, when we have a power of a quotient, each exponent inside the quotient will get multiplied by the outside exponent!

Power of a Quotient

If we raise a quotient to a power, we multiply each exponent in the quotient by the outside power:

$$\left(\frac{m^p}{n^q}\right)^r$$
$$= \frac{m^{p\cdot r}}{n^{q\cdot r}}$$

Example 16

Simplify each expression. Write your final answers in terms of positive exponents.

a. $(2x^2)^3$

b. $\left(\frac{4x^4}{y^7}\right)^5$

c. $(m^3)^4 \cdot m^{-3}$

d. $(a^3 \cdot a^{-2}b^2)^{-2}$

e. $(x^2+2)^2$

f. $\left(\frac{2m^2}{2m^{-2}}\right)^{-1}$

g. $\frac{(x^{-1}y^2)^4}{(x^3y^{-4})^4 \cdot 2y^{-2}}$

Solutions

a. To simplify $(2x^2)^3$ we'll use the Power of a Product Rule. Keep in mind, though, that the leading 2 has an invisible power of 1.

$$\begin{aligned}&(2x^2)^3\\&= (2^1x^2)^3\\&= 2^{1\cdot 3}x^{2\cdot 3} \quad \text{Power of Product}\\&= 2^3x^6\\&= 8x^6\end{aligned}$$

b. We'll actually need both the Power of a Product (because of the numerator) and a Power of a Quotient (because of the fraction).

$$\begin{aligned}&\left(\frac{4x^4}{y^7}\right)^5\\&= \frac{(4x^4)^5}{y^{7\cdot 5}} \quad \text{Power of Quotient}\\&= \frac{4^{1\cdot 5}\cdot x^{4\cdot 5}}{y^{35}} \quad \text{Power of Product}\\&= \frac{4^5x^{20}}{y^{35}}\end{aligned}$$

c. Now that we have a bit more going on, we need to start paying attention to the order of operations a little more. Here, we'll want to attend to the parentheses first, then the rest of the multiplication.

$$\begin{aligned}&(m^3)^4 \cdot m^{-3}\\&= m^{3\cdot 4} \cdot m^{-3} \quad \text{Power of power}\\&= m^{12} \cdot m^{-3}\\&= m^{(12+(-3))} \quad \text{Product Property}\\&= m^9\end{aligned}$$

d. Again in this example, we need to attend to the parentheses first. This could mean simplifying inside the parentheses first, or it could mean using the power of a product rule!

$$
\begin{aligned}
&(a^3 \cdot a^{-2}b^2)^{-2} \\
&= \left(a^{3+(-2)}b^2\right)^{-2} && \text{Product Property} \\
&= (a^1b^2)^{-2} \\
&= a^{1\cdot -2}b^{2\cdot -2} && \text{Power of a Product} \\
&= \mathrm{a}^{-2}b^{-4} \\
&= \frac{1}{a^2b^4} && \text{Negative exponents}
\end{aligned}
$$

e. The expression $(\mathrm{x}^2+2)^2$ is a power of a sum (addition). We CANNOT just raise the x^2 and the 2 to the second power, because ADDITION RUINS EVERYTHING. If we wanted to try to simplify this expression, we'd have to fully distribute it out!

$$
\begin{aligned}
&(\mathrm{x}^2+2)^2 \\
&= (x^2+2)(x^2+2) && \text{Write out exponent} \\
&= x^2 \cdot x^2 + 2\mathrm{x}^2 + 2x^2 + 2 \cdot 2 && \text{Distribute} \\
&= x^{2+2} + 2x^2 + 2x^2 + 4 && \text{Product Property} \\
&= x^4 + 4x^2 + 4 && \text{Combine like terms}
\end{aligned}
$$

f. In the expression $\left(\frac{2m^2}{2m^{-2}}\right)^{-1}$, we once more must attend to the parentheses first. We'll simplify the inside of the parentheses, then apply the outer power and simplify from there.

$$
\begin{aligned}
&\left(\frac{2\mathrm{m}^2}{2\mathrm{m}^{-2}}\right)^{-1} \\
&= \left(\frac{2}{2} \cdot \frac{m^2}{m^{-2}}\right)^{-1} && \text{Fraction multiplication} \\
&= \left(\frac{2}{2} \cdot m^{2-(-2)}\right)^{-1} && \text{Quotient Property} \\
&= (1 \cdot m^4)^{-1} \\
&= m^{4\cdot -1} && \text{Power Property} \\
&= m^{-4} \\
&= \frac{1}{m^4} && \text{Negative exponent}
\end{aligned}
$$

g. This one has a lot going on! We'll take care of the parentheses first by using the Power Property, then we'll slowly simplify from there.

$$\frac{(x^{-1}y^2)^4}{(x^3y^{-4})^4 \cdot 2y^{-2}}$$

$$= \frac{x^{-1\cdot 4}y^{2\cdot 4}}{x^{3\cdot 4}y^{-4\cdot 4} \cdot 2y^{-2}} \qquad \text{Power of Product}$$

$$= \frac{x^{-4}y^8}{x^{12}y^{-16} \cdot 2y^{-2}}$$

$$= \frac{x^{-4}y^8}{2 \cdot x^{12} \cdot y^{-16} \cdot y^{-2}} \qquad \text{Reorder multiplication}$$

$$= \frac{x^{-4}y^8}{2x^{12}y^{-16+(-2)}} \qquad \text{Product Property}$$

$$= \frac{x^{-4}y^8}{2x^{12}y^{-18}}$$

$$= \frac{1}{2} \cdot \frac{x^{-4}}{x^{12}} \cdot \frac{y^8}{y^{-18}} \qquad \text{Fraction multiplication}$$

$$= \frac{1}{2} \cdot x^{-4-12} \cdot y^{8-(-18)} \qquad \text{Quotient Property}$$

$$= \frac{1}{2}x^{-16}y^{26}$$

$$= \frac{1}{2} \cdot \frac{1}{x^{16}}y^{26} \qquad \text{Negative exponent}$$

$$= \frac{y^{26}}{2x^{16}} \qquad \text{Fraction multiplication}$$

Section 0.2 Exercises

Conceptual Questions

1. What are each of the following properties, and how do you use them?
   a. Product Property of Exponents
   b. Quotient Property of Exponents
   c. Zero Exponent
   d. Negative Exponent
   e. Power Property of Exponents
   f. Power of a Product
   g. Power of a Quotient
2. Why can you *not* use the Power of a Product Property to simplify $(x^4 + y^2)^3$?

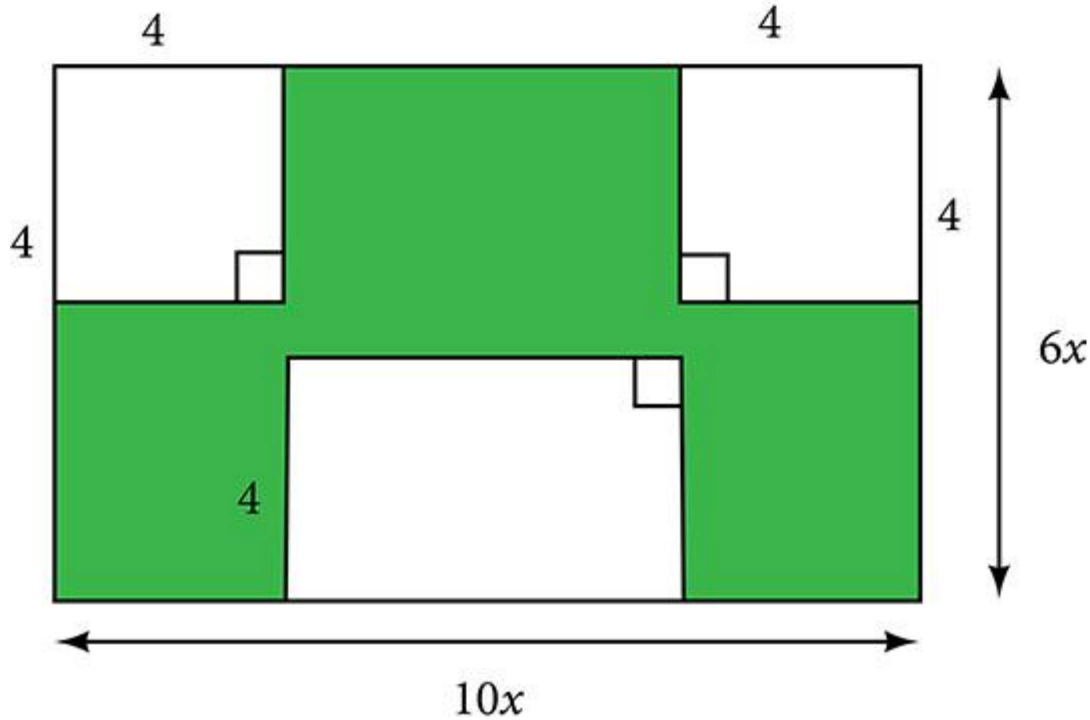
Practice Problems

Simplify each expression using exponent properties (if possible). Write your final answers in terms of only positive exponents.

1. $(3u^{-6}v^6)^{-2}$
2. $\frac{b^6}{b^3}$
3. $2n(6n^4)$
4. $v^3(v^6)^2$
5. $\frac{(vu^4)^2}{v}$
6. $\frac{ab^2}{(b^4)^5}$
7. $\left(\frac{m^{-4} \cdot m^{-4} n^2}{m^6}\right)^6$
8. $\frac{4x^{-2}y^6 \cdot 4x^5}{(6x^3y^6)4y^4}$
9. $5x^3y^4 \cdot 4y^4$
10. $(x^6y^3)^6$
11. $(x^2y^6)(x^5y^2)^3$
12. $\left(\frac{x^5}{x^2 \cdot yx^3}\right)^4$
13. $(m^{-5}n^2)^4 \cdot nm^3$
14. $(4x^3)^4$
15. $\frac{(p^6p^4)^2}{p^5}$
16. $(3x^4)^4$
17. $\frac{4x^6y^5 \cdot 6x^6}{3xy^3}$
18. $(3x + 4y)^3$
19. $\frac{(2xy^3)2y^2}{2x^6}$
20. $\frac{5x^{-5}y^4}{15x^5y^5}$
21. $\frac{(yx^2)^6}{x^{-4}y^{-6} \cdot xy^{-4}}$
22. $(3k^6)^4$
23. $(y^4)^3$
24. $\frac{r^2}{r^3}$
25. $\frac{9x^5}{3x^3}$
26. $\frac{4+x^2}{x^2+8}$
27. $\frac{3uv \cdot 4u^5}{3u^5v^3}$
28. $\frac{(2u^3v^2)^3}{2v^2u^2v^4}$
29. $\frac{6u^{-4}v^6}{24u^3v^4}$
30. $\frac{a^2b^2}{(a^4b^3)^2}$
31. $4u^4v^5 \cdot 2u^4$
32. $\frac{2k^{-6}}{k^2}$
33. $(x \cdot (x^2y^6)^6)^3$
34. $m^5n^3(2m^2n^5 \cdot m^{-2}n^6)^{-2}$

Section 0.3: Factoring

Imagine that we are trying to find the area of a lawn so that we can determine how much grass seed to purchase. The lawn is the green portion in Figure 1.



The area of the entire region can be found using the formula for the area of a rectangle.

$$\begin{aligned} A &= lw \\ &= 10x \cdot 6x \\ &= 60x^2 \; units^2 \end{aligned}$$

The areas of the portions that do not require grass seed need to be subtracted from the area of the entire region. The two square regions each have an area of $A = s^2 = 4^2 = 16$ units^2. The other rectangular region has one side of length $10x - 8$ and one side of length 4, giving an area of $A = lw = 4(10x - 8) = 40x - 32$ units^2. So the region that must be subtracted has an area of $2(16) + 40x - 32 = 40x$ units^2.

The area of the region that requires grass seed is found by subtracting $60x^2 - 40x$ units^2. This area can also be expressed in factored form as $20x(3x - 2)$ units^2. We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

Review: Multiplication

Before we dig into factoring, let's review some multiplication.

Example 1:

Fully expand and simplify.

a. $4x(x+2)$
b. $(x+4)(x+2)$
c. $(2x+1)(x+4)$

Solutions:

a. This is a straightforward case of distribution. We'll multiply the $4x$ by each term inside the parentheses, then simplify:

$$4x(x+2)$$
$$= 4x \cdot x + 4x \cdot 2$$
$$= 4x^2 + 8x$$

b. We'll treat this as distribution, too. This time, we're distributing the whole $x+4$; we'll multiply each term inside the parentheses by $(x+4)$, then simplify:

$$\boxed{(x+4)}(x+2)$$
$$= (x+4) \cdot x + (x+4) \cdot 2$$

To simplify more, we'll distribute x to the first set of parentheses, and distribute 2 to the second set:

$$= x \cdot x + 4 \cdot x + 2 \cdot x + 4 \cdot 2$$
$$= x^2 + 4x + 2x + 8$$
$$= x^2 + 6x + 8$$

In the end, then, we have

$$(x+4)(x+2)$$
$$= x^2 + 6x + 8$$

Notice that the 4 times 2 makes the final 8, and a $4x + 2x$ makes the final $6x$.

c. We'll approach this the same as the previous part, by distributing the entire $2x+1$ to each term in the second parentheses:

$$\boxed{(2x+1)}(x+4)$$
$$= (2x+1)x + (2x+1)4$$
$$= 2x \cdot x + 1 \cdot x + 2x \cdot 4 + 1 \cdot 4$$
$$= 2x^2 + x + 8x + 4$$
$$= 2x^2 + 9x + 4$$

This expanded form doesn't tie back to the original factored form in quite so nice a way as part b, but there is a relationship we can figure out. Notice the middle $1x$ and $8x$ add to the final $9x$, but don't multiply to the final 4. That's because the beginning 2 is also involved. The numbers 1 and 8 multiply to make 8, which is also what we get when we multiply 2 and 4!

Review: Factors

If our goal in this section is to factor, it would be a good idea to know what factors and factoring are. We use the word factor as both a noun and a verb.

Factor (noun)

For real numbers or expressions a and b, if

$$a \cdot b$$
$$= c$$

then a and b are both factors of c.

Example 2

List some factors of each number or expression.

a. 12
b. $6x$
c. $8x^2y$

Solutions:

a. To find factors of 12, we can list the ways we can multiply two numbers to make 12.

$$2 \cdot 6 = 12$$
$$4 \cdot 3 = 12$$
$$1 \cdot 12 = 12$$

For what we're going to do in this unit, it's also useful to consider that

$$-2 \cdot -6 = 12$$
$$-4 \cdot -3 = 12$$
$$-1 \cdot -12 = 12$$

as well. When we're handling integers (positive and negative whole numbers), we'll stick with listing only integer factors. So, our complete list of factors of 12 is

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12.$$

b. The introduction of an x feels a little weird, but we can still work on listing expressions that multiply to make $6x$:

$$2x \cdot 3 = 6x$$
$$2 \cdot 3x = 6x$$
$$6 \cdot x = 6x$$
$$6x \cdot 1 = 6x$$

$$2 \cdot 3 \cdot x = 6x$$
$$1 \cdot 6 \cdot x = 6x$$

If we include all our negative options, the our list of factors looks something like this:

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm x, \pm 2x, \pm 3x, \pm 6x$$

c. For $8x^2y$, we'll need to consider different combinations of the factors of 8, x^2, and y, respectively.

$$8x^2 \cdot y$$
$$x^2 \cdot 8y$$
$$2x^2 \cdot 4y$$
$$4x^2 \cdot 2y$$
$$8x \cdot xy$$
$$x \cdot 8xy$$
$$2x \cdot 4xy$$
$$4x \cdot 2xy$$
$$1 \cdot 8 \cdot x \cdot x \cdot y$$
$$2 \cdot 4 \cdot x \cdot x \cdot y$$

Our list of factors (including negatives) should then be:

$$\pm 1, \pm 2, \pm 4, \pm 8, \pm x, \pm y, \pm x^2, \pm xy, \pm 2x^2,$$
$$\pm 4x^2, \pm 8x^2, \pm 2xy, \pm 4xy, \pm 8xy, \pm 2y, \pm 4y, \pm 8y$$

What we'll be looking for much of the time during this section is common factors and greatest common factors.

Greatest Common Factors (GCFs)

Let c and d be numbers or expressions. If a is a factor of c **and** a factor of d, then a is a **common factor** of c and d.

The **greatest common factor (gcf)** of c and d is the common factor with the largest number and the largest power of each of the variables.

Example 3:

Find the greatest common factor of each pair of numbers or expressions.

a. 8 and 12
b. $8x$ and 12
c. 3 and 4
d. $8x^2$ and $12x$
e. $8x^2y$ and $12x^3y$

Solutions:

a. The factors of 8 are $\pm 1, \pm 2, \pm 4$, and ± 8. The factors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. The biggest factor that 8 and 12 have in common, then, is ± 4.

b. When we through another x into the mix, in this case, it doesn't have a big effect. Because 12 doesn't have any x's as factors, the GCF of $8x$ and 12 is still just ± 4.

c. On the surface, it seems like 3 and 4 don't have any factors in common. Don't forget, though, that $1 \cdot 3 = 3$ and $1 \cdot 4 = 4$, so ± 1 is the greatest common factor!

d. Now things get more interesting. We need to choose common factor with the largest number and the largest power of each of the variables. We know that 8 and 12 have a GCF of ± 4 (that's the largest number), but now we must also consider the x^2 and x portions of each expression. The factors of x^2 are $\pm x^2$, $\pm x$ and ± 1, and the factors of x are $\pm x$ and ± 1, so their GCF is considered to be $\pm x$. That means the GCF of $8x^2$ and $12x$ should be $\pm 4x$.

e. For $8x^2y$ and $12x^3y$, our GCF should include ± 4 (for the largest common number factor), x^2 (for the largest power of x that's a factor of both x^2 and x^3), and a y (the largest power of y). That makes our complete GCF $\pm 4x^2y$.

Now, before we get to our main task of the section, we need to define one more term: the verb form of **factor**.

Factor (verb)
To **factor** a number or expression means to write it in terms of a product or two or more of its factors. That is, if a and b are factors of c, then $a \cdot b$ is a factored form of c.

Our main task now is going to be to factor a bunch of algebraic expressions. We'll build up our skills, one strategy at a time.

Factoring out GCFs

When factoring a polynomial expression, our first step should be to check for and factor out any GCF. We can think of this as "undoing" a distribution.

Example 4

Factor the greatest common factor out of the expression.

$$4x + 8$$

Solution:

Let's first identify the GCF of $4x$ and 8. Looking at the factors of 4 and 8, we find the GCF is 4. Since the 8 term doesn't have any variables, the complete GCF is simply 4. Now, $4x = 4 \cdot x$ and $8 = 4 \cdot 2$, so our expression can be rewritten as

$$4x + 8$$
$$= 4 \cdot x + 4 \cdot 2$$

We can think of this expression as being the result of a distribution process. What was distributed? See that 4 in each term? That must be what we distributed. So if we undo that process, we “pull out” the 4 to get

$$4 \cdot x + 4 \cdot 2$$
$$= 4(x + 2)$$

We can always check our factoring by multiplying out the result. If we expand $4(x + 2)$, we get $4 \cdot x + 4 \cdot 2$, which is $4x + 8$. That means $4(x + 2)$ is a correct factorization!

The final, factored form is $4(x + 2)$.

As we see from the previous example, when we factor out a GCF, we want to do the following.

Factoring out the GCF from an expression

1. Identify the GCF of all terms (considering numbers and variables)
2. Write each term as a product of the GCF and another number or expression.
3. Write the factored expression as the product of the GCF and the sums of the other numbers or expressions (pull out the GCF).

Let’s practice the process!

Example 5

Factor the GCF out of each of the following expressions.

a. $3x^2 - 3x$
b. $6x^3y^3 + 45x^2y^2 + 21xy$
c. $x(b^2 - a) + 6(b^2 - a)$

Solutions:

a. To factor $3x^2 - 3x$, let’s first find the GCF of $3x^2$ and $3x$. The GCF of the numbers is 3; the GCF of the variables is x, so the complete GCF is $3x$. Then we can rewrite the expression, and pull out the GCF:

$$3x^2 - 3x$$
$$= 3x \cdot x - 3x \cdot 1$$
$$= 3x(x - 1)$$

Notice that the second term, $3x$, *was* the GCF. Students often misunderstand and completely leave out the 1 when factoring the $3x$ out of that part of the expression. If we take the time to write out that $3x$ is the same as $3x \cdot 1$, we’ll be less likely to miss that “placeholder” 1 when we factor!

b. Again, let’s start by finding the GCF of $6x^3y^3 + 45x^2y^2 + 21xy$. When we consider 6, 45, and 21, we see that the largest number that is a factor of all three of these is 3. Next, we look at the $x's$; their GCF is x. Similarly, the GCF of the $y's$

is a single y. So, the complete GCF is $3xy$. Now, we'll rewrite the expression so that we see the factors of each individual term, then pull out the GCF.

$$\begin{aligned}&6x^3y^3 + 45x^2y^2 + 21xy\\&= 3xy \cdot 2x^2y^2 + 3xy \cdot 15xy + 3xy \cdot 7\\&= 3xy(2x^2y^2 + 15xy + 7)\end{aligned}$$

c. At first, $x(b^2 - a) + 6(b^2 - a)$ looks very different from our other expressions. The key here is to view $x(b^2 - a)$ and $6(b^2 - a)$ as being the two terms; they're separated by that big plus in the middle. If we look at it like that, then $(b^2 - a)$ is the GCF of the two terms. So, to factor, we pull that $(b^2 - a)$ out:

$$\begin{aligned}&x(b^2 - a) + 6(b^2 - a)\\&= (b^2 - a)(x + 6)\end{aligned}$$

Try it Now

Factor the GCF out of each expression.

a. $7x^2 + 28x$
b. $5xy + 10x^2y$
c. $(x + 4)x - (x + 4)6$

Factoring an Expression in the Form $x^2 + bx + c$

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial $x^2 + 5x + 6$ has a GCF of 1, but it can be written as the product of the factors $(x + 2)$ and $(x + 3)$, since

$$\begin{aligned}&(x + 2)(x + 3)\\&= (x + 2)x + (x + 2)3\\&= x^2 + 2x + 3x + 6\\&= x^2 + 5x + 6.\end{aligned}$$

Notice again, we have this special relationship: the 5 in the middle comes from $2 + 3$, and it also happens that $2 \cdot 3 = 6$, which is the last term of the expression. This shows us a path for factoring expressions in the form $x^2 + bx + c$.

Factoring expressions in the form $x^2 + bx + c$

An expression in the form $x^2 + bx + c$ can be factored if we can find numbers p and q such that

$$p \cdot q = c$$

and

$$p + q = b.$$

If this p and q can be found, then the factored form of the expression is

$$(x + p)(x + q).$$

Example 6

Factor $x^2 + 2x - 15$.

Solution:
We have an expression in the form $x^2 + bx + c$, where $b = 2$ and $c = -15$. So we need to find factors of $c = -15$ which add to make $b = 2$. Let's list factor pairs of c and see if any of them add to b.

Factors of -15	Sum of factors
$1, -15$	$1 + (-15) = -14$
$-1, 15$	$-1 + 15 = 14$
$-3, 5$	$-3 + 5 = 2$
$-5, 3$	$-5 + 3 = -2$

Looks like 3 and -5 is our perfect factor pair, which multiplies to -15 and adds to 2! That means, the factored form of the expression is

$$x^2 + 2x - 15$$
$$= (x + 3)(x - 5)$$

Remember, we can check by multiplying this back out! We should get the original $x^2 + 2x - 15$ if we do.

Example 7

Factor $x^2 + 2x + 3$.

Solution:
The factors of 3 are just 1 and 3. But 1+3 = 4, and we need factors that add to 2. We could try negatives, but then we'd need -1 and -3 since $-1 \cdot -3 = 3$. But $-1 + (-3) = -4$. We can't find a good pair of numbers which multiply to make 3 and add to make 2, so this expression can't be factored (over the integers).

Try it Now:

Factor each expression, if possible.

a. $x^2 - 7x + 6$
b. $x^2 - 5x - 6$
c. $x^2 - 7x - 6$

Factoring by Grouping

Factoring by grouping is a great strategy for when we have four terms in our expression. We'll make groups of terms and factor out each group's GCF. Then, some (artificial) magic will happen.

Example 8

Factor $2x^3 + 4x^2 + 3x + 6$.

Solution:
This expression is not in the nice $x^2 + bx + c$ form; we have four terms! That makes this expression a great candidate for factoring by grouping. We'll make the first two terms into one group, and the last two terms into another group, and factor the GCF out of each group.

$$2x^3 + 4x^2 + 3x + 6$$
$$(2x^3 + 4x^2) + (3x + 6) \qquad \text{Make Groups}$$
$$2x^2(x + 2) + 3(x + 2) \qquad \text{Factor GCF out of each group}$$

Let's pause to make note of a couple of things. First, our two groups did not have the same common factor. The first group, $2x^3 + 4x^2$, had a GCF of $2x^2$ while the second group, $3x + 6$, had a common factor of 3. Then, let's take a step back and look at our expression now. If we think of $2x^2(x + 2)$ as our first term, and $3(x + 2)$ as our second term, we can see they have a GCF of $x + 2$. So, like magic, we can pull out that GCF to finish factoring:

$$2x^2(x + 2) + 3(x + 2)$$
$$= (x + 2)(2x^2 + 3)$$

Okay, the references to magic are silly, but here's the point: we're giving ourselves expressions that factor nicely, to practice factoring. Not all four-term expressions will factor by grouping so nicely. In fact, many expressions won't be factorable at all.

Example 9

Factor each expression.

a. $5x^3 + 15x^2 - 7x - 21$
b. $2x^2 - 2x - 3x + 3$

Solutions:
Before we make our groups, let's rewrite the subtraction, for clarity's sake.

$$5x^3 + 15x^2 + (-7x) + (-21)$$

Now, group, and factor.

$$(5x^3 + 15x^2) + \big(-7x + (-21)\big)$$
$$= 5x^2(x + 3) + (-7)(x + 3)$$

Now we can factor out the common $x + 3$:

$$= (x + 3)\big(5x^2 + (-7)\big)$$
$$= (x + 3)(5x^2 - 7)$$

Consider this example a cautionary tale about negatives! It often helps to rewrite subtraction in terms of addition before attempting to factor.

Once again, let's rewrite that subtraction first, then group and factor.

$$\begin{aligned}&2x^2 - 2x - 3x + 3\\&= 2x^2 + (-2x) + (-3x) + 3\\&= \left(2x^2 + (-2x)\right) + (-3x + 3)\\&= 2x(x - 1) + 3(-x + 1)\end{aligned}$$

Huh, that's not quite working. We have a $x - 1$ factor in our first group, but a $-x + 1$ factor in the second group. Wait a second, though; those are opposites! The signs are just reversed. That means if we factor a *negative* 3 out of the second group, we should have our matching factors.

$$\begin{aligned}&= \left(2x^2 + (-2x)\right) + (-3x + 3)\\&= 2x(x - 1) + (-3)(x - 1)\\&= (x - 1)(2x - 3)\end{aligned}$$

Try it Now

Factor each expression.

a. $9x^3 - 27x^2 + 8x - 24$

b. $3x^2 - 9x - x + 3$

Factoring Expressions in the Form $ax^2 + bx + c$

Now we get to the good stuff. We're going to use all of the previous skills we've been practicing to work on factoring expressions like $2x^2 + 3x + 1$.

Each strategy alone won't work on this expression. If we look for numbers which multiply to make 1 and add to make 3, we're stuck, because $1 \cdot 1 = 1$, but $1 + 1 \neq 3$. Also, we *shouldn't* even use that strategy, because we've got a 2 in front of the x^2! We also can't quite factor by grouping, because we only have three terms.

Wouldn't it be great if we could split up that middle term, then use factoring by grouping? Here's how we can systematically reach that goal.

Factoring expressions in the form $ax^2 + bx + c$
1. List factors of ac
2. Find factors of ac which add to make b. Call those factors p and q.
3. Rewrite the original expression as $ax^2 + px + qx + c$
4. Factor by grouping.

Example 10

Factor $2x^2 + 3x + 1$

Solution:
First, let's name those numbers. This expression is in the form $ax^2 + bx + c$ where $a = 2$, $b = 3$, and $c = 1$.

We need to find factor pairs of $ac = 2 \cdot 1 = 2$ which add to $b = 3$:

Factors of $ac = 2$	Sum
$2,1$	$2 + 1 = 3$
$-2, -1$	$-2 + (-1) = -3$

So $p = 2$ and $q = 1$. Now we rewrite our expression and factor by grouping.

$$\begin{aligned} &2x^2 + 3x + 1 \\ &= 2x^2 + 2x + 1x + 1 \\ &= 2x(x+1) + 1(x+1) \\ &= (x+1)(2x+1) \end{aligned}$$

Let's observe some important pieces of the process in that example. When we rewrote the middle term, we weren't changing the value of the expression, because $2x + 1x$ has the same value as $3x$ if we combine like terms. We wanted to *break apart* the $3x$ into like terms so that we could then use the strategy of factoring by grouping.

Notice also that the second group, $x + 1$, only had a factor of 1 in common. We went ahead and factored that out, so that we'd keep track of that 1 as we continued the factoring process.

Let's practice some more!

Example 11

Factor each expression.

a. $5x^2 + 7x - 6$
b. $2x^2 - 9x + 9$

Solutions:

a. In the expression $5x^2 + 7x - 6$, we have $a = 5$, $b = 7$, and $c = -6$. First, we calculate that $ac = 5(-6) = -30$. That means we're looking for factors of -30 which add to make 7. Let's list factor pairs of -30 until we find two that add to 7.

Sure, there are more factor pairs, but we found one that worked, so we'll stop there! We'll let $p = -3$ and $q = 10$ and continue our process.

Factors of -30	Sum
$-1,30$	$-1 + 30 = 29$
$1, -30$	$1 + (-30) = -29$
$-2,15$	$-2 + 15 = 13$
$2, -15$	$2 + (-15) = -13$
$-3,10$	$-3 + 10 = 7$

$$\begin{aligned}
&5x^2 + 7x - 6 && \\
&= 5x^2 - 3x + 10x - 6 && \text{Rewrite middle term} \\
&= (5x^2 - 3x) + (10x - 6) && \text{group} \\
&= x(5x - 3) + 2(5x - 3) && \text{Factor out each group's GCF} \\
&= (5x - 3)(x + 2) && \text{Factor out GCF } (5x - 3)
\end{aligned}$$

b. For our expression $2x^2 - 9x + 9$, we have $a = 2, b = -9$ and $c = +9$. That means $ac = 2(9) = 18$. We need factors of 18 which add to -9. Seems like -6 and -3 are a good option. Let's factor.

$$\begin{aligned}
&2x^2 - 9x + 9 && \\
&= 2x^2 - 6x - 3x + 9 && \text{Rewrite } -9x \\
&= 2x^2 - 6x + (-3x) + 9 && \text{Rewrite subtraction} \\
&= (2x^2 - 6x) + (-3x + 9) && \text{Group} \\
&= 2x(x - 3) - 3(x - 3) && \text{Factor out each group's GCF}^* \\
&= (x - 3)(2x - 3) && \text{Factor out overall GCF } (x - 3)
\end{aligned}$$

Notice at the $*$, we chose to factor out a -3 from the second group, so that the remaining $x - 3$ factor would match the $x - 3$ in the first group.

Try it Now

Factor each expression.

a. $6x^2 - 5x - 6$
b. $6x^2 + 5x - 6$
c. $4x^2 - 13x + 10$

A Special Case: The Difference of Squares

To understand this special case, let's go back to some multiplication again.

$$\begin{aligned}&(a+b)(a-b)\\&=a^2+ab-ab-b^2\\&=a^2-b^2\end{aligned}$$

What we see here is that the plus/minus pattern in the factors causes the middle terms in the expanded form to zero out (because $ab-ab$ is 0). What we're left with is a perfect square, a^2, minus a perfect square, b^2. This is a difference of squares. When we're factoring, if we notice this a^2-b^2 structure, we can factor it back into $(a+b)(a-b)$.

Factoring a Difference of Squares
An Difference of Squares expression (in the form a^2-b^2) can be factored into $(a+b)(a-b)$.

Example 12

Factor each of the following, if possible.

a. x^2-9
b. $4x^2-25y^4$
c. $4-x^2$
d. $16x^2-9y$

Solutions:

a. When we encounter an expression like this in the wild, the first thing we might notice is that it only has two terms. Then we notice the subtraction. *Then* we notice that each term is a perfect square! Bingo! We can use the difference of squares with $a=x$ and $b=3$.

$$\begin{aligned}&x^2-9\\&=x^2-3^2\\&=(x+3)(x-3)\end{aligned}$$

b. Again we have two terms, we have subtraction, and we have perfect squares. We notice that $4x^2$ is $(2x)^2$ and $25y^4$ is $(5y^2)^2$, so we can use the difference of squares pattern with $a=2x$ and $b=5y^2$:

$$\begin{aligned}&4x^2-25y^4\\&=(2x)^2-(5y^2)^2\\&=(2x+5y^2)(2x-5y^2)\end{aligned}$$

c. The expression $4-x^2$ is again a difference of squares, but we need to be careful about the order of the expression. Since the 4 is first, $a=2$, and since the x^2 is second, $b=x$. This matters in our final factored form!

$$\begin{aligned}&4 - x^2\\&= (2)^2 - x^2\\&= (2 + x)(2 - x)\end{aligned}$$

d. When we examine $16x^2 - 9y$, things seem good at first. We have two terms, we have subtraction, and $16x^2$ is a perfect square. However, $9y$ is NOT a perfect square, since that y isn't squared! So, we can't use the difference of squares pattern on this expression, and as far as factoring goes, we're just stuck.

Combining Strategies

To wrap up this section, we'll look at some factoring exercises which require us to use a combination of strategies. As a general rule, it's good to look for a GCF first, and then factor from there.

Example 13

Factor $8x^2 + 14x + 6$ completely.

Solution:
This expression is in the form $ax^2 + bx + c$, but before we dive right into factoring it, take another look at those numbers. The numbers 8, 14, and 6 have a GCF of 2. We can factor that out first, then factor what remains!

$$\begin{aligned}&8x^2 + 14x + 6\\&= 2 \cdot 4x^2 + 2 \cdot 7x + 2 \cdot 3\\&= 2(4x^2 + 7x + 3)\end{aligned}$$

Now we'll focus on factoring $4x^2 + 7x + 3$, which is in the form $ax^2 + bx + c$. Our ac value is $4 \cdot 3 = 12$, so we need to find factors of 12 which add to make 3. Those factors will be 4 and 3, so we want to rewrite the middle $7x$ as $4x + 3x$, then use factoring by grouping.

$$\begin{aligned}&= 2(4x^2 + 4x + 3x + 3)\\&= 2\big((4x^2 + 4x) + (3x + 3)\big)\\&= 2\big(4x(x + 1) + 3(x + 1)\big)\\&= 2(x + 1)(4x + 3)\end{aligned}$$

Example 14

Factor $-4x^3 + 4x^2 + 3x$ completely.

Solution:
This time our GCF is simply an x, but we'd like to factor out a little more than just an x. Factoring is pretty tricky when the first number is negative (in this case, negative 4), so we can factor out a *negative* x to help us flip that sign around and make our factoring job easier.

$-4x^3 + 4x^2 + 3x$
$= -x \cdot 4x^2 + (-x) \cdot (-4x) + (-x) \cdot (-3)$ Write each expression as $-$ x times other factor
$= -x(4x^2 - 4x - 3)$ Factor out GCF

Now we focus on the $4x^2 - 4x - 3$ factor. We have $a = 4, b = -4$ and $c = -3$, so $ac = -12$. The factors of -12 which add to -4 are -6 and 2, so we'll rewrite the middle $-4x$ as $-6x + 2x$ and factor by grouping:

$$= -x(4x^2 - 6x + 2x - 3)$$
$$= -x\big((4x^2 - 6x) + (2x - 3)\big)$$
$$= -x\big(2x(2x - 3) + 1(2x - 3)\big)$$
$$= -x(2x - 3)(2x + 1)$$

Example 15

Factor, if possible.
a. $50x^2z + 2y^2z$
b. $50x^2z - 2y^2z$

Solutions:
a. Let's hunt for a GCF first. The numbers 50 and 2 have a common factor of 2, and each term also has a z, so the GCF is $2z$.
$$50x^2z + 2y^2z$$
$$= 2z \cdot 25x^2 + 2z \cdot y^2$$
$$= 2z(25x^2 + y^2)$$
Now we look at the $25x^2 + y^2$. We have two terms, and they're both perfect squares, but they're being added not subtracted, so we can't use the difference of squares pattern. None of our other strategies will help either, so this expression is fully factored!

b. This expression is almost the same, but now there's subtraction between the two terms. So, we can still factor out the GCF of $2z$ to get
$$50x^2z - 2y^2z$$
$$= 2z(25x^2 - y^2).$$
Now our leftover factor is a *difference* of squares, so we can factor using that pattern!
$$= 2z((5x)^2 - y^2)$$
$$= 2z(5x + y)(5x - y)$$

Try it Now

Factor each expression completely, if possible.
a. $2x^3 - 4x^2 - 30x$
b. $-15x^2y + 9xy + 6y$
c. $5x^2 - 5$

Section 0.3 Exercises

Conceptual Questions

1. What is a factor? What does it mean *to* factor an expression?
2. If the terms of a expression do not have a GCF, does that mean it is not factorable? Explain.
3. An expression is factorable, but it is not a difference of two squares. Can you factor the expression without finding the GCF?
4. How do we factor an expression in the form $ax^2 + bx + c$?
5. What is the difference of squares pattern? How can we use it to factor?

Practice Problems

For the following exercises, factor the polynomial completely.

1. $7x^2 + 48x - 7$
2. $10h^2 - 9h - 9$
3. $2b^2 - 25b - 247$
4. $9d^2 - 73d + 8$
5. $90v^2 - 181v + 90$
6. $12t^2 + t - 13$
7. $2n^2 - n - 15$
8. $16x^2 - 100$
9. $25y^2 - 196$
10. $121p^2 - 169$
11. $4m^2 - 9$
12. $361d^2 - 81$
13. $324x^2 - 121$
14. $144b^2 - 25c^2$
15. $16a^2 - 8a + 1$
16. $49n^2 + 168n + 144$
17. $121x^2 - 88x + 16$
18. $225y^2 + 120y + 16$
19. $m^2 - 20m + 100$
20. $25p^2 - 120p + 144$
21. $36q^2 + 60q + 25$
22. $x^3 + 216$
23. $27y^3 - 8$

Section 0.4: Solving Linear and Quadratic Equations

The main focus of this book will be the study of various functions. To effectively study these functions, though, we need to have certain algebra skills firmly established. As we wrap up our prerequisite chapter, we will return to some of the ideas from section 0.1, namely, solving equations.

We'll focus here on two types of equations: linear and quadratic. We'll start with some quick definitions, then do a lot of examples!

Linear and Quadratic Equations Overview

Linear Equations

A linear equation in one variable has the following features:

- Contains only one variable
- The variable is only raised to the first power
- The variable is not in a special structure (the denominator of a fraction, a square root, etc).

To solve a linear equation:

- Simplify each side
- Move all terms with variables to the same side
- Isolate the variable

Quadratic Equations

A quadratic equation in one variable has the following features:

- Contains only one variable
- At least one term contains the variable squared, but there are no higher powers of the variable.
- The variable is not in a special structure (the denominator of a fraction, a square root, etc).

To solve a quadratic equation, we may use one of the following strategies:

- Factoring and the Zero Product Property
- Square Root Property
- Quadratic Formula

We'll dig into the different solving strategies soon, but first, let's review the square root property and the quadratic formula. Right now, we're focusing on using these formulas as a tool. If you want a full explanation of where the quadratic formula comes from, check this out.(https://www.mathsisfun.com/algebra/quadratic-equation-derivation.html).

Square Root Property
If $x^2 = k$, then $x = \sqrt{k}$ or $-\sqrt{k}$

Quadratic Formula

To solve a quadratic formula in the form

$$ax^2 + bx + c = 0$$

use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Examples

Example 1

Solve $4(x-3) + 12 = 15 - 5(x+6)$

Solution:

Whenever we solve an equation, we must first ask ourselves what kind of equation it is. Once we know that, we know what strategies we can try to solve the equation.

This equation only has an x, and that x is only raised to the (invisible) first power. That makes this a linear equation. We'll simplify, move the variables to one side, then isolate the variable.

$$\begin{aligned}
4(x-3)+12 &= 15-5(x+6) & \\
4x-12+12 &= 15-5x-30 & \text{Distribute} \\
4x &= -5x-15 & \text{Simplify} \\
4x+5x &= -5x+5x-15 & \text{Add } 5x \text{ to both sides} \\
9x &= -15 & \text{Simplify} \\
\frac{9x}{9} &= -\frac{15}{9} & \text{Divide 9 on both sides} \\
x &= -\frac{15}{9} &
\end{aligned}$$

Example 2

Solve $x^2 + x = 6$

Solution:
First we ask, what kind of equation is this? We see an x^2 and no higher powers of that variable, so this is a quadratic equation. We can use either factoring and the ZPP, or the quadratic formula. This equation isn't a great one for the square root property; we'll talk about that later.

If we want to use factoring and the ZPP, we have to have an equation that is $= 0$. After all, we don't have a 6 Product Property, so it would do us no good to factor the left hand side of the equation while the right hand side is 6!

$$\begin{aligned} x^2 + x &= 6 & \\ x^2 + x - 6 &= 0 & \text{Subtract 6 from both sides} \\ (x+3)(x-2) &= 0 & \text{Factor} \end{aligned}$$

Now we have an equation in the form $ab = 0$, so the Zero Product Property says that $a = 0$, $b = 0$, or both. In this case

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0$$

Each of these is a linear equation; the only thing left to do is isolate x in each equation.

$$\begin{aligned} x + 3 &= 0 \\ x &= -3 \end{aligned}$$

And

$$\begin{aligned} x - 2 &= 0 \\ x &= 2 \end{aligned}$$

This equation has two solutions: $x = 2, -3$

Try it Now

Solve each equation.
a. $x^2 + 10 = -11x$
b. $-2(3x - 1) + x = 14 - x$

These examples show that the format of the equation completely changes the strategy that we use to solve! This is true even within the category of quadratic equations. Let's take a look at some other cases of quadratics.

Example 3

Solve $x^2 = 8$

Solution:
Again, we see an x^2 and no other special powers or features, so this is a quadratic equation. We could try factoring or the quadratic formula, but this quadratic equation is in a special form that allows us to make use of the square root property.

$$x^2 = 8$$
$$x = \sqrt{8}, -\sqrt{8} \quad \text{Square Root Property}$$

Example 4

Solve $x^2 + 1 = -5x$

Solution:
Another quadratic equation (see the x^2?). This one is not a good candidate for the square root property, since we also have that $-5x$ term. Let's move that to the other side so our equation is equal to zero.

$$x^2 + 5x + 1 = 0$$

Now we can try factoring, but we'll get stuck; there are no integer factors of 1 which add to 5! We're left with our last resort: the quadratic formula. Here $a = 1, b = 5$ and $c = 1$, so

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$
$$x = \frac{-5 \pm \sqrt{25 - 4}}{2}$$
$$x = \frac{-5 \pm \sqrt{21}}{2}$$

Note that this actually indicates *two* solutions to the equation, $x = \frac{-5+\sqrt{21}}{2}$ and $x = \frac{-5-\sqrt{21}}{2}$.

Try it Now

Solve each equation

a. $4x^2 + 1 = 8$

b. $9x^2 + 3x - 2 = 0$

Now we've seen many different cases for the types of equations we may need to solve. The next sets of examples will help us to learn to identify the best strategy to use, and will show us some special solution situations.

Example 5

Solve $5x^2 - 1 = 9$

Solution:
We'll get in the habit of asking ourselves a series of questions as we solve.

What type of equation is this?
- Quadratic (x^2)

What strategies may we choose from?
- Factoring and ZPP
- Square root property
- Quadratic Formula

Which strategy is best?
- We see that we have an x^2 term but no x term; that means we should be able to isolate the x^2 and then use the square root property.

Now we know what we're dealing with and we have a plan. Let's solve.

$$\begin{aligned} 5x^2 - 1 &= 9 & & \\ 5x^2 &= 10 & & \text{Add 1 to both sides} \\ x^2 &= 2 & & \text{Divide by 5 on both sides} \\ x &= \sqrt{2}, -\sqrt{2} & & \text{Apply the square root property} \end{aligned}$$

Example 6

Solve $3(x - 4) + 3x = 6(x + 1)$

Solution:
What type of equation is this?
- Linear (x)

What strategies may we choose from?
- Simplify, move x's to same side, isolate x

$$\begin{aligned} 3(x-4) + 3x &= 6(x+1) & & \\ 3x - 12 + 3x &= 6x + 6 & & \text{Distribute} \\ 6x - 12 &= 6x + 6 & & \text{Combine like terms}^* \\ -12 &= 6 & & \text{Subtract } 6x \text{ from both sides} \end{aligned}$$

Wait a second, -12 isn't equal to 6! What happened? We got a false statement, so this equation has no solutions.

You may have even noticed there was a problem at the * step, since it doesn't make sense that $6x - 12$ should be equal to $6x + 6$. Either way, there are no solutions to this equation.

When we're solving an equation, we're **making an assumption that the two sides of the equation are equal**. It's that assumption that allows us to do the same operation to both sides; if we change two equal things in the same way, they should still be equal to one another. We followed that assumption and got a false statement; that doesn't mean our work was wrong, it means **our assumption was wrong**! These two statements never could be equal; the equation has no solutions.

Example 7

Solve $3(x-4)^2 = 15$

Solution:

What type of equation is this?

- Quadratic (we'll have an x^2 term if we were to fully distribute the left hand side).

What strategies may we choose from?

- Factoring and ZPP
- Square root property
- Quadratic Formula

Which strategy is best?

- We see that we have an $(x-4)^2$ term but no x term; that means we should be able to isolate the squared part and then use the square root property. Alternatively, we could multiply out the left hand side and use one of our other strategies, but that's more work than strictly necessary!

$$\begin{aligned} 3(x-4)^2 &= 15 && \\ (x-4)^2 &= 5 && \text{Divide 3 on both sides} \\ \sqrt{(x-4)^2} &= \sqrt{5} && \text{square root both sides} \\ x-4 &= \pm\sqrt{5} && \text{square root property} \end{aligned}$$

This has created two equations for us to solve: $x-4=\sqrt{5}$ and $x-4=-\sqrt{5}$. In each case, we'll isolate x by adding 4 to both sides to create our final solutions: $x = 4+\sqrt{5}, 4-\sqrt{5}$.

Example 8

Solve $-3x^2-5x-2=0$.

Solution:

What type of equation is this?

- Quadratic (x^2)

What strategies may we choose from?

- Factoring and ZPP
- Square root property

- Quadratic Formula

Which strategy is best?
The equation is already in the form $ax^2 + bx + c = 0$, so we're well setup to use either factoring and the ZPP or the quadratic formula. Let's take this as an opportunity to get some factoring practice in.

To make the ac method a bit easier, we'll start by factoring out a GCF of -1:

$$-1(3x^2 + 5x + 2) = 0$$

Now, ac = 6 and b = 5. Factors of ac that add to 5 are 2 and 3. We'll rewrite the middle $5x$ as $2x + 3x$, then factor by grouping.

$$-1(3x^2 + 2x + 3x + 2) = 0$$
$$-1\big(x(3x + 2) + 1(3x + 2)\big) = 0$$
$$-1(3x + 2)(x + 1) = 0$$

Now we apply the ZPP and finish solving

$$3x + 2 = 0 \quad \text{or } x + 1 = 0$$
$$x = -\frac{2}{3}, -1$$

Example 9

Solve $4(x - 1) = 5x - (x + 4)$

Solution:
What type of equation is this?
- Linear (x)

What strategies may we choose from?
- Simplify, move x's to same side, isolate x

$$\begin{aligned} 4(x - 1) &= 5x - (x + 4) \\ 4x - 4 &= 5x - 1(x + 4) \\ 4x - 4 &= 5x - x - 4 && \text{Distribute} \\ 4x - 4 &= 4x - 4 && \text{Simplify} \end{aligned}$$

Something weird has happened again. Of course $4x - 4$ is equal to itself! If we subtract $4x$ from both sides to try to move all the x terms together, then we'll get the even more obvious statement that $-4 = -4$. What does it mean?

This statement will be true for any value of x, so the equation has infinite solutions! In other words, x can be any real number.

Example 10

Solve $x^2 + x + 2 = 0$

Solution:
What type of equation is this?

- Quadratic (x^2)

What strategies may we choose from?

- Factoring and ZPP
- Square root property
- Quadratic Formula

Which strategy is best?
This equation is perfectly setup for either factoring or the quadratic formula. If we try factoring for a while, we'll find we can't proceed that way. Let's use the quadratic formula.

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

$$x = \frac{-1 \pm \sqrt{1 - 8}}{2}$$

$$x = \frac{-1 \pm \sqrt{-7}}{2}$$

That -7 is an issue. The square root of a negative number is imaginary. In this class, we're going to stick with real numbers, so we'll say that this equation has no real solutions.

Summary

Questions for solving equations
What kind of equation is it? What strategies can we use to solve this kind of equation?

Strategies for solving linear equations
• Simplify each side • Move all terms with variables to the same side • Isolate the variable

Strategies for solving quadratic equations

For equations in the form $ax^2 + bx + c = 0$

- Factoring and ZPP

or

- Quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

For equations in the form $(\text{something})^2 = \text{number}$

- Square root property (if $x^2 = k, x = \pm\sqrt{k}$)

No real solutions

When solving a linear equation, if the result is a false statement, there are no real solutions.

When solving a quadratic equation, if we must square root a negative, there are no real solutions.

Section 0.4 Practice Problems

Conceptual Questions

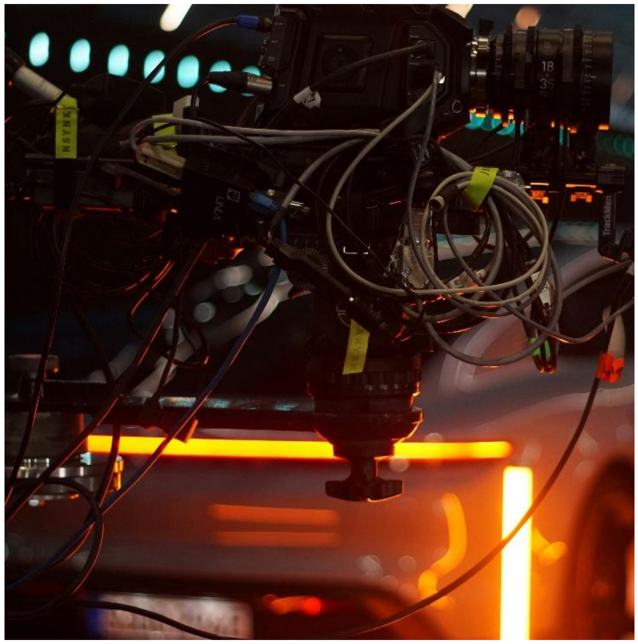
1. What are linear and quadratic equations? What are the differences in how we solve each type of equation?
2. Show that the sum of the two solutions to a quadratic equation is always $-\frac{b}{a}$.
3. When we solve a quadratic equation by factoring, why do we move all terms to one side, having zero on the other side?
4. Describe two scenarios where using the square root property to solve a quadratic equation would be the most efficient method.

Practice Problems

Solve each equation

1. $7x + 2 = 3x - 9$
2. $4x - 3 = 5$
3. $8x^2 - 9 = -6x$
4. $x^2 - 9x = -18$
5. $3(x + 2) - 12 = 5(x + 1)$
6. $3x^2 = 5x - 1$
7. $12 - 5(x + 3) = 2x - 5$
8. $5x^2 = 5x + 30$
9. $\frac{1}{2} - \frac{1}{3}x = \frac{4}{3}$
10. $2x^2 - 8x - 5 = 0$
11. $x^2 + 4x - 21 = 0$
12. $4x^2 = 9$
13. $\frac{x}{3} - \frac{3}{4} = \frac{2x+3}{12}$
14. $(x - 1)^2 = 25$
15. $\frac{2}{3}x + \frac{1}{2} = \frac{31}{6}$
16. $3x^2 = 75$
17. $x^2 + 4x + 2 = 0$
18. $(2x + 1)^2 = 9$
19. $3(2x - 1) + x = 5x + 3$
20. $x^2 = 36$
21. $\frac{2x}{3} - \frac{3}{4} = \frac{x}{6} + \frac{21}{4}$
22. $2x^2 + 14x = 36$
23. $x^2 + x = 4$
24. $4x^2 = 5x$
25. $9x - 5 = -2x^2$
26. $\frac{x+2}{4} - \frac{x-1}{3} = 2$
27. $6x^2 + 5 = -17x$
28. $7x^2 + 3x = 0$
29. $(x - 5)^2 - 4 = 0$
30. $4x^2 - 12x + 8 = 0$

Chapter 1: Functions



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In this chapter, we're investigating the machines we call functions. Our emphasis will be on notation, tables, and graphs; we leave specific functions and their behaviors to later chapters.

Section 1.1 Functions and Function Notation

What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Defining these relationships in a useful and meaningful way takes some consideration. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Function

A rule for a relationship between an input quantity and an output quantity in which:
each input determines exactly one output.

We say "the output is a function of the input."

The inputs are often called the "independent" variable and the outputs are often called the "dependent" variable.

Example 1

In the height and age example above, is height a function of age? Is age a function of height?

In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 18th birthday, I had exactly one height of 69 inches.

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 70 inches, there is more than one output of age since I was 70 inches at the age of 20 and 21.

Example 2

At a coffee shop, the menu consists of items and their prices. Is price a function of the item? Is the item a function of the price?

We could say that price is a function of the item, since each input of an item has one output of a price corresponding to it. We could not say that item is a function of price, since two items might have the same price.

Example 3

In many classes the overall percentage you earn in the course corresponds to a decimal grade point. Is decimal grade a function of percentage? Is percentage a function of decimal grade?

For any percentage earned, there would be a decimal grade associated, so we could say that the decimal grade is a function of percentage. That is, if you input the percentage, your output would be a decimal grade. Percentage may or may not be a function of decimal grade, depending upon the teacher's grading scheme. With some grading systems, there are a range of percentages that correspond to the same decimal grade.

Sometimes in a relationship each input corresponds to exactly one output, AND every output corresponds to exactly one input. We call this kind of relationship a **one-to-one function**.

One-to-One Function
A one-to-one function is a function for which each output corresponds to exactly one input.

From Example 3, *if* each unique percentage corresponds to one unique decimal grade point and each unique decimal grade point corresponds to one unique percentage then it is a one-to-one function.

Try it Now

Let's consider bank account information.

1. Is your balance a function of your bank account number?
   (if you input a bank account number does it make sense that the output is your balance?)

2. Is your bank account number a function of your balance?
   (if you input a balance does it make sense that the output is your bank account number?)

Function Notation

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable *h* to represent height and we could use the descriptive variable *a* to represent age.

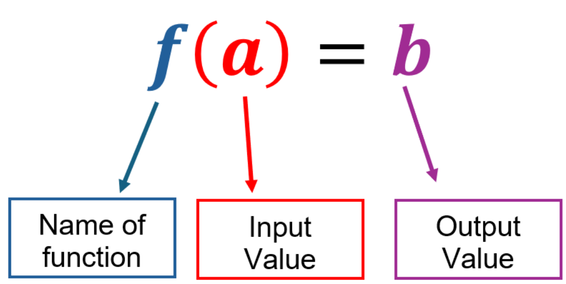
"height is a function of age"	if we name the function *f* we write
"*h* is *f* of *a*"	or more simply
$h = f(a)$	we could instead name the function *h* and write
$h(a)$	which is read "*h* of *a*"

Remember we can use any variable to name the function; the notation *h(a)* shows us that *h* depends on *a*. The value "*a*" must be put into the function "*h*" to get a result. Be careful - the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication! It does *not* mean "h times a").

Function Notation

The notation output $= f(input)$ defines a function named *f*. This would be read "output is *f* of input"

The parentheses in function notation do NOT indicate multiplication! The parentheses merely surround the input to the function named f.



Example 4

Introduce function notation to represent a function that takes as input the name of a month, and gives as output the number of days in that month.

Solution:
The number of days in a month is a function of the name of the month, so if we name the function *f*, we could write "$days = f(month)$" or $d = f(m)$. If we simply name the function *d*, we could write $d(m)$.

For example, *d*(March) = 31, since March has 31 days. The notation *d(m)* reminds us that the number of days, *d* the output, is dependent on the name of the month, *m* the input.

Example 5

A function *N* = *f(y)* gives the number of police officers, *N*, in a town in year *y*. What does $f(2005) = 300$ tell us?

Solution:
When we read $f(2005) = 300$, we see the input quantity is 2005, which is a value for the input quantity of the function, the year (*y*). The output value is 300, the number of police officers (*N*), a value for the output quantity. Remember *N=f(y)*. This tells us that in the year 2005 there were 300 police officers in the town.

Tables as Functions

Functions can be represented in many ways: with words (as we did in the last few examples), tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values.

In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1: This table represents the input, number of the month (January = 1, February = 2, and so on) while the output is the number of days in that month. This represents everything we know about the months & days for a given year (that is not a leap year)

(input) Month number, *m*	1	2	3	4	5	6	7	8	9	10	11	12
(output) Days in month, *D*	31	28	31	30	31	30	31	31	30	31	30	31

Table 2: The table below defines a function *Q = g(n)*. Remember this notation tells us *g* is the name of the function that takes the input *n* and gives the output *Q*.

n	1	2	3	4	5
Q	8	6	7	6	8

Table 3: This table represents the age of children in years and their corresponding heights. This represents just some of the data available for height and ages of children.

(input) *a*, age in years	5	5	6	7	8	9	10
(output) *h*, height inches	40	42	44	47	50	52	54

Example 6

Which of these tables define a function (if any)? Are any of them one-to-one?

Input	Output
2	1
5	3
8	6

Input	Output
-3	5
0	1
4	5

Input	Output
1	0
5	2
5	4

Solutions:
The first and second tables define functions. In both, each input corresponds to exactly one output. The third table does not define a function since the input value of 5 corresponds with two different output values.
Only the first table is one-to-one; it is both a function, and each output corresponds to exactly one input. Although table 2 is a function, because each input corresponds to exactly one output, each output does not correspond to exactly one input so this function is not one-to-one. Table 3 is not even a function and so we don't even need to consider if it is a one-to-one function.

Try it Now

If each percentage earned translated to one letter grade, would this be a function? Is it one-to-one?

Solving and Evaluating Functions:

When we work with functions, there are two typical things we do: evaluate and solve. Evaluating a function is what we do when we know an input and use the function to determine the corresponding output. *Evaluating will always produce one result, since each input of a function corresponds to exactly one output.*

Solving equations involving a function is what we do when we know an output, and use the function to determine the inputs that would produce that output. Solving a function could produce more than one solution, since different inputs can produce the same output.

Functions: Solving vs. Evaluating

If we know an input and use the function to determine an output, we are **evaluating** the function.

If we want to know what input produces an output, we are **solving** a function.

Example 7

Using the table shown, where $Q=g(n)$

a. Evaluate $g(3)$
b. Solve $g(n) = 6$

n	1	2	3	4	5
Q	8	6	7	6	8

Solutions:

a. Evaluating $g(3)$ (read: "g of 3") means that we need to determine the output value, Q, of the function g given the input value of n=3. Looking at the table, we see the output corresponding to n=3 is Q=7, allowing us to conclude $g(3) = 7$.

b. Solving $g(n) = 6$ means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

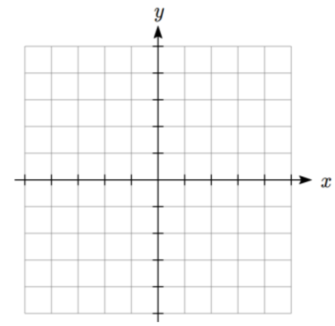
When we input 2 into the function g, our output is $Q = 6$

When we input 4 into the function g, our output is also $Q = 6$

Try it Now

Using the function in Example 7, evaluate $g(4)$

Graphs as Functions

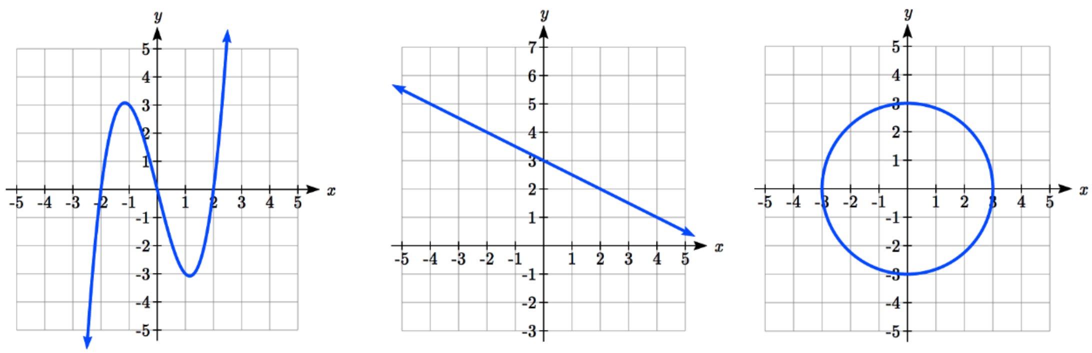


Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

The most common graph has *y* on the vertical axis and *x* on the horizontal axis, and we say *y* is a function of *x*, or *y* = *f(x)* when the function is named *f*.

Example 8

Which of these graphs defines a function *y=f(x)*? Which of these graphs defines a one-to-one function?



Solution:
Looking at the three graphs above, the first two define a function *y=f(x)*, since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the *y*-value of the graph. The 3rd graph does not define a function *y=f(x)* since some input values, such as *x*=2, correspond with more than one output value.

Graph 1 is not a one-to-one function. For example, the output value 3 has two corresponding input values, -1 and 2.3

Graph 2 is a one-to-one function; each input corresponds to exactly one output, and every output corresponds to exactly one input.

Graph 3 is not even a function so there is no reason to even check to see if it is a one-to-one function.

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input, and therefore the graph is not a function.

Vertical Line Test
If a vertical line touches the graph of an equation more than once, then the graph does not represent a function. In this case, we would say the graph "fails the vertical line test." This is because the vertical line is showing a location where an input corresponds to more than one output.

Once you have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line crosses the graph more than once, then the graph does not define a one-to-one function.

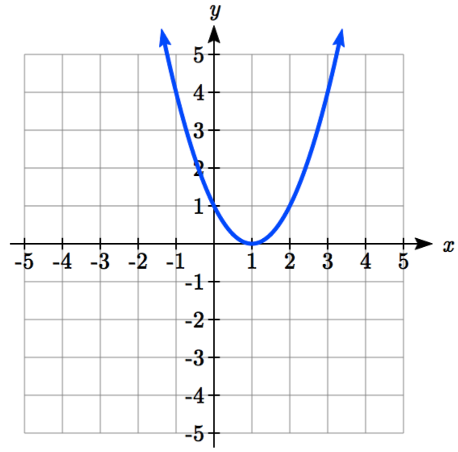
Horizontal Line Test
If a horizontal line touches the graph of a function more than once, then the graph does not represent a one-to-one function. In this case, we would say the graph "fails the horizontal line test." This is because the horizontal line is showing a location where an output corresponds to more than one input.

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 9

Given the graph of *f(x)*

a. Evaluate $f(2)$
b. Solve $f(x) = 4$



Solutions:

a. To evaluate $f(2)$, we find the input of $x = 2$ on the horizontal axis. Moving up to the graph gives the point (2, 1), giving an output of *y*=1. $f(2) = 1$.

b. To solve $f(x) = 4$, we find the value 4 on the vertical axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: (-1,4) and (3,4). These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$

This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4.

Notice that while the graph in the previous example is a function, finding two input values for the output value of 4 shows us that this function is not one-to-one.

Try it Now

Using the graph from example 9, evaluate $f(1)$ and solve $f(x) = 1$.

Formulas as Functions

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 10

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$ if possible.

Solution:

To express the relationship in this form, we need to be able to write the relationship where *p* is a function of *n*, which means writing it as *p* = [something involving *n*]. So, we'll work on isolating the p in the equation.

$2n + 6p = 12$ subtract 2n from both sides

$6p = 12 - 2n$ divide both sides by 6 and simplify

$p = \frac{12-2n}{6}$

$p = \frac{12}{6} - \frac{2n}{6}$ Having rewritten the formula as *p*=,

$p = 2 - \frac{1}{3}n$

we can now express *p* as a function:

$$p = f(n) = 2 - \frac{1}{3}n$$

It is important to note that not every relationship can be expressed as a function with a formula.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations - no further solving is required. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed into machines, whereas solving things is much more challenging.

Example 11

Express the relationship $x^2 + y^2 = 1$ as a function $y = f(x)$ if possible.

Solution:

If we try to solve for *y* in this equation:

$$y^2 = 1 - x^2$$
$$y = \pm\sqrt{1 - x^2}$$

We end up with two outputs corresponding to the same input (the positive root and the negative root), so this relationship cannot be represented as a single function $y = f(x)$.

As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

Example 12

Given the function $f(x) = x^2 - 3x + 2$, evaluate each of the following.

a. $f(2)$
b. $f(-2)$
c. $f(a)$
d. $f(a+h)$
e. $f(a+h) - f(a)$

Solutions:

In this example, we're trying to get used to working with function notation algebraically. When we're evaluating our function, it's useful to think of the variable as just a big box where we're going to dump our inputs:

$$f(x) = x^2 - 3x + 2$$

becomes

$$f(\quad) = (\quad)^2 - 3(\quad) + 2$$

So, when we're evaluating our function, we'll put identical copies of the input into each box.

a. Starting with

$$f(\quad) = (\quad)^2 - 3(\quad) + 2$$

we'll evaluate $f(2)$ by replacing each and every input box with a 2.

$$f(\mathbf{2}) = (\mathbf{2})^2 - 3(\mathbf{2}) + 2$$
$$= 4 - 6 + 2$$
$$= -2 + 2$$
$$= 0$$

So, $f(2) = 0$

b. This time, we'll replace each box with -2. This is where it becomes very useful to always put our inputs in a set of parentheses!

$$f(-2) = (-2)^2 - 3(-2) + 2$$
$$= 4 + 6 + 2$$
$$= 12$$

That means $f(-2) = 12$.

c. To find $f(a)$, we'll follow the same procedure as before: we'll replace each input box with a. We just won't be able to simplify much after that!

$$f(a) = (a)^2 - 3(a) + 2$$
$$= a^2 - 3a + 2$$

So $f(a) = a^2 - 3a + 2$

Now, we want to fill each of our input boxes with $(a+h)$. We'll have to be careful with our algebra after that!

$$f(\quad) = (\quad)^2 - 3(\quad) + 2$$

becomes

$$f(\boldsymbol{a+h}) = (\boldsymbol{a+h})^2 - 3(\boldsymbol{a+h}) + 2$$

Remember, that $(a+h)^2$ means $(a+h)(a+h)$, so we have a lot of distributing to do!

$$= (a+h)(a+h) + (-3)(a+h) + 2$$
$$= a^2 + ah + ah + h^2 + (-3a) + (-3h) + 2$$
$$= a^2 + 2ah + h^2 - 3a - 3h + 2$$

d. Finally we come to $f(a+h) - f(a)$. We already know that $f(a) = a^2 - 3a + 2$, and $f(a+h) = a^2 + 2ah + h^2 - 3a - 3h + 2$, so we'll put those together to build our full answer:

$$f(a+h) - f(a)$$
$$= a^2 + 2ah + h^2 - 3a - 3h + 2 - (a^2 - 3a + 2)$$
$$= a^2 + 2ah + h^2 - 3a - 3h + 2 - a^2 + 3a - 2$$
$$= a^2 - a^2 - 3a + 3a + 2 - 2 + 2ah + h^2 - 3h$$
$$= 2ah + h^2 - 3h$$

Try it Now

Given the function $g(x) = 5 - x^2$, evaluate the following:

a. $g(3)$
b. $g(-3)$
c. $g(a+h)$
d. $g(a)$
e. $g(a+h) - g(a)$

Example 13

Given the function $k(t) = 3(t-4)$

a. Evaluate *k*(2)
b. Solve *k(t)* = 1

Solutions:

a. "Evaluate *k*(2)" means we want to find the output when the input is *t* = 2. We plug in the input value 2 into the formula wherever we see the input variable *t*, then simplify

$$k(2)$$
$$= 3(2-4)$$
$$= 3(-2)$$
$$= -6$$

So $k(2) = -6$.

b. "Solve *k(t)* = 1" means we want to find the input *t* that will give us an output of 1. We set the formula for *k(t)* equal to 1, and solve for the input

$1 = 3(t - 4)$ Set formula equal to 1
$1 = 3t - 12$ distribute
$1 + 12 = 3t - 12 + 12$ add 12 to both sides
$13 = 3t$ Simplify
$\frac{13}{3} = \frac{3t}{3}$ divide by 3 on both sides
$t = \frac{13}{3}$ simplify

Example 14

Given the function $h(p) = p^2 + 2p$

a. Evaluate $h(4)$
b. Solve $h(p) = 3$

Solutions:

a. To evaluate *h*(4) we substitute the value 4 for the input variable *p* in the given function.

$$h(4)$$
$$= (4)^2 + 2(4)$$
$$= 16 + 8$$
$$= 24$$

b. To solve $h(p) = 3$, we'll set our formula equal to 3 and solve. That means we want to solve $p^2 + 2p = 3$. That's a quadratic equation! Let's see if we can factor and use the Zero Product Property.

$p^2 + 2p = 3$
$p^2 + 2p - 3 = 3 - 3$ subtract 3 from both sides
$p^2 + 2p - 3 = 0$
$(p - 1)(p + 3) = 0$ factor

By the zero product property since $(p - 1)(p + 3) = 0$, either $p - 1 = 0$ or $p + 3 = 0$ (or both of them equal 0) and so we solve both equations for *p*, finding $p = -3$ from the first equation and $p = 1$ from the second equation.

This gives us the solution: $h(p) = 3$ when $p = 1$ or $p = -3$.

We found two solutions in this case, which tells us this function is not one-to-one.

Try it Now

Given the function $g(m) = 4 - (2 + m)$

a. Evaluate *g*(5)
b. Solve *g(m)* = 2

Graphing Functions from Formulas

There will be many times when we want to analyze various features of functions. Sometimes these features are easier to spot and understand if we can start with a graph of a function. For that reason, we want to work on the skill of creating graphs from function formulas.

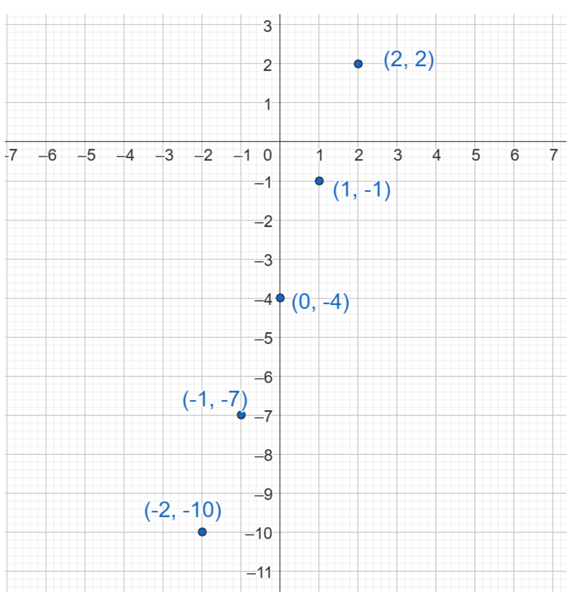
Example 15

Create a graph for the function $f(x) = 3x - 4$

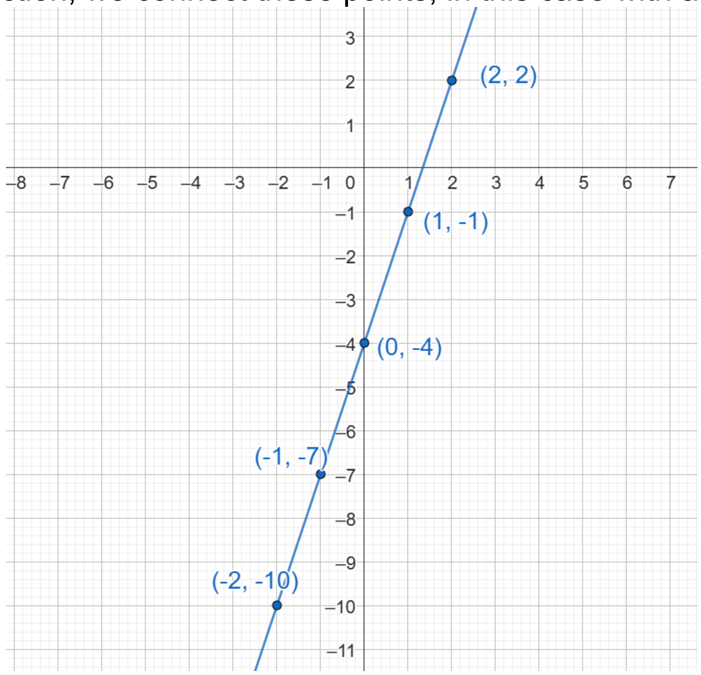
Solution:
To create a graph for this function, we want to describe a bunch of input/output relationships. To do this, we'll arbitrarily choose a list of input values, say x-values from -2 to 2. We'll evaluate the function at each of these inputs to find the corresponding outputs, then plot the points that we've determined.

x	$f(x)$	Point
-2	$f(-2) = 3(-2) - 4 = -10$	$(-2, -10)$
-1	$f(-1) = 3(-1) - 4 = -7$	$(-1, -7)$
0	$f(0) = 3(0) - 4 = -4$	$(0, -4)$
1	$f(1) = 3(1) - 4 = -1$	$(1, -1)$
2	$f(2) = 3(2) - 4 = 2$	$(2,2)$

We didn't *have* to pick x-values from -2 to 2. We could have chose 1.5, or 0.9, or 57. We're just making a choice that is a) fairly easy to calculate and b) creates enough points to get an idea of what the graph looks like. Currently, if we graph these points, we see the following:



These are only *some* of the points that form the function though. To represent the graph of the complete function, we connect these points, in this case with a straight line:



A big part of College Algebra is being able to quickly understand functions and their features. We'll start learning ways to recognize ways to identify a function's expected shape and other features; for now, we'll choose enough x-values to get an idea of the function's shape and sketch the graph as best we can.

Example 16

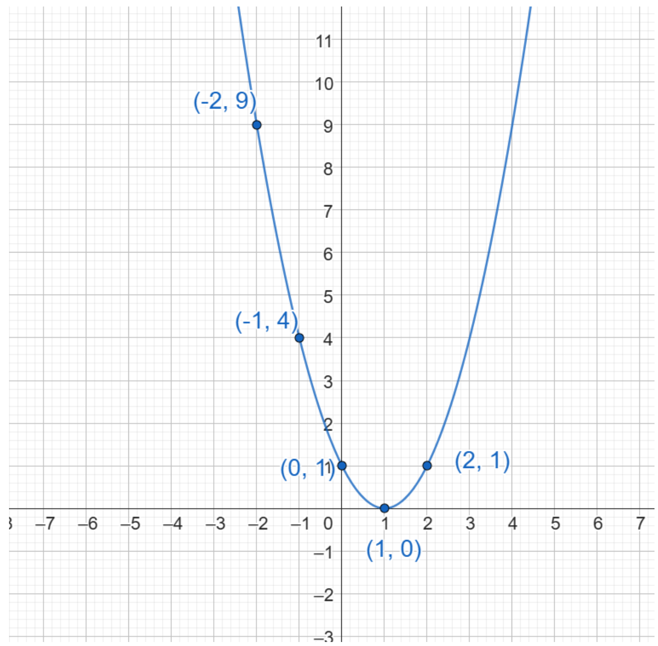
Create a graph for the function $f(x) = (x-1)^2$

Solution:

Again, we'll choose a set of x-values to substitute in, so we can create a group of points to graph. We can again work with values from -2 to 2.

x	$f(x)$	Point
-2	$f(-2) = (-2-1)^2 = (-3)^2 = 9$	$(-2,9)$
-1	$f(-1) = (-1-1)^2 = (-2)^2 = 4$	$(-1,4)$
0	$f(0) = (0-1)^2 = (-1)^2 = 1$	$(0,1)$
1	$f(1) = (1-1)^2 = 0^2 = 0$	$(1,0)$
2	$f(2) = (2-1)^2 = 1^2 = 1$	$(2,1)$

When we graph these points, we don't see a perfectly straight line anymore. In fact, we see a curved shape like this:



Try it Now

Graph the function $f(x) = \sqrt{x-1}$. Be careful about the x-values you choose!

Important Topics of this Section
Definition of a function
Input (independent variable)
Output (dependent variable)
Definition of a one-to-one function
Function notation
Descriptive variables
Functions in words, tables, graphs & formulas
Vertical line test
Horizontal line test
Evaluating a function at a specific input value
Solving a function given a specific output value
Sketching graphs of functions

Section 1.1 Exercises

Conceptual Questions

1. What is a function?
2. What does it mean for a relation to be one-to-one? Can something be one-to-one but not a function?
3. How do we evaluate functions from tables? From graphs? From formulas?
4. How do we solve functions from tables? From graphs? From formulas?
5. What does the vertical line test determine? Why does it work?
6. What does the horizontal line test determine? Why does it work?
7. How can we sketch a graph of a function?

Practice Problems

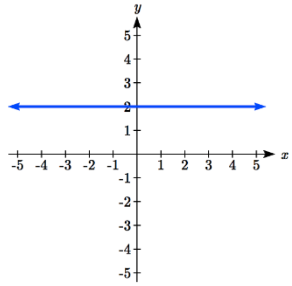
1. The amount of garbage, *G*, produced by a city with population *p* is given by $G = f(p)$. *G* is measured in tons per week, and *p* is measured in thousands of people.
    a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function *f*.
    b. Explain the meaning of the statement $f(5) = 2$.

2. The number of cubic yards of dirt, *D*, needed to cover a garden with area *a* square feet is given by $D = g(a)$.
    a. A garden with area 5000 ft^2 requires 50 cubic yards of dirt. Express this information in terms of the function *g*.
    b. Explain the meaning of the statement $g(100) = 1$.

3. Let $f(t)$ be the number of ducks in a lake *t* years after 1990. Explain the meaning of each statement:
    a. $f(5) = 30$ b. $f(10) = 40$

4. Let $h(t)$ be the height above ground, in feet, of a rocket *t* seconds after launching. Explain the meaning of each statement:
    a. $h(1) = 200$ b. $h(2) = 350$

5. Select all of the following graphs which represent y as a function of x.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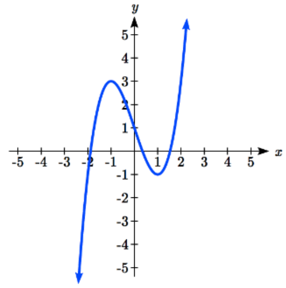
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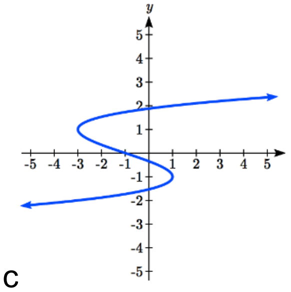
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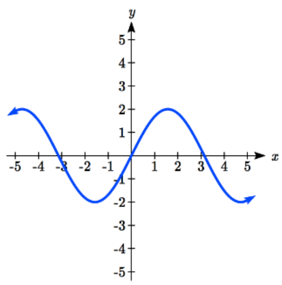
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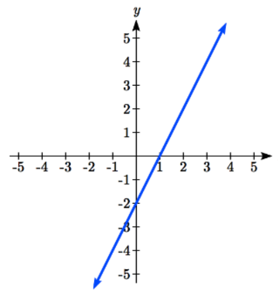
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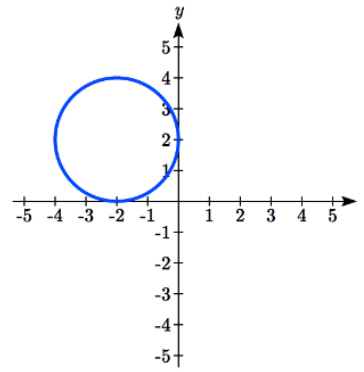
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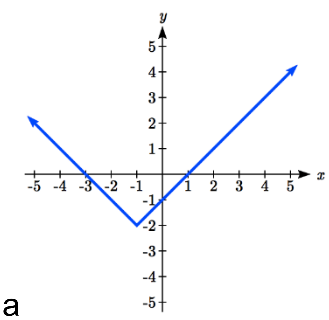
6. Select all of the following graphs which represent y as a function of x.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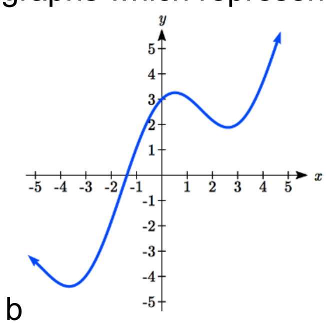
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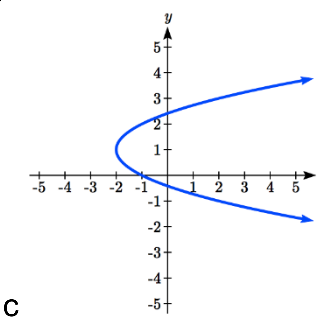
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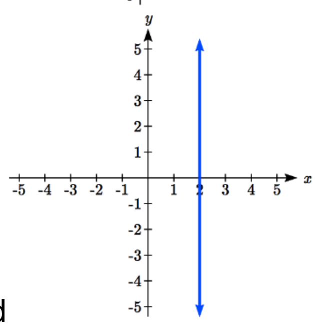
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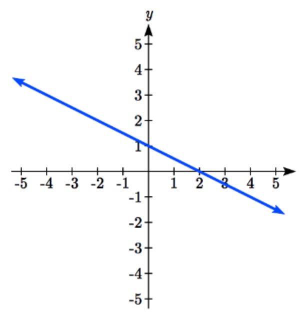
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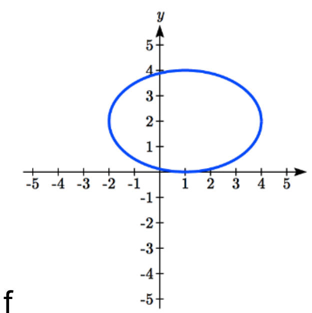
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7. Select all of the following tables which represent y as a function of x.

a.

x	5	10	15
y	3	8	14

b.

x	5	10	15
y	3	8	8

c.

x	5	10	10
y	3	8	14

8. Select all of the following tables which represent *y* as a function of *x*.

a.

x	2	6	13
y	3	10	10

b.

x	2	6	6
y	3	10	14

c.

x	2	6	13
y	3	10	14

9. Select all of the following tables which represent *y* as a function of *x*.

a.

x	***y***
0	-2
3	1
4	6
8	9
3	1

b.

x	***y***
-1	-4
2	3
5	4
8	7
12	11

c.

x	***y***
0	-5
3	1
3	4
9	8
16	13

d.

x	***y***
-1	-4
1	2
4	2
9	7
12	13

10. Select all of the following tables which represent *y* as a function of *x*.

a.

x	***y***
-4	-2
3	2
6	4
9	7
12	16

b.

x	***y***
-5	-3
2	1
2	4
7	9
11	10

c.

x	***y***
-1	-3
1	2
5	4
9	8
1	2

d.

x	***y***
-1	-5
3	1
5	1
8	7
14	12

11. Select all of the following tables which represent *y* as a function of *x* **and** are one-to-one.

a.

x	3	8	12
y	4	7	7

b.

x	3	8	12
y	4	7	13

c.

x	3	8	8
y	4	7	13

12. Select all of the following tables which represent *y* as a function of *x* **and** are one-to-one.

a.

x	2	8	8
y	5	6	13

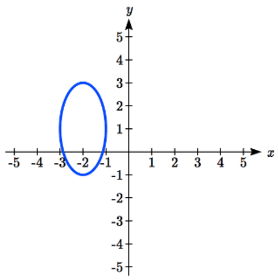
b.

x	2	8	14
y	5	6	6

c.

x	2	8	14
y	5	6	13

13. Select all of the following graphs which are **one-to-one functions**.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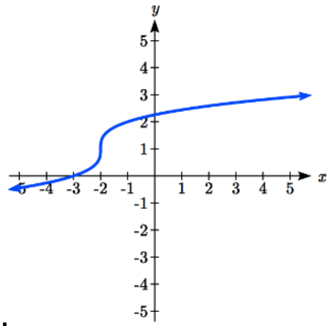
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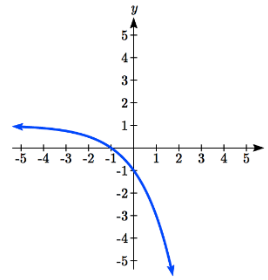
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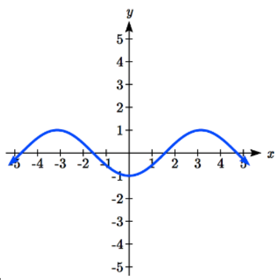
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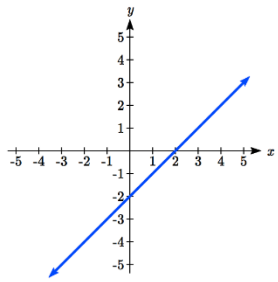
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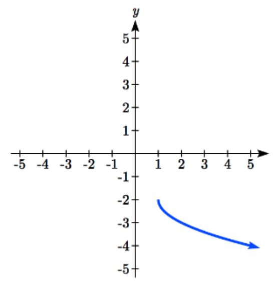
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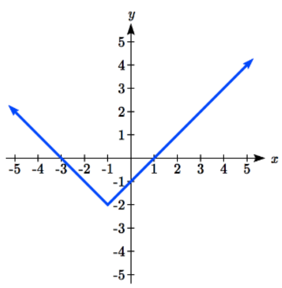
14. Select all of the following graphs which are **one-to-one functions**.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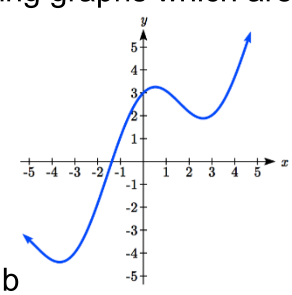
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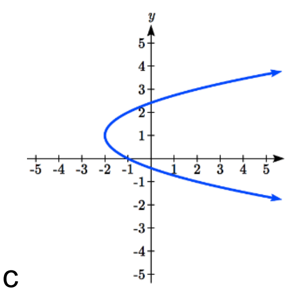
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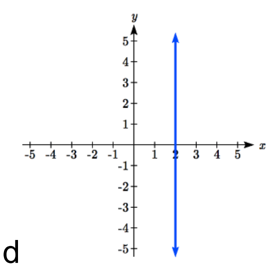
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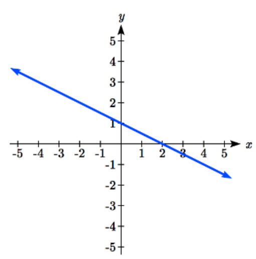
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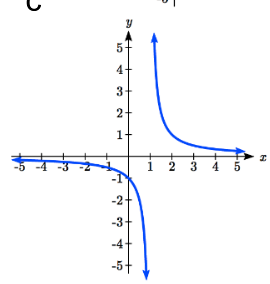
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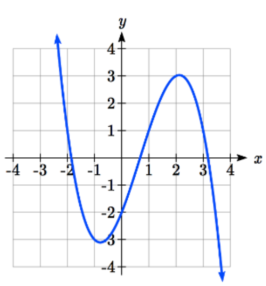
Given each function $f(x)$ graphed, evaluate $f(1)$ and $f(3)$

15.



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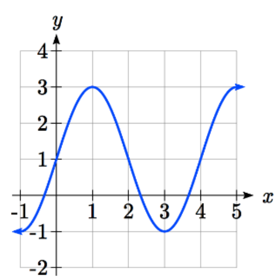
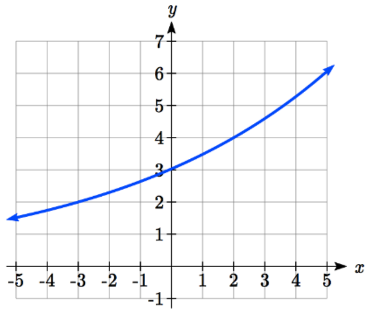
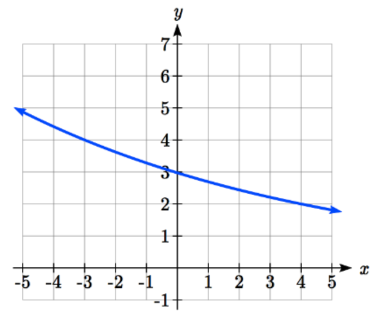


17. Given the function $g(x)$ graphed here,
    a. Evaluate $g(2)$
    b. Solve $g(x) = 2$



18. Given the function $f(x)$ graphed here.
    a. Evaluate $f(4)$
    b. Solve $f(x) = 4$



19. Based on the table below,
    a. Evaluate $f(3)$ b. Solve $f(x) = 1$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

20. Based on the table below,
    a. Evaluate $f(8)$ b. Solve $f(x) = 7$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	62	8	7	38	86	73	70	39	75	34

For each of the following functions, evaluate: $\boldsymbol{f(-2)}$, $\boldsymbol{f(-1)}$, $\boldsymbol{f(0)}$ $\boldsymbol{f(0)}$, $\boldsymbol{f(1)}$, $\boldsymbol{f(2)}$, $\boldsymbol{f(a+h)}$, *and* $\boldsymbol{f(a+h) - f(a)}$.

21. $f(x) = 4 - 2x$
22. $f(x) = 8 - 3x$
23. $f(x) = 8x^2 - 7x + 3$
24. $f(x) = 6x^2 - 7x + 4$
25. $f(x) = -x^3 + 2x$
26. $f(x) = 5x^4 + x^2$
27. $f(x) = 3 + \sqrt{x+3}$
28. $f(x) = 4 - \sqrt[3]{x-2}$
29. $f(x) = (x-2)(x+3)$
30. $f(x) = (x+3)(x-1)^2$
31. $f(x) = \frac{x-3}{x+1}$
32. $f(x) = \frac{x-2}{x+2}$
33. $f(x) = 2^x$
34. $f(x) = 3^x$

35. Suppose $f(x) = x^2 + 8x - 4$. Compute the following:
   a. $f(-1) + f(1)$ b. $f(-1) - f(1)$

36. Suppose $f(x) = x^2 + x + 3$. Compute the following:
   a. $f(-2) + f(4)$ b. $f(-2) - f(4)$

37. Let $f(t) = 3t + 5$
   a. Evaluate $f(0)$ b. Solve $f(t) = 0$

38. Let $g(p) = 6 - 2p$
   a. Evaluate $g(0)$ b. Solve $g(p) = 0$

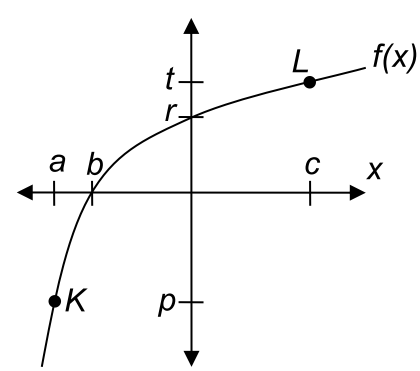
Sketch a graph for each function by creating a table of values.

39. $f(x) = 4x + 1$
40. $g(x) = x^2 - 1$
41. $h(x) = \sqrt{x} + 2$
42. $k(x) = 2(x - 1)$

43. Sketch a reasonable graph for each of the following functions. [UW]
   a. Distance of your big toe from the ground as you ride your bike for 10 seconds.
   b. Your height above the water level in a swimming pool after you dive off the high board.

44. The percentage of dates and names you'll remember for a history test, Sketch a reasonable graph for each of the following functions. [UW]
   a. Height of a person depending on age.
   b. Height of the top of your head as you jump on a pogo stick for 5 seconds.
   c. The amount of postage you must put on a first class letter, depending on the weight of the letter.
   c. depending on the time you study.

45. Using the graph shown,
   a. Evaluate $f(c)$
   b. Solve $f(x) = p$
   c. Suppose $f(b) = z$. Find $f(z)$
   d. What are the coordinates of points *L* and *K*?



46. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function $s = d(t)$ which keeps track of Dave's distance s from Padelford Hall at time t. Take distance units to be "feet" and time units to be "minutes." Assume Dave's path to Gould Hall is long a straight line which is 2400 feet long. [UW]

a. Dave leaves Padelford Hall and walks at a constant spend until he reaches Gould Hall 10 minutes later.

b. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.

c. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.

d. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.

e. Dave leaves Padelford heading for Gould Hall at the same instant Angela leaves Gould Hall heading for Padelford Hall. Both walk at a constant speed, but Angela walks twice as fast as Dave. Indicate a plot of "distance from Padelford" vs. "time" for the both Angela and Dave.

f. Suppose you want to sketch the graph of a new function s = g(t) that keeps track of Dave's distance s from Gould Hall at time t. How would your graphs change in (a)-(e

Section 1.2 Domain and Range

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between circumference and height of a tree as it grows.

Circumference, *c*	1.7	2.5	5.5	8.2	13.7
Height, *h*	24.5	31	45.2	54.6	92.1

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function.

Domain and Range

Domain: The set of possible input values to a function

Range: The set of possible output values of a function

Example 1

Using the tree table above, determine a reasonable domain and range.

Solution:
We could combine the data provided with our own experiences and reason to approximate the domain and range of the function $h = f(c)$.For the domain, possible values for the input circumference *c*, it doesn't make sense to have negative values, so $c > 0$. We could make an educated guess at a maximum reasonable value, or look up that the maximum circumference measured is about 119 feet[2]. With this information, we would say a reasonable domain is $0 < c \leq 119$ feet.

Similarly for the range, it doesn't make sense to have negative heights, and the maximum height of a tree could be looked up to be 379 feet, so a reasonable range is $0 < h \leq 379$ feet.

[2] http://en.wikipedia.org/wiki/Tree, retrieved July 19, 2010

Example 2

When sending a letter through the United States Postal Service, the price depends upon the weight of the letter[3], as shown in the table below. Determine the domain and range.

Letters	
Weight not Over	Price
1 ounce	\$0.58
2 ounces	\$0.78
3 ounces	\$0.98
3.5 ounces	\$1.18

Solution:
Suppose we notate Weight by *w* and Price by *p*, and set up a function named *P*, where Price, *p* is a function of Weight, *w*. $p = P(w)$.

Since acceptable weights are 3.5 ounces or less, and negative weights don't make sense, the domain would be $0 < w \leq 3.5$. Technically 0 could be included in the domain, but logically it would mean we are mailing nothing, so it doesn't hurt to leave it out.

Since possible prices are from a limited set of values, we can only define the range of this function by listing the possible values. The range is p = \$0.58, \$0.78, \$0.98, or \$1.18.

Try it Now

The population of a small town in the year 1960 was 100 people. Since then, the population has grown to 1400 people reported during the 2010 census. Choose descriptive variables for your input and output and use interval notation to write the domain and range.

[3] https://pe.usps.com/text/dmm300/Notice123.htm, retrieved July 7, 2022

Notation

In the previous examples, we used inequalities to describe the domain and range of the functions. This is one way to describe intervals of input and output values, but is not the only way. Let us take a moment to discuss notation for domain and range.

Using inequalities, such as $0 < c \leq 163$, $0 < w \leq 3.5$, and $0 < h \leq 379$ imply that we are interested in all values between the low and high values, including the high values in these examples.

However, occasionally we are interested in a specific list of numbers like the range for the price to send letters, $p = \$0.58, \$0.78, \$0.98, or\ \$1.18$. These numbers represent a set of specific values: $\{0.58, 0.78, 0.98, 1.18\}$.

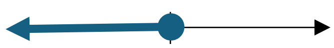
Representing values as a set, or giving instructions on how a set is built, leads us to another type of notation to describe the domain and range.

Suppose we want to describe the values for a variable *x* that are 10 or greater, but less than 30. In inequalities, we would write $10 \leq x < 30$.

When describing domains and ranges, we sometimes extend this into **set-builder notation**, which would look like this: $\{x|10 \leq x < 30\}$. The curly brackets {} are read as "the set of", and the vertical bar | is read as "such that", so altogether we would read $\{x|10 \leq x < 30\}$ as "the set of *x*-values such that 10 is less than or equal to *x* and *x* is less than 30."

When describing ranges in set-builder notation, we could similarly write something like , or if the output had its own variable, we could use it. So for our tree height example above, we could write for the range $\{h|0 < h \leq 379\}$. In set-builder notation, if a domain or range is not limited, we could write $\{t|t \text{ is a real number}\}$, or $\{t|t \in \mathbb{R}\}$, read as "the set of *t*-values such that *t* is an element of the set of real numbers.

A more compact alternative to set-builder notation is **interval notation**, in which intervals of values are referred to by the starting and ending values. Curved parentheses are used for "strictly less than," and square brackets are used for "less than or equal to." Since infinity is not a number, we can't include it in the interval, so we always use curved parentheses with ∞ and -∞. The table below will help you see how inequalities correspond to set-builder notation and interval notation:

Inequality	**Picture**	**Set Builder Notation**	**Interval notation**
$x > b$		$\{x \mid x > b\}$	(b, ∞)
$x \geq b$		$\{x \mid x \geq b\}$	$[b, \infty)$
$x < a$		$\{x \mid x < a\}$	$(-\infty, a)$
$x \leq a$		$\{x \mid x \leq a\}$	$(-\infty, a]$
$b < x < a$		$\{x \mid b < x < a\}$	(b, a)
$b \leq x < a$		$\{x \mid b \leq x < a\}$	$[b, a)$
$b < x \leq a$		$\{x \mid b < x \leq a\}$	$(b, a]$
$b \leq x \leq a$		$\{x \mid b \leq x \leq a\}$	$[b, a]$

To combine two intervals together, using inequalities or set-builder notation we can use the word “or”. In interval notation, we use the union symbol, ∪, to combine two unconnected intervals together.

Example 3

Describe the intervals of values shown on the line graph below using set builder and interval notations.

Solution:

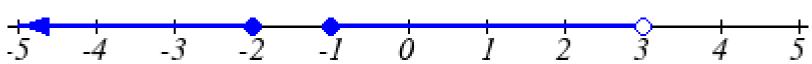
To describe the values, x, that lie in the intervals shown above we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

As an inequality it is: $1 \leq x \leq 3$ or $x > 5$
In set builder notation:$\{x|1 \leq x \leq 3$ or $x > 5\}$
In interval notation: $[1,3] \cup (5, \infty)$

Remember when writing or reading interval notation:
Using a square bracket [means the start value is included in the set.
Using a parenthesis (means the start value is not included in the set.

Try it Now

Given the following interval, write its meaning in words, set builder notation, and interval notation.



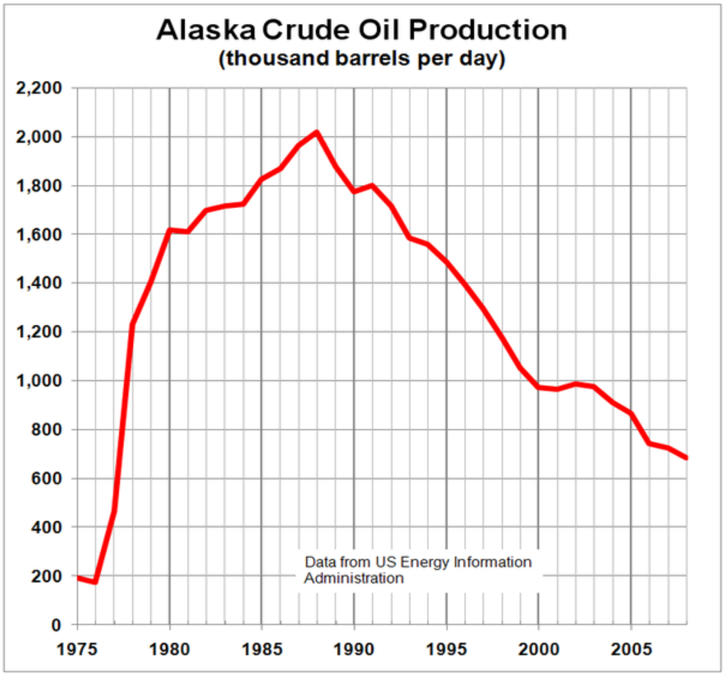
Domain and Range from Graphs

We can also talk about domain and range based on graphs. Since domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the graph. Remember that input values are almost always shown along the horizontal axis of the graph. Likewise, since range is the set of possible output values, the range of a graph we can see from the possible values along the vertical axis of the graph.

Be careful – if the graph continues beyond the window on which we can see the graph, the domain and range might be larger than the values we can see.

Example 4

Determine the domain and range of the graph below.



Solution:
In the graph above[4], the input quantity along the horizontal axis appears to be "year", which we could notate with the variable *y*. The output is "thousands of barrels of oil per day", which we might notate with the variable *b*, for barrels. The graph would likely continue to the left and right beyond what is shown, but based on the portion of the graph that is shown to us, we can determine the domain is $1975 \le y \le 2008$, and the range is approximately $180 \le b \le 2010$.

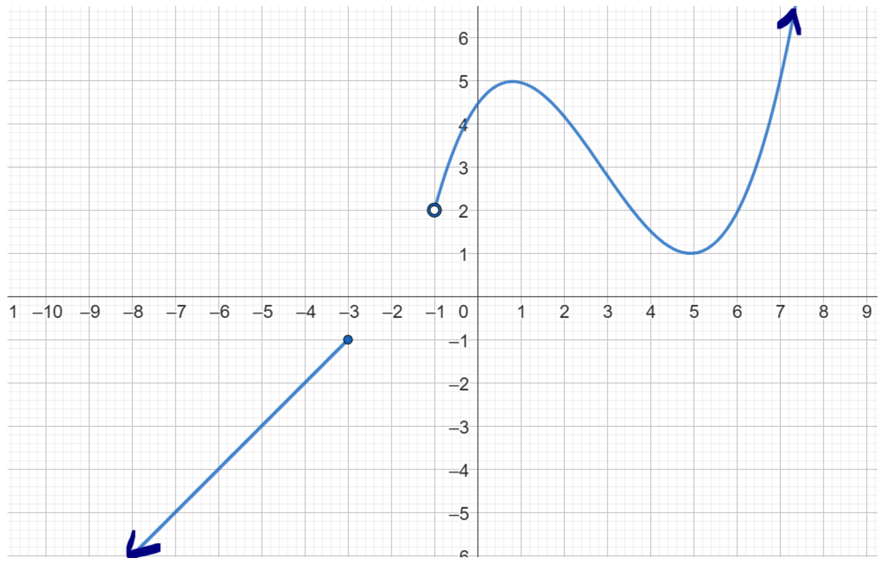
In interval notation, the domain would be [1975, 2008] and the range would be about [180, 2010]. For the range, we have to approximate the smallest and largest outputs since they don't fall exactly on the grid lines.

Remember that, as in the previous example, *x* and *y* are not always the input and output variables. Using descriptive variables is an important tool to remembering the context of the problem.

[4] http://commons.wikimedia.org/wiki/File:Alaska_Crude_Oil_Production.PNG, CC-BY-SA, July 19, 2010

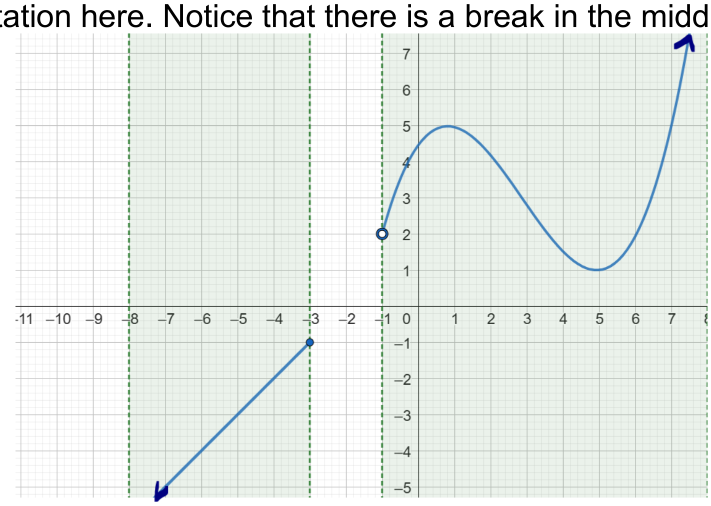
Example 5

Determine the domain and range of the graph below.



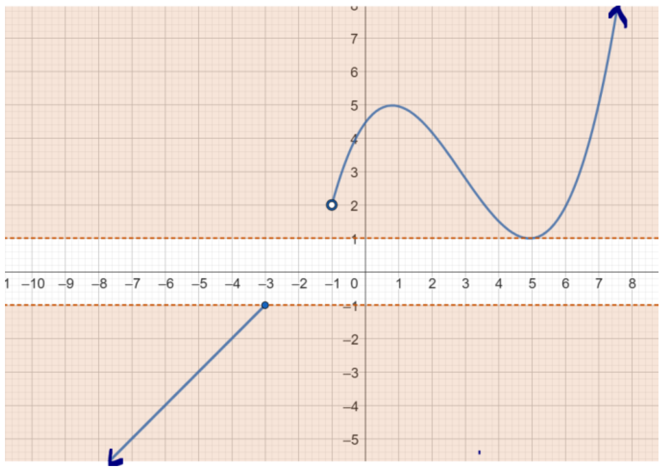
Solution:

For the domain, we want to list all the possible x-values from left to right. We'll use interval notation here. Notice that there is a break in the middle of the graph:



So the function is using x-values from the far left up to -3 (with a closed dot), and from -1 (with an open dot) to the far right. That makes the domain $(-\infty, -3] \cup (-1, \infty)$.

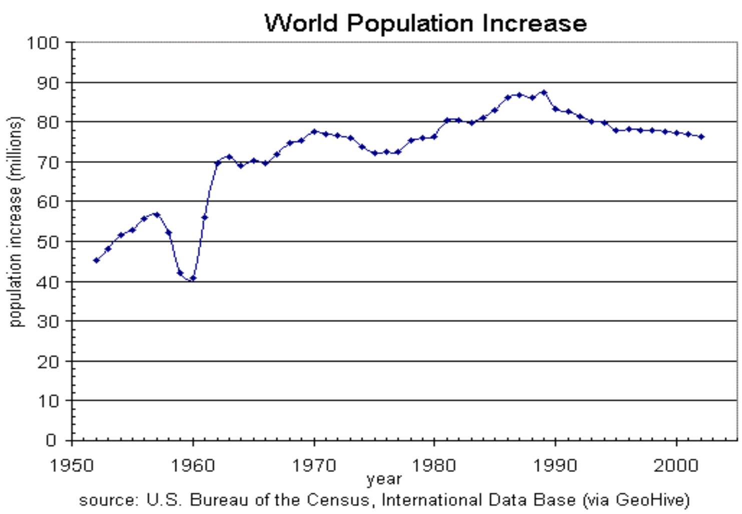
For the range, we want to list all possible y-values from bottom to top. Again, there's a break in the graph:



The function uses y-values from the bottom up to -1 (with a closed dot), then from 1 (with an implied closed dot) to the top. That makes the range $(-\infty, -1] \cup [1, \infty)$.

Try it Now

Given the graph below write the domain and range in interval notation



Piecewise Functions

From your previous studies, you may be familiar with the absolute value function $f(x) = |x|$. Students tend to this of this function as the one that makes things positive.

With a domain of all real numbers and a range of values greater than or equal to 0, the absolute value can be defined as the magnitude or modulus of a number, a real number value regardless of sign, the size of the number, or the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value the output is unchanged (stays positive)
$f(x) = x$ if $x \geq 0$

If we input a negative value the sign must change from negative to positive.
$f(x) = -x$ if $x < 0$, since multiplying a negative value by -1 makes it positive.

Since this requires two different processes or pieces, the absolute value function is often called the most basic piecewise defined function.

Piecewise Function

A **piecewise function** is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x) = \begin{cases} \text{formula 1} & \text{if} \quad \text{domain to use formula 1} \\ \text{formula 2} & \text{if} \quad \text{domain to use formula 2} \\ \text{formula 3} & \text{if} \quad \text{domain to use formula 3} \end{cases}$$

Example 6

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people, or a fixed \$50 fee for 10 or more people in the group. Set up a function relating the number of people, *n*, to the cost, *C*.

Solution:
To set up this function, two different formulas would be needed. *C* = 5*n* would work for *n* values under 10, and *C* = 50 would work for values of *n* ten or greater. Notating this:

$$C(n) = \begin{cases} 5n, & \text{if } 0 < n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Example 7

A prepaid cell phone company uses the function below to determine the cost, *C*, in dollars for *g* gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & if \quad 0 < g < 2 \\ 25 + 10(g-2) & if \quad g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.

Solution:
To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which piece of domain our input falls in. Since 1.5 is less than 2, we use the first formula, giving $C(1.5) = \$25$.

The find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we'll use the second formula. $C(4) = 25 + 10(4 - 2) = \$45$.

Example 8

Sketch a graph of the function

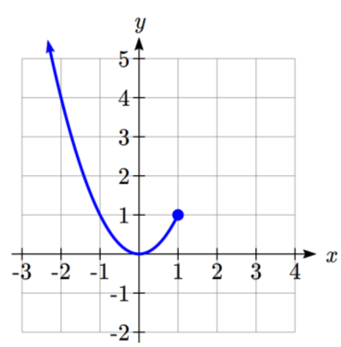
$$f(x) = \begin{cases} x^2 & if \quad x \le 1 \\ 3 & if \quad 1 < x \le 2 \\ 6 - x & if \quad x > 2 \end{cases}$$

Solution:
In the previous section, to sketch a graph of a function, we chose some x-values to substitute into the formula to figure out y-values, and therefore determine points. We can take the same approach here, but we'll need to make sure we use the right piece of the function for our chosen x-values.

.
If we start with x-values of -2, -1, 0, and 1, we'll use the first piece of the function, since that is defined for a domain of $x \le 1$:

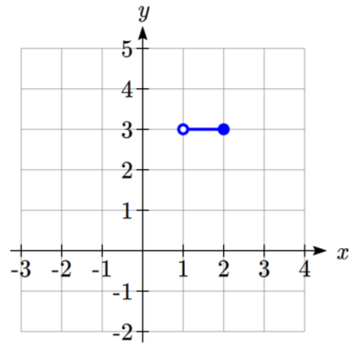
x-value	$f(x)$	Point
-2	$f(-2) = (-2)^2 = 4$	$(-2,4)$
-1	$f(-1) = (-1)^2 = 1$	$(-1,1)$
0	$f(0) = (0)^2 = 0$	$(0,0)$
1	$f(1) = (1)^2 = 1$	$(1,1)$

These points will make the following graph. Note the solid dot at (1,1), since the domain of this piece includes $x = 1$.



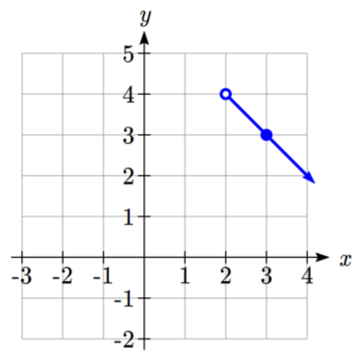
For the next part, we'll choose x-values on the domain $1 < x \le 2$ and use the corresponding formula, $f(x) = 3$. We'll use $x = 1$ and $x = 2$, although the point we graph for $x = 1$ will be represented with an open circle, since x is strictly greater than 1 on this domain.

x-value	$f(x)$	Point
1	$f(1) = 3$	$(1,3)$
2	$f(2) = 3$	$(2,3)$

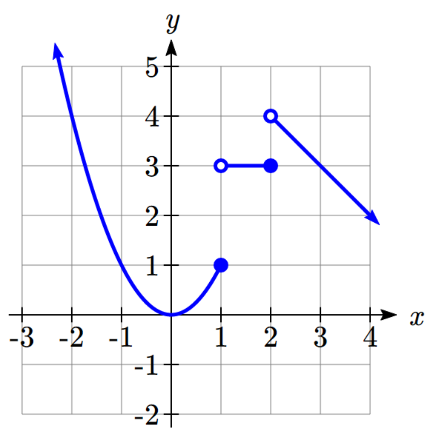


For the third function, you may recognize this as a linear equation from your previous coursework. If you remember how to graph a line using slope and intercept, you can do that. Otherwise, we could calculate a couple values, plot points, and connect them with a line.

At *x* = 2, *f*(2) = 6 – 2 = 4. We place an open circle here.
At *x* = 3, *f*(3) = 6 – 3 = 3. Connect these points with a line.



Now that we have each piece individually, we combine them onto the same graph:



Try it Now

4. At Pierce College during the 2009-2010 school year tuition rates for in-state residents were \$89.50 per credit for the first 10 credits, \$33 per credit for credits 11-18, and for over 18 credits the rate is \$73 per credit[5]. Write a piecewise defined function for the total tuition, *T*, at Pierce College during 2009-2010 as a function of the number of credits taken, *c*. Be sure to consider a reasonable domain and range.

Important Topics of this Section
Definition of domain
Definition of range
Inequalities
Interval notation
Set builder notation
Domain and Range from graphs
Piecewise defined functions

[5] https://www.pierce.ctc.edu/dist/tuition/ref/files/0910_tuition_rate.pdf, retrieved August 6, 2010

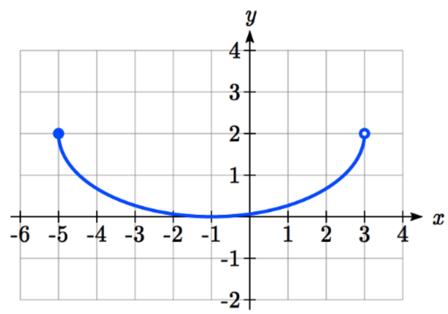
Section 1.2 Exercises

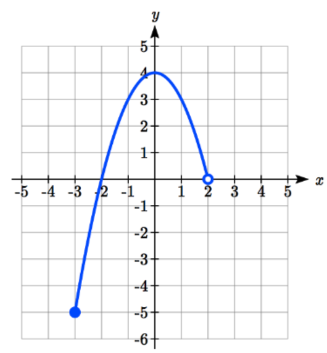
Conceptual Questions

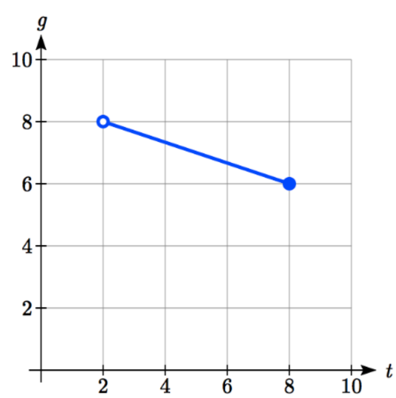
1. What are the domain and range of a function?
2. How can we find a domain from a table? From a graph?
3. How can we find a range from a table? From a graph?
4. How does interval notation work?
5. How do we read a piecewise function?
6. How can we use the formula for a piecewise function to create a graph?

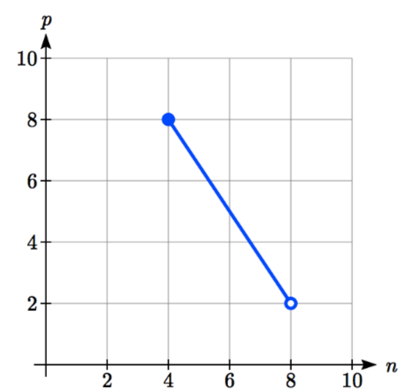
Practice Problems

Write the domain and range of the function as an inequality and using interval notation.

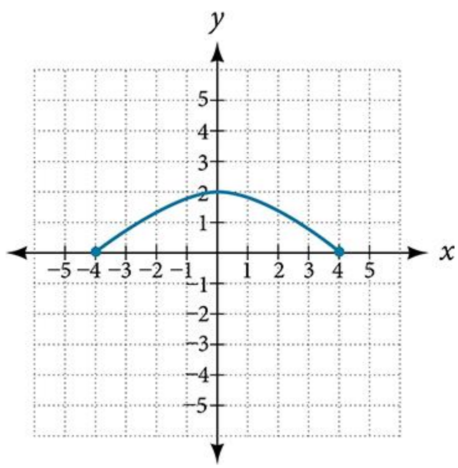
1. 

2. 

3. 

4. 

Write the domain and range of the function as an inequality and using interval notation.



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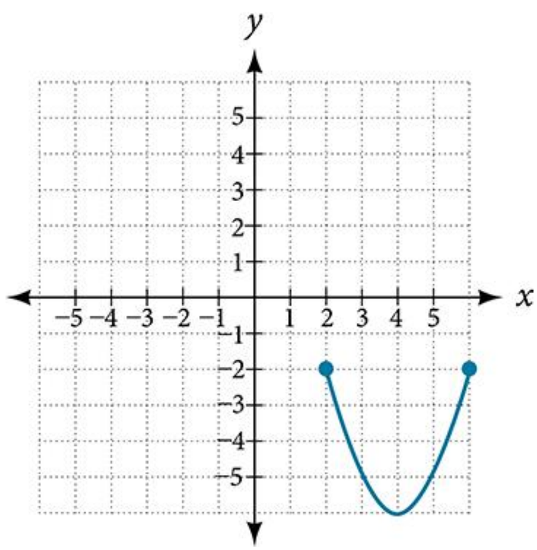

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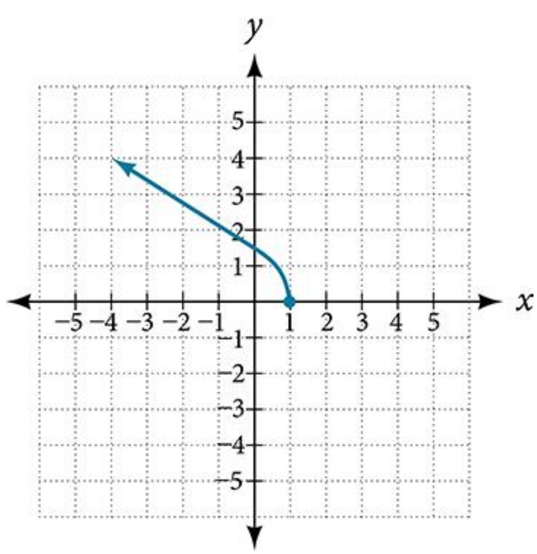

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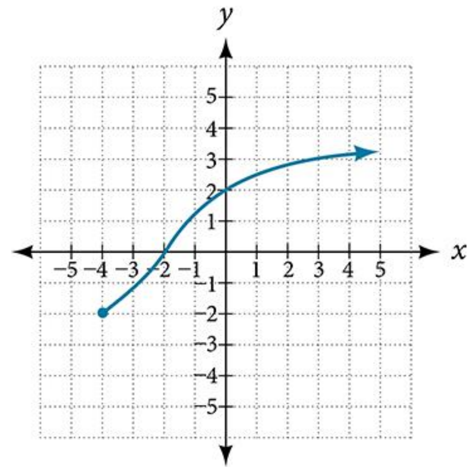

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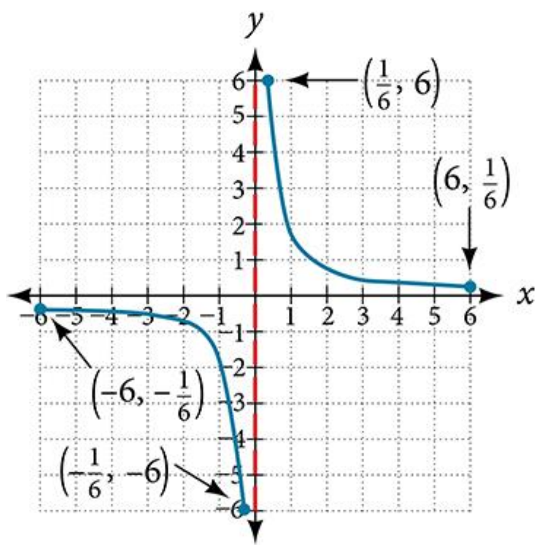

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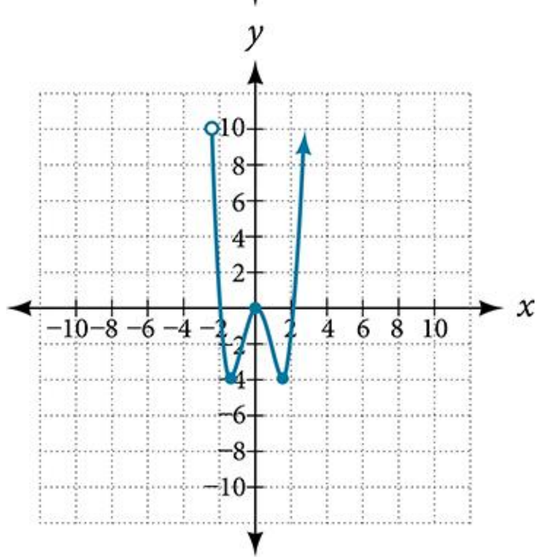
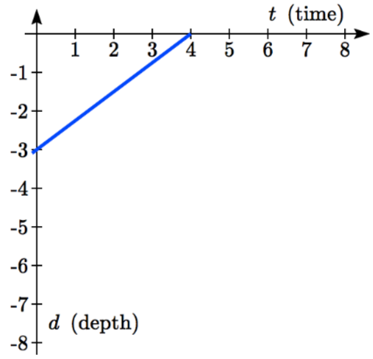
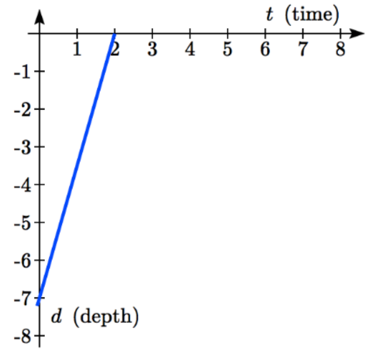


10.

Suppose that you are holding your toy submarine under the water. You release it and it begins to ascend. The graph models the depth of the submarine as a function of time, stopping once the sub surfaces. What is the domain and range of the function in the graph?

11. 

12. 

Given each function, evaluate: $f(-1)$, $f(0)$, $f(2)$, $f(4)$

13. $f(x) = \begin{cases} 7x+3 & if \quad x<0 \\ 7x+6 & if \quad x \ge 0 \end{cases}$

14. $f(x) = \begin{cases} 4x-9 & if \quad x<0 \\ 4x-18 & if \quad x \ge 0 \end{cases}$

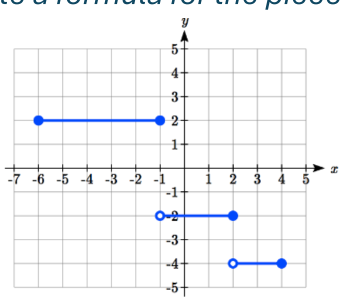
15. $f(x) = \begin{cases} x^2-2 & if \quad x<2 \\ 4+|x-5| & if \quad x \ge 2 \end{cases}$

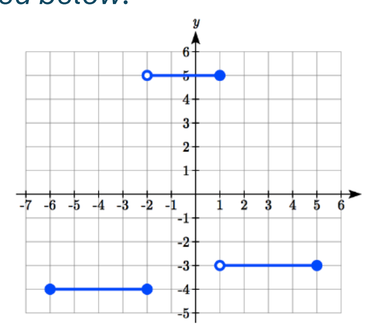
16. $f(x) = \begin{cases} 4-x^3 & if \quad x<1 \\ \sqrt{x+1} & if \quad x \ge 1 \end{cases}$

17. $f(x) = \begin{cases} 5x & if \quad x<0 \\ 3 & if \quad 0 \le x \le 3 \\ x^2 & if \quad x>3 \end{cases}$

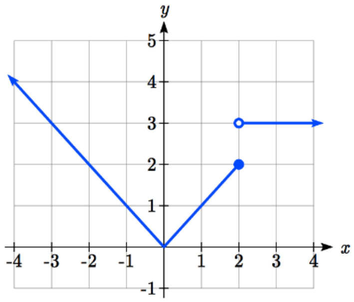
18. $f(x) = \begin{cases} x^3+1 & if \quad x<0 \\ 4 & if \quad 0 \le x \le 3 \\ 3x+1 & if \quad x>3 \end{cases}$

Write a formula for the piecewise function graphed below.

19. 

20. 

Write a formula for the piecewise function graphed below.

21. 

Sketch a graph of each piecewise function

22. $f(x) = \begin{cases} |x| & if \quad x < 2 \\ 5 & if \quad x \geq 2 \end{cases}$

23. $f(x) = \begin{cases} 4 & if \quad x < 0 \\ \sqrt{x} & if \quad x \geq 0 \end{cases}$

24. $f(x) = \begin{cases} x^2 & if \quad x < 0 \\ x + 2 & if \quad x \geq 0 \end{cases}$

25. $f(x) = \begin{cases} x + 1 & if \quad x < 1 \\ x^3 & if \quad x \geq 1 \end{cases}$

26. $f(x) =$
$\begin{cases} 3 & if \quad x \leq -2 \\ -x + 1 & if \quad -2 < x \leq 1 \\ 3 & if \quad x > 1 \end{cases}$

27. $f(x) = \begin{cases} -3 & if \quad x \leq -2 \\ x - 1 & if \quad -2 < x \leq 2 \\ 0 & if \quad x > 2 \end{cases}$

Section 1.3: Features of Functions

Suppose we have a function, $h(x) = -9.8x^2 + 110x + 20$, which models the height in meters, h, of a pumpkin being trebucheted at time x in seconds. If we know this function, can we figure out other interesting things about the situation, like the pumpkin's initial height when it is released by the trebuchet? Or the time at which the pumpkin will hit the ground?

A punkin chunkin event at Ramstein Air Force Base.
Public Domain. https://www.dvidshub.net/about/copyright

Intercepts

One of the ways we can go about answering some of these questions is by looking into the intercepts of the function. We'll want to find where the graph of our function intersects the axes; these places will often have real-life interpretations.

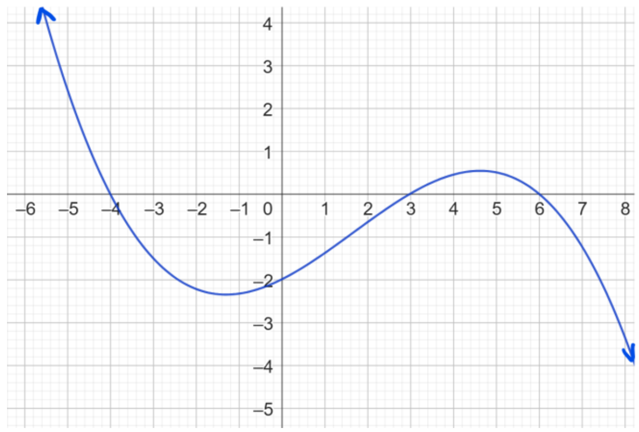
Let's start by looking at a general function and work our way up to our punkin chunkin problem.

X-Intercepts
The x-intercepts, or horizontal intercepts, of a function occur when the graph of the function touches the x-axis.

Y-Intercepts
The y-intercept, or vertical intercept, of a function occurs when the graph of the function touches the y-axis.

Example 1

Given the graph of $f(x)$, find the x- and y-intercepts. Give your answers as ordered pairs.



Solution:

To look for x-intercepts, we'll list the places where our graph intersects the horizontal, or x-axis. In this case, that happens when $x = -4$, 3, and 6. We want to list the ordered pairs, or (x, y) points for each intercept. Notice that the height at each x-intercept is 0, so our list of x-intercepts is $(-4,0)$, $(3,0)$, and $(6,0)$.

There is only one place where our graph intersects the vertical, or y-axis, and that is when $y = -2$. The x-value here is 0, so our y-intercept is $(0, -2)$.

That graphical example illuminates an important point: we see x-intercepts when $y = 0$, and we see y-intercepts when $x = 0$. That gives us a way of finding intercepts when we don't have a graph in front of us!

Detecting Intercepts
To find an $\boldsymbol{x}$-intercept of $\boldsymbol{f(x)}$, solve the equation $f(x) = 0$.
To find a $\boldsymbol{y}$-intercept of $\boldsymbol{f(x)}$, evaluate $f(0)$.

Example 2

Given the following table for $f(x)$, find the x- and y-intercepts.

x	$f(x)$
-6	0
-3	1
0	4
3	0
6	-4

Solution:
To find the x-intercepts, we'll want to solve for when $f(x) = 0$. We see that $f(x)$=0 when $x = -6$ and when $x = 3$, so our x-intercepts are at $(-6,0)$ and $(3,0)$.

To find the y-intercept, we'll evaluate $f(0)$. On the table, we see that when we have an input of $x = 0$, the output is 4, so $f(0) = 4$ and our y-intercept is $(0,4)$.

Example 3

Given that $f(x) = x(x+2)(x-4)$, find the x- and y-intercepts.

Solution:
Once again, to find the x-intercepts, we'll solve for when $f(x) = 0$. That creates the equation $x(x+2)(x-4) = 0$. We can use the Zero Product Property to solve for $x = 0, -2,$ and 4. So, the x-intercepts are $(0,0)$, $(-2,0)$, $(4,0)$.

For the y-intercepts, we'll evaluate $f(0)$.

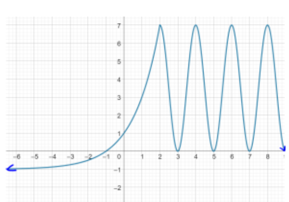
$$\begin{aligned} f(0) &= 0(0+2)(0-4) \\ &= 0 \cdot 2 \cdot -4 \\ &= 0. \end{aligned}$$

That makes the y-intercept $(0,0)$.

Try it Now

Find the x and y-intercepts for each function.

a.



b.

x	$f(x)$
-8	0
-4	1
0	2
4	3
8	4

c. $f(x) = 6x - 12$

Now we return to our flying pumpkin. If the function $h(x) = -9.8x^2 + 110x + 20$ models the height h in meters of the pumpkin at times x seconds, then we can figure out some cool information about the situation with our intercepts!

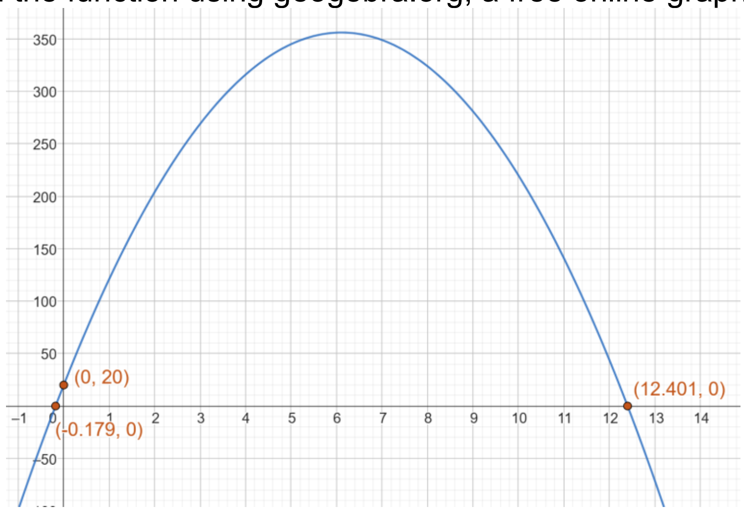
Example 3

A pumpkin is launched from a trebuchet. The function $h(x) = -9.8x^2 + 110x + 20$ represents the pumpkin's height, h, in meters at time x, in seconds.

a. What will the x-intercept tell us about this situation?
b. What will the y-intercept tell us about this situation?
c. Graph the function using technology. Find and interpret the intercepts.

Solution:

a. To find the x-intercept, we'll solve the equation $h(x) = 0$. So, that will tell us when the pumpkin's height (h) is 0. Finding the x-intercept will tell us the time at which the pumpkin hits the ground.
b. To find the y-intercept, we'll evaluate $h(0)$. That means we'll find the height, h, when $x = 0$. This value will give us the pumpkin's starting height, when it is released from the trebuchet.
c. We'll graph the function using geogebra.org, a free online graphing calculator.



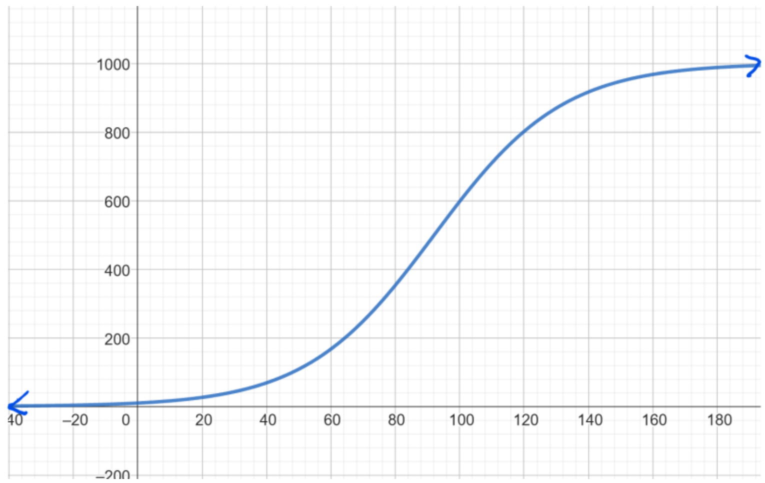
After adjusting the window and using the intercept tool, we see there are *two* x-intercepts, (-0.179, 0) and (12.401, 0). Only the second intercept makes sense in this situation, though, because our model doesn't account for negative time. So, we see that the pumpkin hits the ground at 12.401 seconds.

The y-intercept is (0, 20), so the pumpkin's starting height was 20 meters.

We can also see a local maximum at the graph at around 6 seconds and a little over 350 meters. That's the maximum height the pumpkin will reach!

End Behavior and Asymptotes

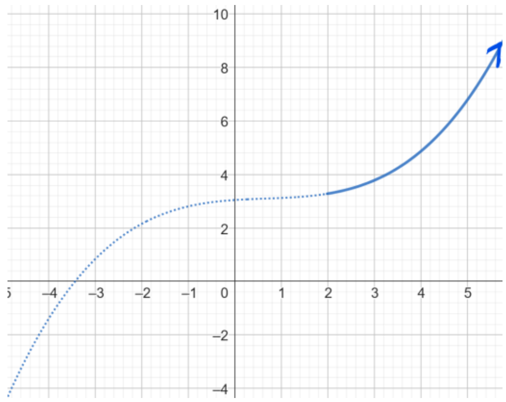
In many situations, we want to describe long term behavior, or end behavior, of a quantity. Suppose the function f below models the spread of a rumor online in thousands of people, $f(x)$, at hour x.



At first (the y-intercept), hardly anyone has heard the rumor. Then, there is sharp uptake in the 60 to 120 hour range. Then, the rumor spread levels out again, getting near a level of 1,000,000 people as time goes on. As we look out to that extreme right side of the graph, we're examining the graph's **end behavior**.

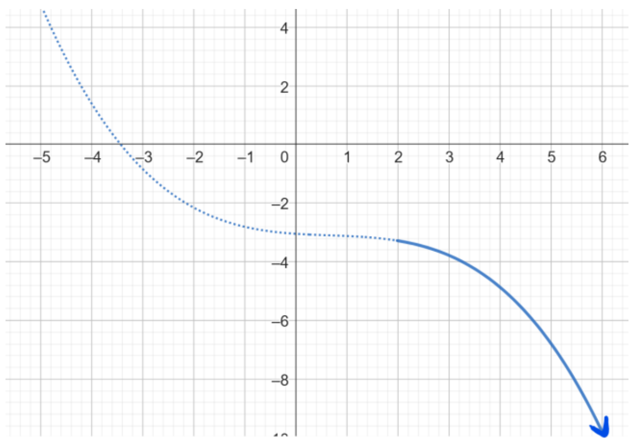
End Behavior
The **end behavior**, or **long-run behavior**, of a function describes what happens to the function value (height) on the extreme right or extreme left side of the graph.

Right side end behavior
To describe right side end behavior, we examine what happens to the height on the extreme right side of the graph. To indicate that we're looking at the right side of the graph, we say that $x \to \infty$, since we are looking at extremely large values of x. To describe the height, we may say one of the following:



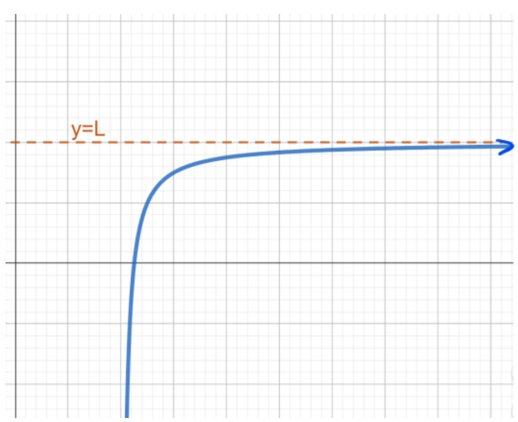
As $x \to \infty, f(x) \to \infty$

This indicates that on the far right (as $x \to \infty$), the height of the function grows without bound ($f(x) \to \infty$).



As $x \to \infty, f(x) \to -\infty$

This indicates that on the far right (as $x \to \infty$), the height of the function grows very large **and negative** without bound ($f(x) \to -\infty$).



As $x \to \infty, f(x) \to L$

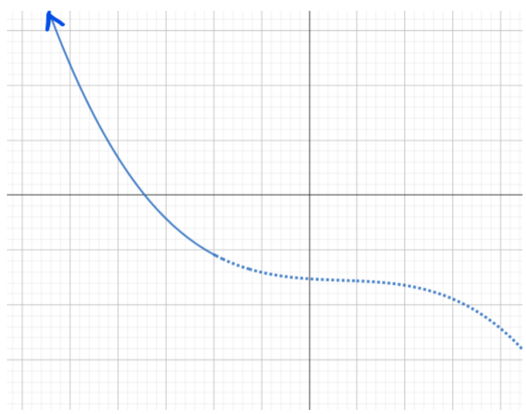
This indicates that on the far right (as $x \to \infty$), the height of the function grows closer and closer to a height of L ($f(x) \to L$).

Left side end behavior

To describe left side end behavior, we examine what happens to the height on the extreme left side of the graph.

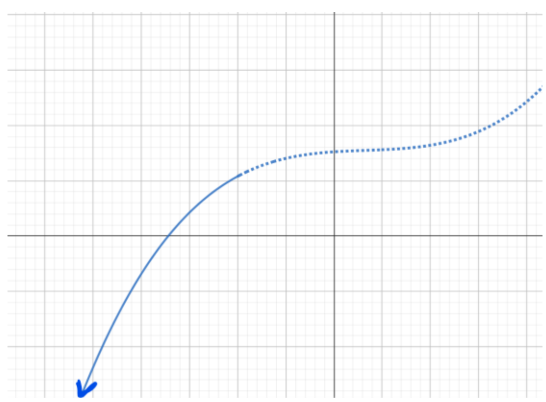
To indicate that we're looking at the right side of the graph, we say that $x \to -\infty$, since we are looking at extremely large **and negative** values of x.

To describe the height, we may say one of the following:



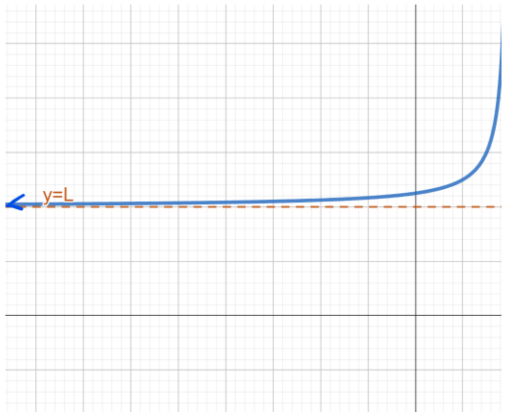
As $x \to -\infty, f(x) \to \infty$

This indicates that on the far right (as $x \to -\infty$), the height of the function grows without bound ($f(x) \to \infty$).



As $x \to -\infty, f(x) \to -\infty$

This indicates that on the far right (as $x \to -\infty$), the height of the function grows very large **and negative** without bound ($f(x) \to -\infty$).



As $x \to -\infty, f(x) \to L$

This indicates that on the far right (as $x \to -\infty$), the height of the function grows closer and closer to a height of L ($f(x) \to L$).

We have a special name for the times when the end behavior approaches a specific height, rather than growing without bound. We call these heights the **horizontal asymptotes** of a function.

Horizontal Asymptote

The horizontal line $y = L$ is a horizontal asymptote of $f(x)$ if:

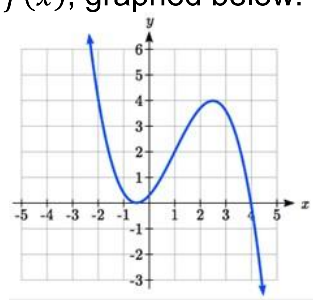
As $x \to \infty, f(x) \to L$

or

As $x \to -\infty, f(x) \to L$

Example 4

Describe the end behavior of $f(x)$, graphed below:



Solution:

Since on the far right side of the graph, the arrow points down, we see that the height values are growing very large and negative. Therefore

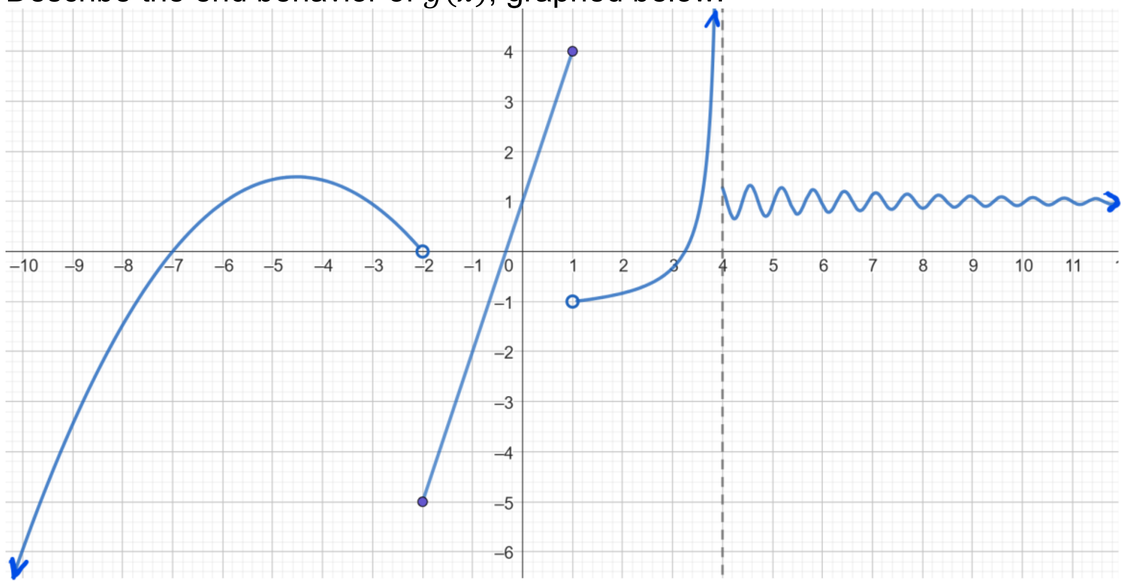
$$\text{As } x \to \infty, f(x) \to -\infty.$$

On the far left side of the graph, the arrow points up. That tells us that the heights are growing very large, so

$$\text{As } x \to \infty, f(x) \to \infty$$

Example 5

Describe the end behavior of $g(x)$, graphed below:



Solution:

There's a lot going on in this function, but most of it is a distraction to the task at hand. We only need to look at the height of the function on the far right and the far left sides of the graph.

On the far right, the function oscillates, but the oscillations get smaller and smaller. They are getting closer and closer to a height of 1, so

$$\text{As } x \to \infty, g(x) \to 1.$$

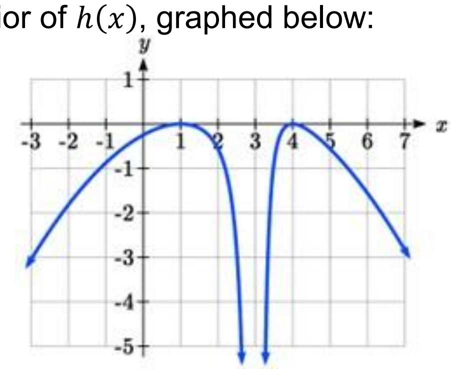
So, there is a horizontal asymptote at $y = 1$.

On the far left, the function goes downward. The heights grow very large and negative, so

$$\text{As } x \to -\infty, g(x) \to -\infty.$$

Try it Now:

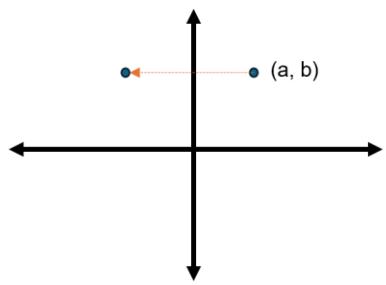
Describe the end behavior of $h(x)$, graphed below:



Symmetry

Another feature of functions that often interests mathematicians is symmetry. Symmetry, roughly speaking, means that a function is reflected across a line or point, so that it is the same on opposite sides of that reflection. At this time, we will mainly focus on y-axis and origin symmetry.

Let's examine a single point and how it reflects first. We'll look at point (a, b) and reflect it across the y-axis.



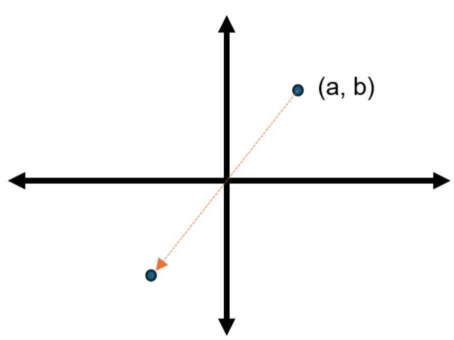
We reflect that point across the y-axis. It's now at the same height, but its y-value is the opposite. That makes the reflected point $(-a, b)$.

If we turn to function notation, for the original point we see $f(a) = b$, and for the reflected point $f(-a) = b$. In other words, $f(a) = f(-a)$

If a function is symmetric about the y-axis, then every point on the function will be reflected in this way.

Y-axis symmetry
A function is symmetric about the y-axis if $f(x) = f(-x)$ for every x in the domain. Functions with y-axis symmetry are said to be **even** functions.

Another important type of symmetry is origin symmetry. We'll begin again with a single point:



The reflected point has an opposite x and an opposite y-coordinate. That means (a, b) becomes $(-a, -b)$ when it is reflected across the origin.

In function notation, the original point $f(a) = b$, becomes $f(-a) = -b$, so in a function with this type of symmetry, $f(-a) = -f(a)$.

Origin Symmetry

A function is symmetric about the origin if $f(-x) = -f(x)$ for every x in the domain.

Functions with origin symmetry are said to be **odd.**

Example 6

Determine whether each function is even, odd, or neither.

a. $f(x) = x^2 + 1$
b. $f(x) = x^3 + x$
c. $f(x) = x^3 + 1$

Solution:
We'll want to check each function for symmetry. To do so, we'll compare $f(x)$ to $f(-x)$.

a. We know that $f(x) = x^2 + 1$. Now let's examine $f(-x)$:

$$\begin{aligned} f(-x) &= (-x)^2 + 1 \\ &= -x \cdot -x + 1 \\ &= x^2 + 1 \\ &= f(x) \end{aligned}$$

Our calculations show that $f(-x) = f(x)$, so the function has y-axis symmetry. It is an even function.

b. Here, we start with $f(x) = x^3 + x$. We'll first check $f(-x)$:

$$\begin{aligned} f(-x) &= (-x)^3 + (-x) \\ &= -x \cdot -x \cdot -x + (-x) \\ &= -x^3 + (-x) \\ &= -(x^3 + x) \\ &= -f(x) \end{aligned}$$

That means $f(-x) = -f(x)$, so the function has origin. This function is odd.

c. Finally, we investigate $f(x) = x^3 + 1$. Again, we want to know what $f(-x)$ looks like:

$$\begin{aligned} f(-x) &= (-x)^3 + 1 \\ &= -x^3 + 1 \end{aligned}$$

Well that's…nothing. It's not the same as $f(x)$, and it's not the opposite of $f(x)$. That means this function is neither even nor odd.

Try it Now

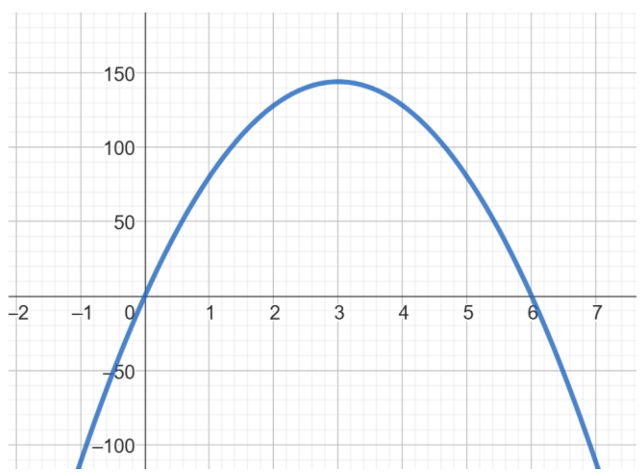
Determine whether each function is even, odd, or neither:

a. $f(x) = x^5 - x^3$
b. $f(x) = x^5 - x^2$
c. $f(x) = x^4 - x^6$

Inequalities

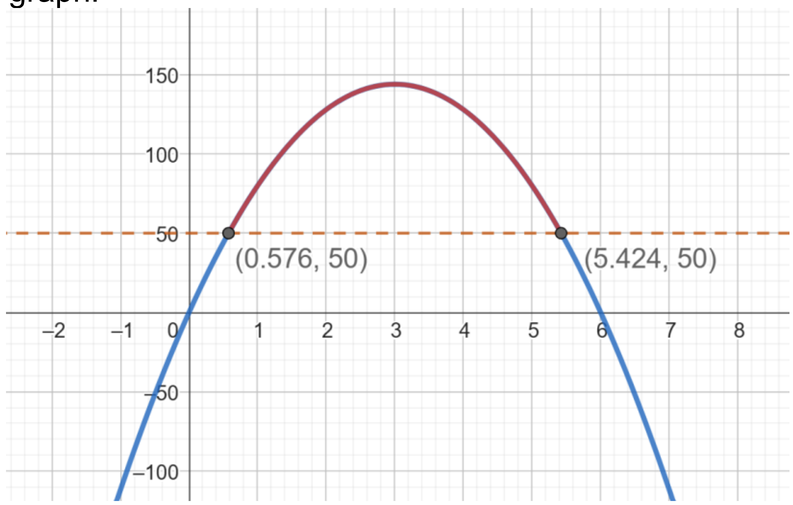
Let's think about another flying object, this time a ball being launched from a device designed to throw balls for overactive dogs. An equation that might describe the height of such a ball is $f(x) = -16x^2 + 96x$, where f is the height in feet and x is the time in seconds. We're interested in figuring out how long the ball is in the air. In other words, we want to know for what x-values $f(x) \geq 0$.

On a graph, we want to find the interval of x-values when the graph is *above* the x-axis:



We see that the heights of the function are greater than or equal to 0 between $x = 0$ and $x = 6$. In other words, the solution to $-16x^2 + 96x \geq 0$ is $[0,6]$.

We could also use the function to determine how long the ball is above 50 feet. For that, we're solving $f(x) \geq 50$, or $-16x^2 + 96x \geq 50$. That would be in this red region of the graph:



So, the ball is above a height of 50 feet from 0.576 to 5.424 seconds from launch, and the solution to the inequality $-16x^2 + 96x \geq 50$ is $[0.576, 5.424]$.

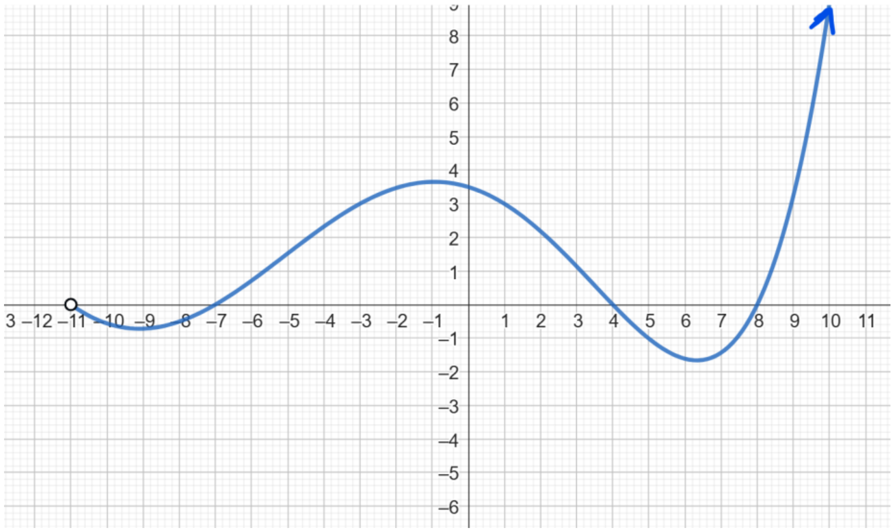
Solution to an inequality

The solution to an inequality in the form $f(x) > a$ or $f(x) \geq a$ is an interval of x-values which describes the region where the function $f(x)$ has a height that is higher than (or equal to) a.

The solution to an inequality in the form $f(x) < b$ or $f(x) \leq b$ is an interval of x-values which describes the region where the function $f(x)$ has a height that is lower than (or equal to) b.

Example 7

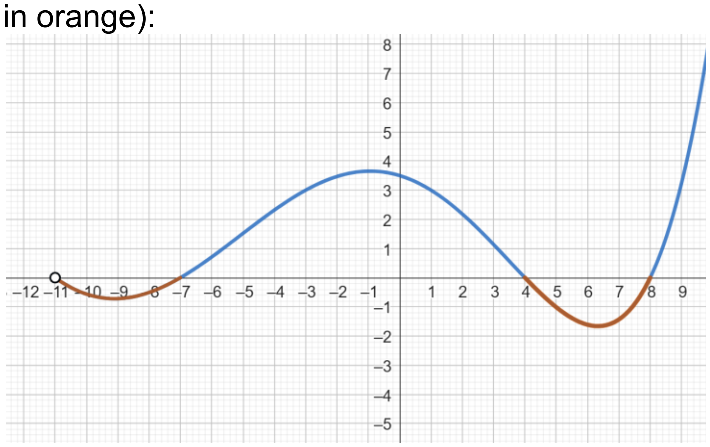
Use the graph of $f(x)$ to solve the following inequalities:



a. $f(x) \leq 0$
b. $f(x) > 3$

Solutions:

a. To solve $f(x) \leq 0,$ we want to find where the function has a height of 0, or a height lower than 0 (that is, negative). That is in the following regions (highlighted in orange):

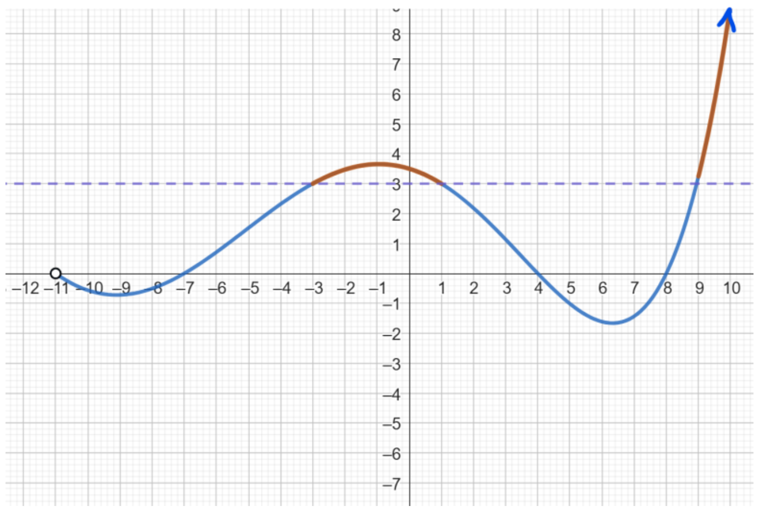


Our solution to the inequality, then, are the x-value intervals which describe this region:

$$(-11, -7] \cup [4,8].$$

Notice that -7, 4, and 8 have brackets, because our inequality states we want $f(x)$ to be less than *or equal to* 0. The value -11 must have parentheses, because the function has an open circle at that x-location.

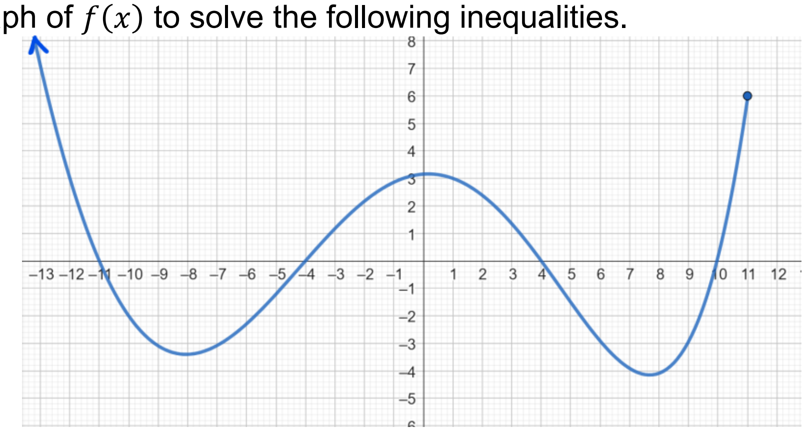
b. To solve $f(x) > 3$, it's useful to plot a line at a height of 3. Then we can more easily spot the regions where the graph of $f(x)$ has heights that are taller than 3.



Since our inequality specifies we want heights strictly greater than (not equal to) 3, our solution is $(-3, 1) \cup (9, \infty)$.

Try it Now

Use the graph of $f(x)$ to solve the following inequalities.



a. $f(x) < -3$
b. $f(x) \geq 0$

Important Topics in this Section
x- and y-intercepts
End behavior
Horizontal Asymptotes
Symmetry
Solving inequalities graphically

Section 1.3 Exercises

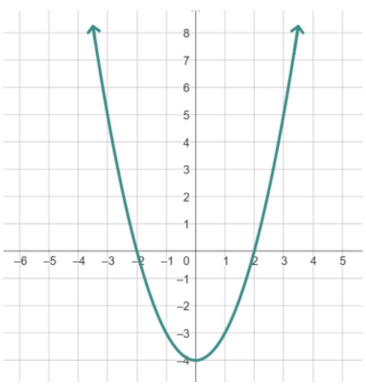
Conceptual Questions

1. What are intercepts?
2. How can we find intercepts on a graph? From a formula?
3. What types of real world situations might require us to find an x-intercept? A y-intercept?
4. What is end behavior? How do we describe it?
5. What is symmetry? How can we find it in a graph? From a formula?
6. How can we solve inequalities from a graph of a function?

Practice Problems

For each function graphed below, describe a) the intercepts, b) the end behavior c) whether the function is even, odd, or neither. Dotted lines will indicate the presence of asymptotes.

1.



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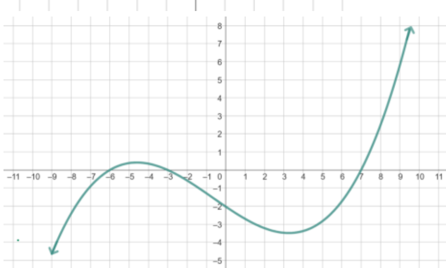
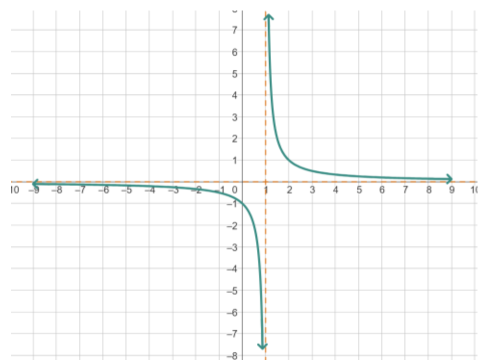
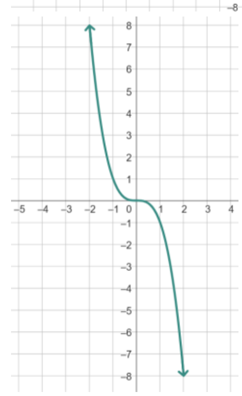
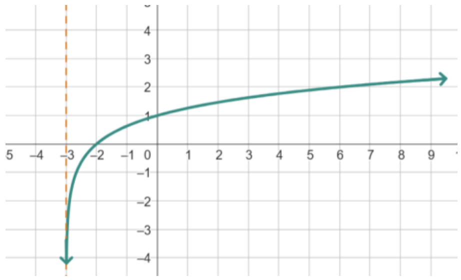


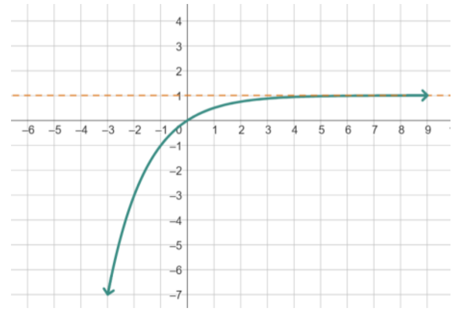
3.



4.



5. 

6. 

Determine whether each function is even, odd, or neither. Justify your answer.

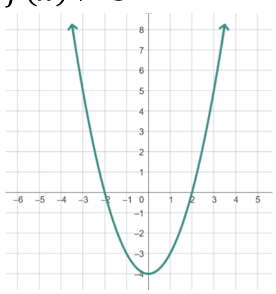
7. $f(x) = 3x^4$
8. $g(x) = \sqrt{x}$
9. $h(x) = \frac{1}{x} + 3x$
10. $f(x) = (x - 2)^2$
11. $g(x) = 2x^4$
12. $h(x) = 2x - x^3$

Given each function, determine the $\boldsymbol{x}$- and $\boldsymbol{y}$- intercepts.

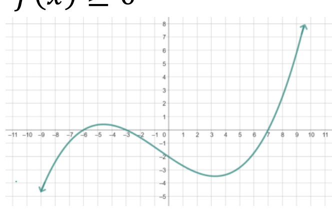
13. $f(x) = -3x + 6$
14. $g(x) = \frac{1}{2}x - 4$
15. $h(x) = 8(x + 3) - 2$
16. $f(x) = 7x^2$
17. $g(x) = (x - 2)^2 + 4$
18. $h(x) = 2x^2 + 3x + 1$

Given each graph of $\boldsymbol{f}(\boldsymbol{x})$, solve the indicated inequality.

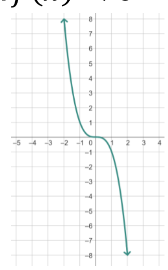
19. $f(x) > 5$



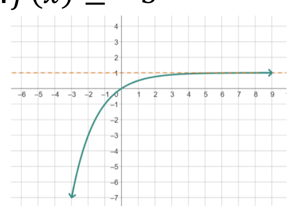
20. $f(x) \geq 0$



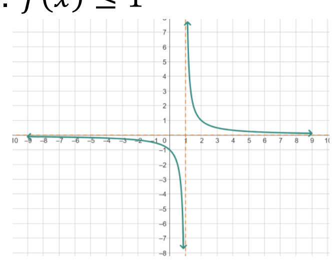
21. $f(x) < 0$



22. $f(x) \leq -3$



23. $f(x) \leq 1$



Section 1.4 Rates of Change and Behavior of Graphs

Since functions represent how an output quantity varies with an input quantity, it is natural to ask about the rate at which the values of the function are changing.

For example, the function *C(t)* below gives the average cost, in dollars, of a gallon of gasoline *t* years after 2000.

t	2	3	4	5	6	7	8	9
C(t)	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

If we were interested in how the gas prices had changed between 2002 and 2009, we could compute that the cost per gallon had increased from $1.47 to $2.14, an increase of $0.67. While this is interesting, it might be more useful to look at how much the price changed *per year*. You are probably noticing that the price didn't change the same amount each year, so we would be finding the **average rate of change** over a specified amount of time.

The gas price increased by $0.67 from 2002 to 2009, over 7 years, for an average of $\frac{\$0.67}{7 years} \approx 0.096$ dollars per year. On average, the price of gas increased by about 9.6 cents each year.

Rate of Change

A **rate of change** describes how the output quantity changes in relation to the input quantity. The units on a rate of change are "output units per input units"

Some other examples of rates of change would be quantities like:
- A population of rats increases by 40 rats per week
- A barista earns $9 per hour (dollars per hour)
- A farmer plants 60,000 onions per acre
- A car can drive 27 miles per gallon
- A population of grey whales decreases by 8 whales per year
- The amount of money in your college account decreases by $4,000 per quarter

Average Rate of Change

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

Average rate of change $= \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

The unit label on an average rate of change is $\frac{\text{output units}}{\text{input units}}$, that is output units per input units.

Example 1

Using the cost-of-gas function from earlier, find the average rate of change between 2007 and 2009

Solution:
From the table, in 2007 the cost of gas was \$2.64. In 2009 the cost was \$2.14.

The input (years) has changed by 2. The output has changed by \$2.14 - \$2.64 = -0.50. The average rate of change is then $\frac{-\$0.50}{2 \text{ years}} = -0.25$ dollars per year

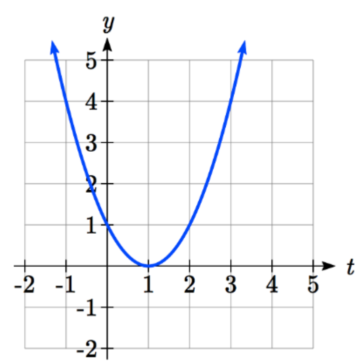
Try it Now

Using the same cost-of-gas function, find the average rate of change between 2003 and 2008

Notice that in the last example the change of output was *negative* since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

Example 2

Given the function $g(t)$ shown here, find the average rate of change on the interval
$[0,3]$.



Solution:
At $t = 0$, the graph shows $g(0) = 1$
At $t = 3$, the graph shows $g(3) = 4$

The output has changed by 3 while the input has changed by 3, giving an average rate of change of:

$$\frac{4-1}{3-0} = \frac{3}{3} = 1$$

Example 3

On a road trip, after picking up your friend who lives 10 miles away, you decide to record your distance from home over time. Find your average speed over the first 6 hours.

t (hours)	0	1	2	3	4	5	6	7
D(t) (miles)	10	55	90	153	214	240	292	300

Here, your average speed is the average rate of change.
You traveled 282 miles in 6 hours, for an average speed of
$\frac{292-10}{6-0} = \frac{282}{6} =$ 47 miles per hour

We can more formally state the average rate of change calculation using function notation.

Average Rate of Change using Function Notation

Given a function $f(x)$, the average rate of change on the interval $[a, b]$is

$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{f(b)-f(a)}{b-a}$

Example 4

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval $[2, 4]$.

We can start by computing the function values at each endpoint of the interval

$$f(2) = 2^2 - \frac{1}{2} = 4 - \frac{1}{2} = \frac{7}{2}$$
$$f(4) = 4^2 - \frac{1}{4} = 16 - \frac{1}{4} = \frac{63}{4}$$

Now computing the average rate of change

Average rate of change = $\frac{f(4)-f(2)}{4-2} = \frac{\frac{63}{4}-\frac{7}{2}}{4-2} = \frac{\frac{49}{4}}{2} = \frac{49}{8}$

Try it Now

Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval $[1, 9]$.

Example 5

The magnetic force *F*, measured in Newtons, between two magnets is related to the distance between the magnets *d*, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the magnets is increased from 2 cm to 6 cm.

Solution:

We are computing the average rate of change of $F(d) = \frac{2}{d^2}$ on the interval$[2,6]$.

Average rate of change = $\frac{F(6)-F(2)}{6-2}$Evaluating the function

$$\frac{F(6)-F(2)}{6-2}$$

$$= \frac{\frac{2}{6^2}-\frac{2}{2^2}}{6-2} \qquad \text{Evaluating}$$

$$= \frac{\frac{2}{36}-\frac{2}{4}}{4} \qquad \text{Simplifying}$$

$$= \frac{\frac{-16}{36}}{4} \qquad \text{Combining the numerator terms}$$

$$= -\frac{1}{9}$$

This tells us the magnetic force decreases, on average, by 1/9 Newtons per centimeter over this interval.

Example 6

Find the average rate of change of $g(t) = t^2 + 3t + 1$on the interval $[0, a]$. Your answer will be an expression involving a.

Solution:

Using the average rate of change formula

$$\frac{g(a)-g(0)}{a-0}$$

$=\dfrac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0}$ Evaluating the function

$=\dfrac{a^2+3a+1-1}{a-0}$ Simplify

$=\dfrac{a^2+3a}{a}$ Simplify

$=\dfrac{a(a+3)}{a}$ Factor

$=a+3$ Cancel common factor

This result tells us the average rate of change between $t = 0$ and any other point $t = a$. For example, on the interval [0, 5], the average rate of change would be 5+3 = 8.

Example 7

Find the average rate of change of $f(x) = 5x - x^2$ on the interval $[a, a+h]$.

Solution:

We'll use the average rate of change formula:

$$\frac{f(a+h)-f(a)}{a+h-a}$$

Let's calculate each piece of the numerator first.

$f(a+h) \quad = 5(a+h)-(a+h)^2$ Substitute $a+h$ in for the x

$= 5a+5h-(a+h)(a+h)$ Simplify

$= 5a+5h-(a^2+ah+ah+h^2)$ Simplify

$= 5a+5h-a^2-2ah-h^2$ Distribute the negative

and

$f(a) \quad = 5a-a^2$ Substitute a in for the x

ao

$f(a+h)-f(a) \quad = 5a+6h-a^2-2ah-h^2-(5a-a^2)$

$= 5a+6h-a^2-2ah-h^2-5a+a^2$ Distribute negative

$= 6h-2ah-h^2$ Simplify

Now we'll finish evaluating the formula:

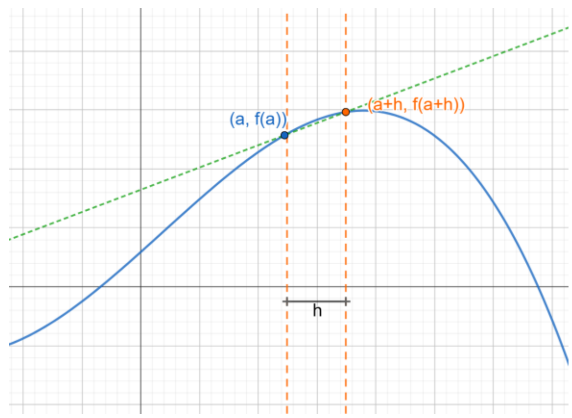
$$\frac{f(a+h)-f(a)}{a+h-a} = \frac{6h-2ah-h^2}{a+h-a}$$

$$= \frac{h(6-2a-h)}{h}$$

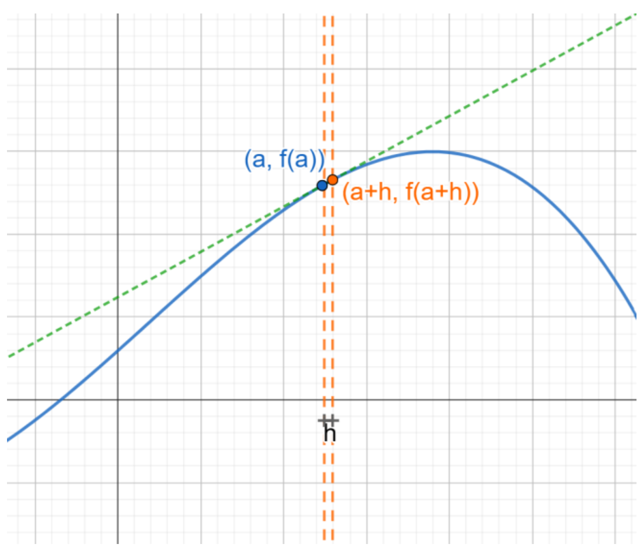
$$= 6-2a-h$$

Now, why would anyone ever want to calculate a rate of change like the one in the last example? One big question in math is “Can we find an *exact* rate of change at a specific moment in time?” This is called an instantaneous rate of change, and it’s going to be a big idea in your calculus class someday. The calculation above can help us to calculate an instantaneous rate of change.

Here's the idea: we want to find an instantaneous rate of change at the x-value of a. We’ll start by finding an average rate of change on a small interval starting at a:



If we calculate $\frac{f(a+h)-f(a)}{a+h-a}$, that’s the slope of the green line, called the **secant line**. If we make h very tiny, then the point through $x = a + h$ will move close to $x = a$:



Once those points are practically on top of each other (as we let h get closer and closer to 0), the slope of the secant line becomes a pretty good estimate of the instantaneous rate of change.

Try it Now

Find the average rate of change of $f(x) = x^3 + 2$ on the interval $[a, a + h]$.

Graphical Behavior of Functions

As part of exploring how functions change, it is interesting to explore the graphical behavior of functions.

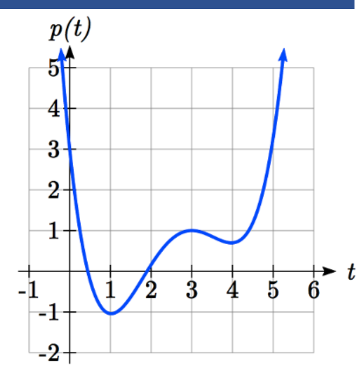
Increasing/Decreasing

A function is **increasing** on an interval if the function values increase as the inputs increase, that is, the function is moving uphill as we travel along it from left to right. More formally, a function is increasing if $f(b) > f(a)$ for any two input values a and b in the interval with $b > a$. The average rate of change of an increasing function is **positive.**

A function is **decreasing** on an interval if the function values decrease as the inputs increase, that is, the function is moving downhill as we travel along it from left to right. More formally, a function is decreasing if $f(b) < f(a)$ for any two input values a and b in the interval with $b > a$. The average rate of change of a decreasing function is **negative.**

Example 8

Given the function *p(t)* graphed here, on what intervals does the function appear to be increasing?



Solution:

The function appears to be increasing from t = 1 to t = 3, and from t = 4 on.

In interval notation, we would say the function appears to be increasing on the interval $(1,3)$ and the interval $(4, \infty)$.

Notice in the last example that we used open intervals (intervals that don't include the endpoints) since the function is neither increasing nor decreasing at t = 1, 3, or 4.

Local Extrema

A point where a function changes from increasing to decreasing is called a **local maximum**. It is the highest point in its neighborhood.

A point where a function changes from decreasing to increasing is called a **local minimum.** It is the lowest point in its neighborhood.

Together, local maxima and minima are called the **local extrema**, or local extreme values, of the function.

Example 9

Using the cost of gasoline function from the beginning of the section, find an interval on which the function appears to be decreasing. Estimate any local extrema using the table.

t	2	3	4	5	6	7	8	9
C(t)	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

Solution:
It appears that the cost of gas increased from t = 2 to t = 8. It appears the cost of gas decreased from t = 8 to t = 9, so the function appears to be decreasing on the interval (8, 9).

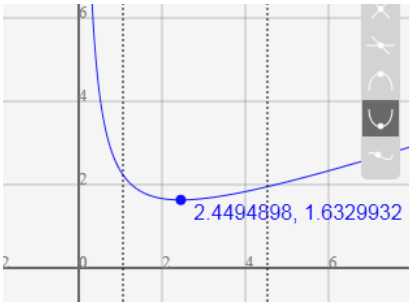
Since the function appears to change from increasing to decreasing at t = 8, there is local maximum at t = 8.

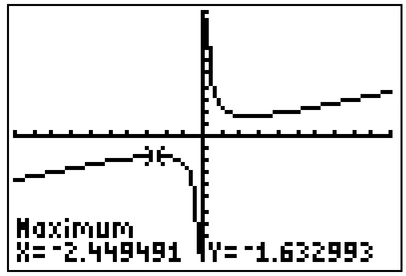
Example 10

Use a graph to estimate the local extrema of the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Use these to determine the intervals on which the function is increasing.

Solution:
Using technology to graph the function, it appears there is a local minimum somewhere between $x = 2$ and $x = 3$, and a symmetric local maximum somewhere between $x = -3$ and $x = -2$.

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Below are screen images from two different technologies, showing the estimate for the local maximum and minimum.





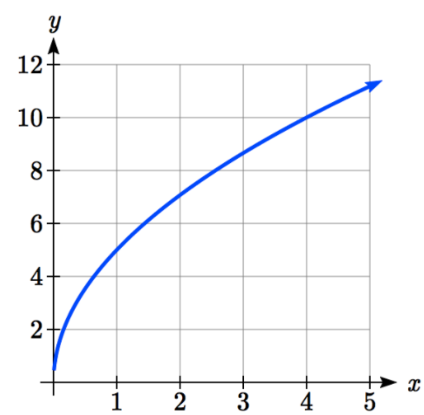
Based on these estimates, the function is increasing on the intervals $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that while we expect the extrema to be symmetric, the two different technologies agree only up to 4 decimals due to the differing approximation algorithms used by each.

Try it Now

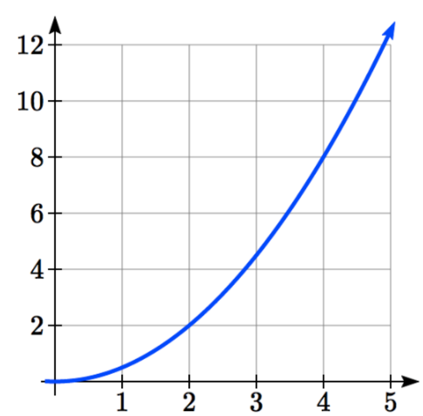
Use a graph of the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Concavity

The total sales, in thousands of dollars, for two companies over 4 weeks are shown.



Company A



Company B

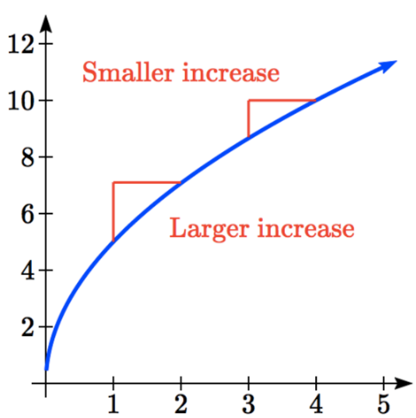
As you can see, the sales for each company are increasing, but they are increasing in very different ways. To describe the difference in behavior, we can investigate how the average rate of change varies over different intervals. Using tables of values, Company A

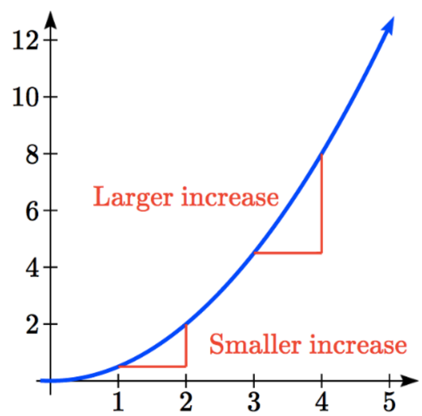
Week	Sales	Rate of Change
0	0	
		5
1	5	
		2.1
2	7.1	
		1.6
3	8.7	
		1.3
4	10	

Company B

Week	Sales	Rate of Change
0	0	
		0.5
1	0.5	
		1.5
2	2	
		2.5
3	4.5	
		3.5
4	8	

From the tables, we can see that the rate of change for company A is *decreasing*, while the rate of change for company B is *increasing*.





When the rate of change is getting smaller, as with Company A, we say the function is **concave down**. When the rate of change is getting larger, as with Company B, we say the function is **concave up**.

Concavity

A function is **concave up** if the rate of change is increasing.

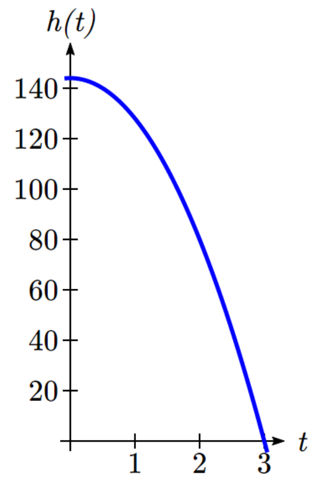
A function is **concave down** if the rate of change is decreasing.

A point where a function changes from concave up to concave down or vice versa is called an **inflection point.**

Example 11

An object is thrown from the top of a building. The object's height in feet above ground after *t* seconds is given by the function $h(t) = 144 - 16t^2$ for $0 \le t \le 3$. Describe the concavity of the graph.

Solution:
Sketching a graph of the function, we can see that the function is decreasing. We can calculate some rates of change to explore the behavior.



t	*h(t)*	Rate of Change
0	144	
		-16
1	128	
		-48
2	80	
		-80
3	0	

Notice that the rates of change are becoming more negative, so the rates of change are *decreasing*. This means the function is concave down.

Example 12

The value, *V*, of a car after *t* years is given in the table below. Is the value increasing or decreasing? Is the function concave up or concave down?

t	0	2	4	6	8
V(t)	28000	24342	21162	18397	15994

Solution:
Since the values are getting smaller, we can determine that the value is decreasing. We can compute rates of change to determine concavity.

t	0	2	4	6	8
V(t)	28000	24342	21162	18397	15994
Rate of change	-1829	-1590	-1382.5	-1201.5	

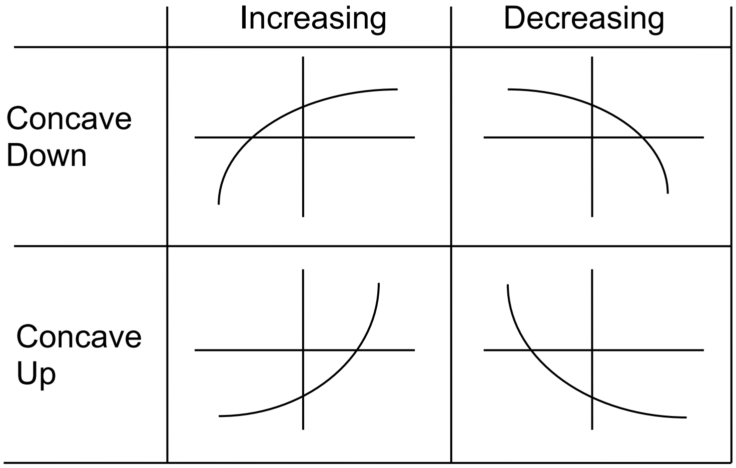
Since these values are becoming less negative, the rates of change are *increasing*, so this function is concave up.

Try it Now

Is the function described in the table below concave up or concave down?

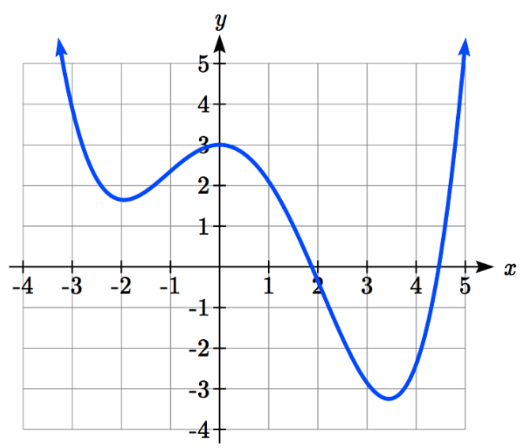
x	0	5	10	15	20
g(x)	10000	9000	7000	4000	0

Graphically, concave down functions bend downwards like a frown, and concave up function bend upwards like a smile.



Example 13

Estimate from the graph shown the intervals on which the function is concave down and concave up.



Solution:
On the far left, the graph is decreasing but concave up, since it is bending upwards. It begins increasing at *x* = -2, but it continues to bend upwards until about *x* = -1.

From *x* = -1 the graph starts to bend downward, and continues to do so until about *x* = 2. The graph then begins curving upwards for the remainder of the graph shown.

From this, we can estimate that the graph is concave up on the intervals $(-\infty, -1)$ and $(2, \infty)$, and is concave down on the interval $(-1, 2)$. The graph has inflection points at *x* = -1 and *x* = 2.

Try it Now

Using the graph of $f(x) = x^3 - 6x^2 - 15x + 20$, estimate the intervals on which the function is concave up and concave down.

Important Topics of This Section
Rate of Change
Average Rate of Change
Calculating Average Rate of Change using Function Notation
Increasing/Decreasing
Local Maxima and Minima (Extrema)
Inflection points
Concavity

Section 1.4 Exercises

Conceptual Questions

1. What is a rate of change? How do we find an average rate of change for a function?
2. How do we label the units of a rate of change?
3. What is the Difference Quotient used for?
4. What does it mean for a function to be increasing or decreasing? How is that related to rate of change?
5. What are local maxima or minima? How are they connected to rate of change?
6. What is concavity? How is it related to rate of change?

Practice Problems

1. The table below gives the annual sales (in millions of dollars) of a product. What was the average rate of change of annual sales…
   a) Between 2001 and 2002?   b) Between 2001 and 2004?

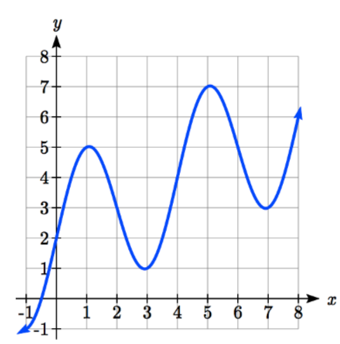
year	1998	1999	2000	2001	2002	2003	2004	2005	2006
sales	201	219	233	243	249	251	249	243	233

2. The table below gives the population of a town, in thousands. What was the average rate of change of population…
   a) Between 2002 and 2004?   b) Between 2002 and 2006?

year	2000	2001	2002	2003	2004	2005	2006	2007	2008
population	87	84	83	80	77	76	75	78	81

3. Based on the graph shown, estimate the average rate of change from x = 1 to x = 4.

4. Based on the graph shown, estimate the average rate of change from x = 2 to x = 5.



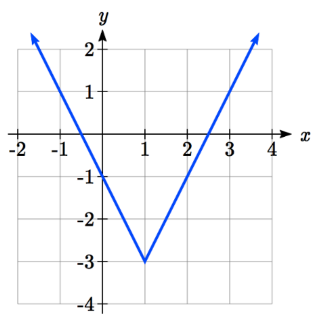
Find the average rate of change of each function on the interval specified.

5. $f(x) = x^2$ on [1, 5]
6. $q(x) = x^3$ on [-4, 2]
7. $g(x) = 3x^3 - 1$ on [-3, 3]
8. $h(x) = 5 - 2x^2$ on [-2, 4]
9. $k(t) = 6t^2 + \frac{4}{t^3}$ on [1, 3]
10. $p(t) = \frac{t^2-4t+1}{t^2+3}$ on [-3, 1]

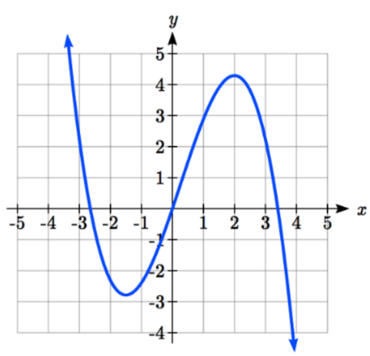
Find the average rate of change of each function on the interval specified. Your answers will be expressions involving a parameter (b or h).

11. $f(x) = 4x^2 - 7$ on [1, *b*]
12. $g(x) = 2x^2 - 9$ on [4, *b*]
13. $h(x) = 3x + 4$ on [2, 2+*h*]
14. $k(x) = 4x - 2$ on [3, 3+*h*]
15. $a(t) = \frac{1}{t+4}$ on [9, 9+*h*]
16. $b(x) = \frac{1}{x+3}$ on [1, 1+*h*]
17. $j(x) = 3x^3$ on [1, 1+*h*]
18. $r(t) = 4t^3$ on [2, 2+*h*]
19. $f(x) = 2x^2 + 1$ on [*x*, *x*+*h*]
20. $g(x) = 3x^2 - 2$ on [*x*, *x*+*h*]

For each function graphed, estimate the intervals on which the function is increasing and decreasing.



21.



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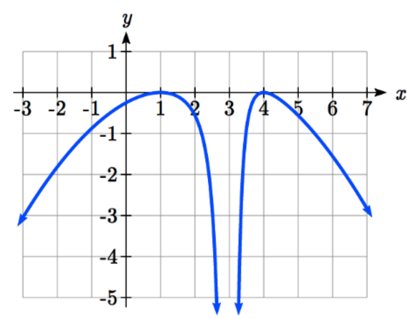
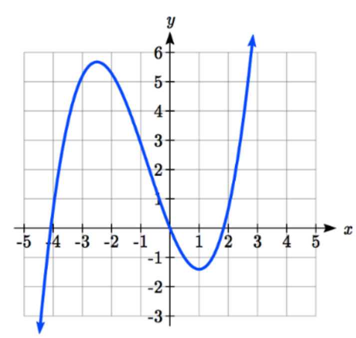




24.

For each table below, select whether the table represents a function that is increasing or decreasing, and whether the function is concave up or concave down.

25.

x	***f(x)***
1	2
2	4
3	8
4	16
5	32

26.

x	***g(x)***
1	90
2	80
3	75
4	72
5	70

27.

x	***h(x)***
1	300
2	290
3	270
4	240
5	200

28.

x	***k(x)***
1	0
2	15
3	25
4	32
5	35

29.

x	***f(x)***
1	-10
2	-25
3	-37
4	-47
5	-54

30.

x	***g(x)***
1	-200
2	-190
3	-160
4	-100
5	0

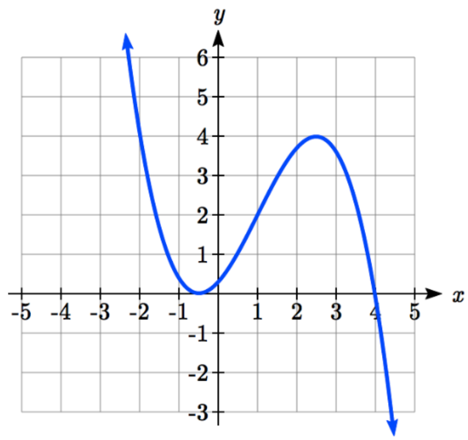
31.

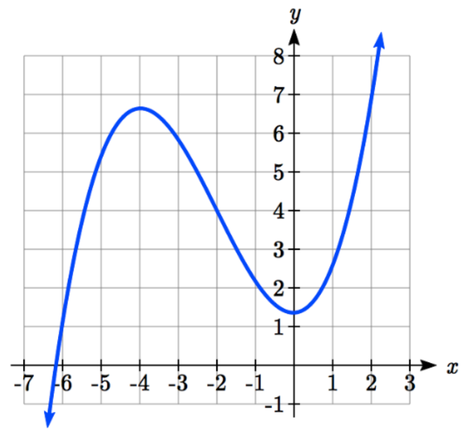
x	***h(x)***
1	-100
2	-50
3	-25
4	-10
5	0

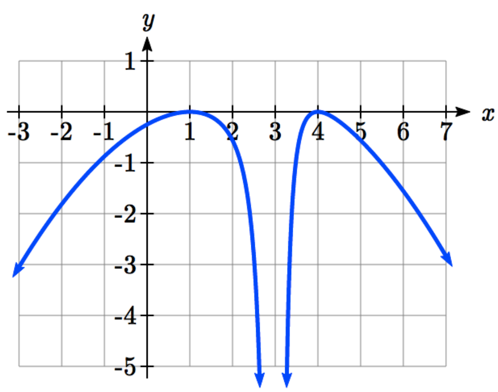
32.

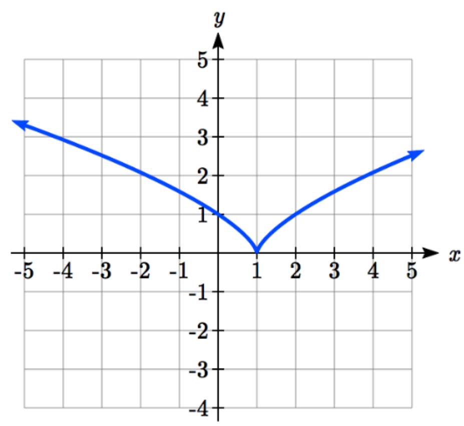
x	***k(x)***
1	-50
2	-100
3	-200
4	-400
5	-900

For each function graphed, estimate the intervals on which the function is concave up and concave down, and the location of any inflection points.

33. 

34. 

35. 

36. 

Use a graph to estimate the local extrema and inflection points of each function, and to estimate the intervals on which the function is increasing, decreasing, concave up, and concave down.

37. $f(x) = x^4 - 4x^3 + 5$

38. $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

39. $g(t) = t\sqrt{t+3}$

40. $k(t) = 3t^{2/3} - t$

41. $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

42. $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

Section 1.5 Composition of Functions

Suppose we wanted to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The temperature depends on the day, and the cost depends on the temperature. Using descriptive variables, we can notate these two functions.

The first function, *C(T),* gives the cost *C* of heating a house when the average daily temperature is *T* degrees Celsius, and the second, *T(d)*, gives the average daily temperature on day *d* of the year in some city. If we wanted to determine the cost of heating the house on the 5th day of the year, we could do this by linking our two functions together, an idea called composition of functions. Using the function *T(d)*, we could evaluate *T*(5) to determine the average daily temperature on the 5th day of the year. We could then use that temperature as the input to the *C(T)* function to find the cost to heat the house on the 5th day of the year: *C(T(5)).*

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. We write $f\big(g(x)\big)$, and read this as "f of g of x" or "f composed with g at x".

An alternate notation for composition uses the composition operator: $\circ$
$(f \circ g)(x)$ is read "f of g of x" or "f composed with g at x", just like $f\big(g(x)\big)$.

Notice that $f\big(g(x)\big)$ DOES NOT mean f times g. It means $g(x)$ is the input into the $f(x)$ function!

Example 1

Suppose $c(s)$ gives the number of calories burned doing *s* sit-ups, and $s(t)$ gives the number of sit-ups a person can do in *t* minutes. Interpret $c\big(s(3)\big)$.

Solution:
When we are asked to interpret, we are being asked to explain the meaning of the expression in words. The inside expression in the composition is $s(3)$. Since the input to the *s* function is time, the 3 is representing 3 minutes, and $s(3)$ is the number of sit-ups that can be done in 3 minutes. Taking this output and using it as the input to the $c(s)$ function will gives us the calories that can be burned by the number of sit-ups that can be done in 3 minutes.

Note that it is not important that the same variable be used for the output of the inside function and the input to the outside function. However, it *is* essential that the units on the output of the inside function match the units on the input to the outside function, if the units are specified.

Example 2

Suppose $f(x)$ gives miles that can be driven in x hours, and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution:
The expression $g(y)$ takes miles as the input and outputs a number of gallons. The function $f(x)$ is expecting a number of hours as the input; trying to give it a number of gallons as input does not make sense. Remember the units must match, and number of gallons does not match number of hours, so the expression $f(g(y))$ is meaningless.

The expression $f(x)$takes hours as input and outputs a number of miles driven. The function $g(y)$ is expecting a number of miles as the input, so giving the output of the $f(x)$function (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will give the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Try it Now

In a department store you see a sign that says 50% off clearance merchandise, so final cost C depends on the clearance price, p, according to the function $C(p)$. Clearance price, p, depends on the original discount, d, given to the clearance item, $p(d)$. Interpret $C(p(d))$.

Composition of Functions using Tables and Graphs

When working with functions given as tables and graphs, we can look up values for the functions using a provided table or graph, as discussed in section 1.1. We start evaluation from the provided input, and first evaluate the inside function.

We can then use the output of the inside function as the input to the outside function. To remember this, always work from the inside out.

Example 3

Using the tables below, evaluate $f(g(3))$ and $g(f(4))$

x	*f(x)*
1	6
2	8
3	3
4	1

x	*g(x)*
1	3
2	5
3	2
4	7

Solutions:

To evaluate $f(g(3))$, we start from the inside with the value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g: $g(3) = 2$. We can then use that result as the input to the *f* function, so $g(3)$ is replaced by the equivalent value 2 and we can evaluate $f(2)$. Then using the table that defines the function f, we find that $f(2) = 8$.

$$f(g(3))$$
$$= f(2)$$
$$= 8$$

To evaluate $g(f(4))$, we first evaluate the inside expression $f(4)$ using the first table: $f(4) = 1$. Then using the table for g we can evaluate:

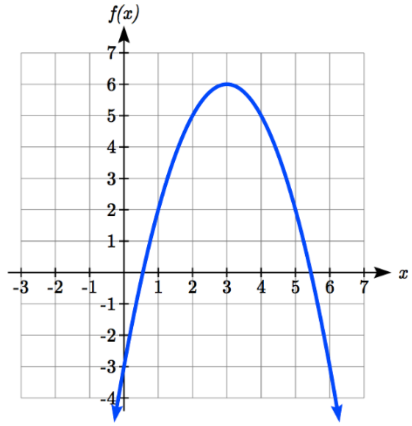
$$g(f(4))$$
$$= g(1)$$
$$= 3$$

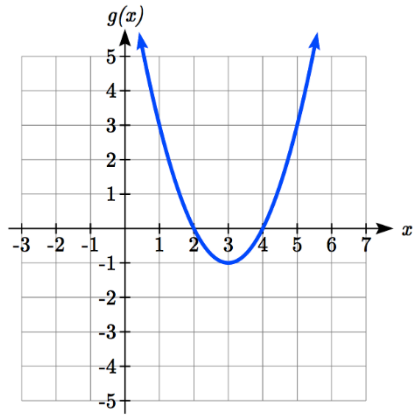
Try it Now

Using the tables from the example above, evaluate $f(g(1))$ and $g(f(3))$.

Example 4

Using the graphs below, evaluate $f(g(1))$.





Solutions:
To evaluate $f(g(1))$, we again start with the inside evaluation. We evaluate $g(1)$ using the graph of the $g(x)$ function, finding the input of 1 on the horizontal axis and finding the output value of the graph at that input. Here, $g(1) = 3$.

Using this value as the input to the *f* function, $f(g(1)) = f(3)$. We can then evaluate this by looking to the graph of the $f(x)$ function, finding the input of 3 on the horizontal axis, and reading the output value of the graph at this input. $f(3) = 6$, so $f(g(1)) = 6$.

Try it Now

Using the graphs from the previous example, evaluate $g(f(2))$.

Composition using Formulas

When evaluating a composition of functions where we have either created or been given formulas, the concept of working from the inside out remains the same. First, we evaluate the inside function using the input value provided, then use the resulting output as the input to the outside function.

Example 5

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution:
Since the inside evaluation is $h(1)$ we start by evaluating the $h(x)$ function at 1:

$$h(1)$$
$$= 3(1) + 2$$
$$= 5$$

Then $f(h(1)) = f(5)$, so we evaluate the $f(t)$ function at an input of 5:

$$f(h(1))$$
$$= f(5)$$
$$= 5^2 - 5$$
$$= 20$$

Try it Now

Using the functions from the example above, evaluate $h(f(-2))$.

In the previous example, we immediately evaluated the inner function at a given number. Alternatively, we could find an expression for a composition of functions. If we want to find a formula for *f(g(x))*, we can start by writing out the formula for *g(x)*. We can then evaluate the function *f(x)* at that expression. We can then use the resulting formula to evaluate the composition at any value we like (in the domain).

Example 6

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, find a formula for $f(h(x))$. Then use the formula to evaluate $f(h(1))$ and $f(h(2))$.

Solution:

To evaluate $f(h(x))$, we can first write the formula for $h(x)$:

$$h(x) = 3x + 2$$

That's what we want to input into f:

$$f(h(x)) = f(3x + 2)$$

Now we evaluate $f(3x + 2)$ by substituting in $(3x + 2)$ for the input of f:

$$f(3x + 2) = (3x + 2)^2 - (3x + 2)$$
$$= (3x + 2)(3x + 2) - (3x + 2)$$
$$= 9x^2 + 6x + 6x + 4 - 3x - 2$$
$$= 9x^2 + 9x + 2$$

This gives us a pretty formula for $f(h(x)) = 9x^2 + 9x + 2$. That means, if we want to calculate $f(h(1))$, we can use 1 as the input to our formula:

$$f(h(1)) = 9 \cdot 1^2 + 9 \cdot 1 + 2$$
$$= 20$$

$$f(h(2)) = 9 \cdot 2^2 + 9 \cdot 2 + 2$$
$$= 56$$

Example 7

Let $f(x) = x^2$ and $g(x) = \frac{1}{x} - 2x$, find $f(g(x))$ and $g(f(x))$.

Solution:
To find $f(g(x))$, we start by evaluating the inside, writing out the formula for $g(x)$.

$$g(x) = \frac{1}{x} - 2x$$

We then use the expression $\left(\frac{1}{x} - 2x\right)$ as input for the function f.

$$f(g(x)) = f\left(\frac{1}{x} - 2x\right)$$

We then evaluate the function $f(x)$ using the formula for $g(x)$ as the input.
Since $f(x) = x^2$, $f\left(\frac{1}{x} - 2x\right) = \left(\frac{1}{x} - 2x\right)^2$
This gives us the formula for the composition: $f(g(x)) = \left(\frac{1}{x} - 2x\right)^2$.

Likewise, to find $g(f(x))$, we evaluate the inside, writing out the formula for $f(x)$

$$g(f(x)) = g(x^2)$$

Now we evaluate the function $g(x)$ using x^2 as the input.

$$g(f(x)) = \frac{1}{x^2} - 2x^2$$

Try it Now

Let $f(x) = x^3 + 3x$ and $g(x) = \sqrt{x}$, find $f(g(x))$ and $g(f(x))$.

Example 8

A city manager determines that the tax revenue, R, in millions of dollars collected on a population of p thousand people is given by the formula $R(p) = 0.03p + \sqrt{p}$, and that the city's population, in thousands, is predicted to follow the formula $p(t) = 60 + 2t + 0.3t^2$, where t is measured in years after 2010. Find a formula for the tax revenue as a function of the year.

Solution:
Since we want tax revenue as a function of the year, we want year to be our initial input, and revenue to be our final output. To find revenue, we will first have

to predict the city population, and then use that result as the input to the tax function. So we need to find $R(p(t))$. Evaluating this,

$$\begin{aligned}&R(p(t))\\&= R(60 + 2t + 0.3t^2)\\&= 0.03(60 + 2t + 0.3t^2) + \sqrt{60 + 2t + 0.3t^2}\end{aligned}$$

This composition gives us a single formula which can be used to predict the tax revenue during a given year, without needing to find the intermediary population value.

For example, to predict the tax revenue in 2017, when $t = 7$ (because t is measured in years after 2010),

$$\begin{aligned}&R(p(7))\\&= 0.03(60 + 2(7) + 0.3(7)^2) + \sqrt{60 + 2(7) + 0.3(7)^2}\\&\approx 12.079 \text{ million dollars}\end{aligned}$$

Decomposing Functions

In some cases, it is desirable to decompose a function – to write it as a composition of two simpler functions.

Example 9

Write $f(x) = 3 + \sqrt{5 - x^2}$ as the composition of two functions.

Solution:
We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that $5 - x^2$ is the inside of the square root. We could then decompose the function as:

$$h(x) = 5 - x^2$$
$$g(x) = 3 + \sqrt{x}$$

We can check our answer by recomposing the functions:

$$g(h(x)) = g(5 - x^2) = 3 + \sqrt{5 - x^2}$$

Note that this is not the only solution to the problem. Another non-trivial decomposition would be $h(x) = x^2$ and $g(x) = 3 + \sqrt{5 - x}$

Try it Now

Find functions f and g such that $h(x) = \frac{1}{x^2+4}$ can be written as $h(x) = f\big(g(x)\big)$.

Important Topics of this Section
Definition of Composition of Functions
Compositions using:
Words
Tables
Graphs
Equations
Domain of Compositions
Decomposition of Functions

Section 1.5 Exercises

Conceptual Questions

1. What does the notation $f\left(g(x)\right)$ mean? What about $f(f(x))$?

Practice Problems

Given each pair of functions, calculate $\boldsymbol{f\left(g(0)\right)}$ and $\boldsymbol{g\left(f(0)\right)}$.

1. $f(x) = 4x + 8,\ g(x) = 7 - x^2$
2. $f(x) = 5x + 7,\ g(x) = 4 - 2x^2$
3. $f(x) = \sqrt{x+4},\ g(x) = 12 - x^3$
4. $f(x) = \frac{1}{x+2},\ g(x) = 4x + 3$

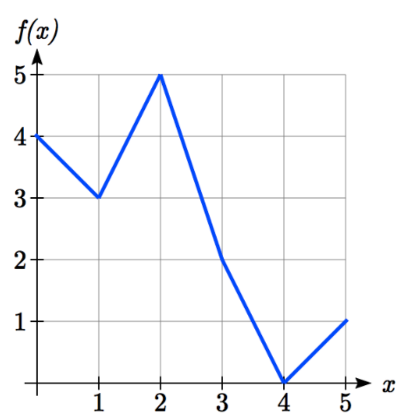
Use the table of values to evaluate each expression

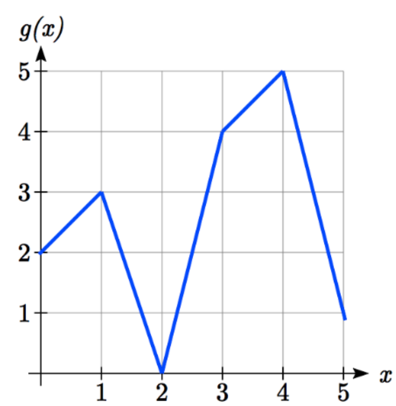
5. $f\left(g(8)\right)$
6. $f\left(g(5)\right)$
7. $g\left(f(5)\right)$
8. $g\left(f(3)\right)$
9. $f\left(f(4)\right)$
10. $f\left(f(1)\right)$
11. $g\left(g(2)\right)$
12. $g\left(g(6)\right)$

$\boldsymbol{x}$	$\boldsymbol{f(x)}$	$\boldsymbol{g(x)}$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

Use the graphs to evaluate the expressions below.

13. $f\left(g(3)\right)$
14. $f\left(g(1)\right)$
15. $g\left(f(1)\right)$
16. $g\left(f(0)\right)$
17. $f\left(f(5)\right)$
18. $f\left(f(4)\right)$
19. $g\left(g(2)\right)$
20. $g\left(g(0)\right)$





For each pair of functions, find $\boldsymbol{f\left(g(x)\right)}$ and $\boldsymbol{g\left(f(x)\right)}$. Simplify your answers.

21. $f(x) = \frac{1}{x-6},\ g(x) = \frac{7}{x} + 6$
22. $f(x) = \frac{1}{x-4},\ g(x) = \frac{2}{x} + 4$
23. $f(x) = x^2 + 1,\ g(x) = \sqrt{x+2}$
24. $f(x) = \sqrt{x} + 2,\ g(x) = x^2 + 3$
25. $f(x) = |x|,\ g(x) = 5x + 1$

26. $f(x) = \sqrt[3]{x}$, $g(x) = \frac{x+1}{x^3}$

27. If $f(x) = x^4 + 6$, $g(x) = x - 6$ and $h(x) = \sqrt{x}$, find $f\left(g\left(h(x)\right)\right)$

28. If $f(x) = x^2 + 1$, $g(x) = \frac{1}{x}$ and $h(x) = x + 3$, find $f\left(g\left(h(x)\right)\right)$

29. The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost, $C(x)$ is the cost of producing x items. To determine the cost of production when the price is \$6, you would do which of the following:
    a. Evaluate $D\left(C(6)\right)$
    b. Evaluate $C\left(D(6)\right)$
    c. Solve $D\left(C(x)\right) = 6$
    d. Solve $C\left(D(p)\right) = 6$

30. The function $A(d)$ gives the pain level on a scale of 0-10 experienced by a patient with d milligrams of a pain reduction drug in their system. The milligrams of drug in the patient's system after t minutes is modeled by $m(t)$. To determine when the patient will be at a pain level of 4, you would need to:
    a. Evaluate $A\left(m(4)\right)$
    b. Evaluate $m\left(A(4)\right)$
    c. Solve $A\left(m(t)\right) = 4$
    d. Solve $m\left(A(d)\right) = 4$

31. The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where *t* is the time in hours.
    a. Find the composite function $N\left(T(t)\right)$
    b. Find the bacteria count after 4 hours

32. The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.
    a. Find the composite function $r\left(V(t)\right)$
    b. Find the radius after 20 seconds

Find functions $\boldsymbol{f(x)}$ and $\boldsymbol{g(x)}$ so the given function can be expressed as $h(x) = f\left(g(x)\right)$.

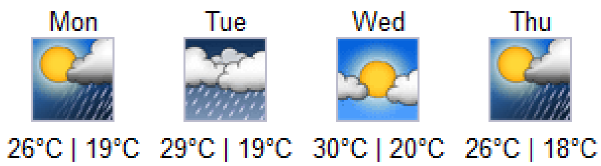
33. $h(x) = (x + 2)^2$
34. $h(x) = (x - 5)^3$
35. $h(x) = \frac{3}{x-5}$
36. $h(x) = \frac{4}{(x+2)^2}$
37. $h(x) = 3 + \sqrt{x - 2}$
38. $h(x) = 4 + \sqrt[3]{x}$

39. Let $f(x)$ be a linear function, with form $f(x) = ax + b$ for constants a and b. [UW]
    a. Show that $f\big(f(x)\big)$ is a linear function
    b. Find a function $g(x)$ such that $g\big(g(x)\big) = 6x - 8$
40. Let $f(x) = \frac{1}{2}x + 3$ [UW]
    a. Sketch the graphs of $f(x)$, $f\big(f(x)\big)$, $f\Big(f\big(f(x)\big)\Big)$ on the interval $-2 \le x \le 10$.
    b. Your graphs should all intersect at the point (6, 6). The value $x = 6$ is called a fixed point of the function $f(x)$ since $f(6) = 6$; that is, 6 is fixed - it doesn't move when *f* is applied to it. Give an explanation for why 6 is a fixed point for any function $f\Big(f\big(f(\ldots f(x)\ldots)\big)\Big)$.
    c. Linear functions (with the exception of $f(x) = x$) can have at most one fixed point. Quadratic functions can have at most two. Find the fixed points of the function $g(x) = x^2 - 2$.
    d. Give a quadratic function whose fixed points are $x = -2$ and $x = 3$.

41. A car leaves Seattle heading east. The speed of the car in mph after *m* minutes is given by the function $C(m) = \frac{70m^2}{10+m^2}$. [UW]
    a. Find a function $m = f(s)$ that converts seconds s into minutes m. Write out the formula for the new function $C\big(f(s)\big)$; what does this function calculate?
    b. Find a function $m = g(h)$ that converts hours h into minutes m. Write out the formula for the new function $C\big(g(h)\big)$; what does this function calculate?
    c. Find a function $z = v(s)$ that converts mph *s* into ft/sec *z*. Write out the formula for the new function $v(C(m)$; what does this function calculate?

Section 1.6 Inverse Functions

A fashion designer is travelling to Milan for a fashion show. He asks his assistant, Betty, what 75 degrees Fahrenheit is in Celsius, and after a quick search on Google, she finds the formula $C = \frac{5}{9}(F - 32)$. Using this formula, she calculates $\frac{5}{9}(75 - 32) \approx 24$degrees Celsius. The next day, the designer sends his assistant the week's weather forecast for Milan, and asks her to convert the temperatures to Fahrenheit.



At first, Betty might consider using the formula she has already found to do the conversions. After all, she knows her algebra, and can easily solve the equation for *F* after substituting a value for *C*. For example, to convert 26 degrees Celsius, she could write:

$$26 = \frac{5}{9}(F - 32)$$

$$26 \cdot \frac{9}{5} = F - 32$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79$$

After considering this option for a moment, she realizes that solving the equation for each of the temperatures would get awfully tedious, and realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one which takes the Celsius temperature and outputs the Fahrenheit temperature. This is the idea of an inverse function, where the input becomes the output and the output becomes the input.

Inverse Function
If $f(a) = b$, then a function $g(x)$ is an **inverse** of f if $g(b) = a$. The inverse of $f(x)$ is typically notated$f^{-1}(x)$, which is read "f inverse of x", so equivalently, if $f(a) = b$ then $f^{-1}(b) = a$.

Important: The raised -1 used in the notation for inverse functions is simply a notation, and does not designate an exponent or power of -1.

Example 1

If for a particular function, $f(2) = 4$, what do we know about the inverse?

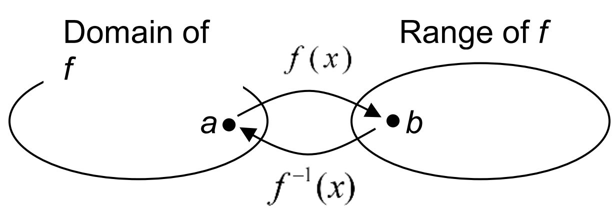
Solution:
The inverse function reverses which quantity is input and which quantity is output, so if $f(2) = 4$, then $f^{-1}(4) = 2$.

Alternatively, if you want to re-name the inverse function $g(x)$, then $g(4) = 2$.

Try it Now

Given that $h^{-1}(6) = 2$, what do we know about the original function $h(x)$?

Notice that the original function and the inverse function *undo* each other. If $f(a) = b$, then $f^{-1}(b) = a$, returning us to the original input. More simply put, if you compose these functions together you get the original input as your answer.
$f^{-1}(f(a)) = a$ and $f(f^{-1}(b)) = b$



Since the outputs of the function f are the inputs to f^{-1}, the range of f is also the domain of f^{-1}. Likewise, since the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}.

Basically, like how the input and output values switch, the domain & ranges switch as well. But be careful, because sometimes a function doesn't even have an inverse function, or only has an inverse on a limited domain. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, since a square "undoes" a square root, but it is only the inverse of $f(x)$ on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

Example 2

The function $f(x) = 2^x$ has domain $(-\infty, \infty)$and range $(0, \infty)$, what would we expect the domain and range of f^{-1} to be?

Solution:
We would expect f^{-1} to swap the domain and range of f, so f^{-1} would have domain$(0, \infty)$ and range $(-\infty, \infty)$.

Example 3

A function $f(t)$ is given as a table below, showing distance in miles that a car has traveled in *t* minutes. Find and interpret $f^{-1}(70)$

t (minutes)	30	50	70	90
f(t) (miles)	20	40	60	70

Solution:
The inverse function takes an output of f and returns an input for f. So in the expression$f^{-1}(70)$, the 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function *f*, 90 minutes, so$f^{-1}(70) = 90$. Interpreting this, it means that to drive 70 miles, it took 90 minutes.

Alternatively, recall the definition of the inverse was that if $f(a) = b$ then $f^{-1}(b) = a$. By this definition, if you are given $f^{-1}(70) = a$ then you are looking for a value *a* so that$f(a) = 70$. In this case, we are looking for a t so that$f(t) = 70$, which is when $t = 90$.

Try it Now

Using the table below

t (minutes)	30	50	60	70	90
f(t) (miles)	20	40	50	60	70

Find and interpret the following

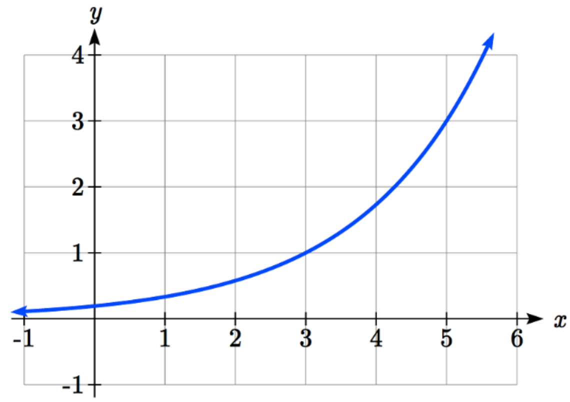
a. $f(60)$
b. $f^{-1}(60)$

Example 4

A function $g(x)$ is given as a graph below. Find $g(3)$ and $g^{-1}(3)$

Solution:

To evaluate $g(3)$, we find 3 on the horizontal axis and find the corresponding output value on the vertical axis. The point (3, 1) tells us that $g(3) = 1$.



To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means $g(x) = 3$. By looking for the output value 3 on the vertical axis we find the point (5, 3) on the graph, which means $g(5) = 3$, so by definition
$g^{-1}(3) = 5$.

Try it Now

Using the graph in Example 4 above

a. find $g^{-1}(1)$
b. estimate $g^{-1}(4)$

Example 5

Returning to our designer's assistant, find a formula for the inverse function that gives Fahrenheit temperature given a Celsius temperature.

Solution:

A quick Google search would find the inverse function, but alternatively, Betty might look back at how she solved for the Fahrenheit temperature for a specific Celsius value, and repeat the process in general

$$C = \frac{5}{9}(F - 32)$$

$$C \cdot \frac{9}{5} = F - 32$$

$$F = \frac{9}{5}C + 32$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32)$$

Then

$$F = h^{-1}(C) = \frac{9}{5}C + 32$$

In this case, we introduced a function h to represent the conversion since the input and output variables are descriptive, and writing C^{-1}could get confusing.

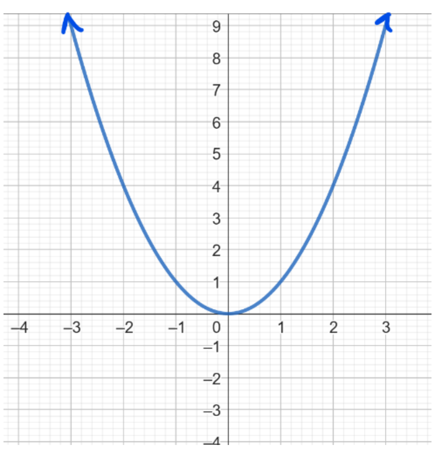
It is important to note that not all functions will have an inverse function. Since the inverse $f^{-1}(x)$ takes an output of f and returns an input of f, in order for f^{-1} to itself be a function, then each output of f (input to f^{-1}) must correspond to exactly one input of f (output of f^{-1}) in order for f^{-1} to be a function. You might recall that this is the definition of a one-to-one function.

Properties of Inverses
In order **for a function to have an inverse**, it must be a one-to-one function.

In some cases, it is desirable to have an inverse for a function even though the function is not one-to-one. In those cases, we can often limit the domain of the original function to an interval on which the function *is* one-to-one, then find an inverse only on that interval.

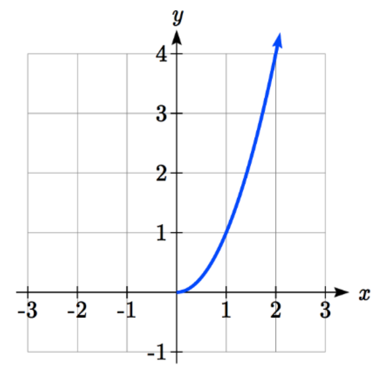
Example 6

The quadratic function $h(x) = x^2$ is not one-to-one, as pictured below. Find a domain on which this function is one-to-one, and find the inverse on that domain.



Solution:
The graph clearly fails the horizontal line test. We want to choose a portion of the function that still gives us all possible output values (from $[0, \infty)$), but leaves us with a one-to-one portion of the graph.

We can limit the domain to $[0, \infty)$ as pictured here, and find an inverse on this limited domain.



You may have already guessed that since we undo a square with a square root, the inverse of $h(x) = x^2$ on this domain is $h^{-1}(x) = \sqrt{x}$.

You can also solve for the inverse function algebraically. If $h(x) = x^2$, we can introduce the variable y to represent the output values, allowing us to write $y = x^2$. To find the inverse we solve for the input variable

To solve for x we take the square root of each side.

$$\sqrt{y} = \sqrt{x^2}$$

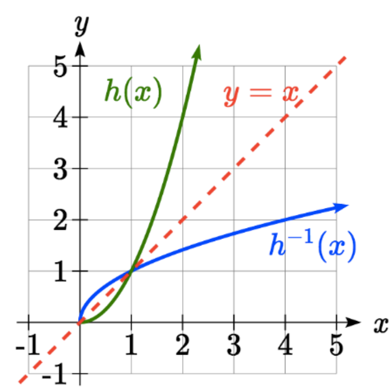
and get

$$\sqrt{y} = |x|,$$

so

$$x = \pm\sqrt{y}.$$

We have restricted x to being non-negative, so we'll use the positive square root, $x = \sqrt{y}$ or $h^{-1}(y) = \sqrt{y}$. In cases like this where the variables are not descriptive, it is common to see the inverse function rewritten with the variable x: $h^{-1}(x) = \sqrt{x}$. Rewriting the inverse using the variable *x* is often required for graphing inverse functions using calculators or computers.



Note that the domain and range of the square root function do correspond with the range and domain of the quadratic function on the limited domain. In fact, if we graph $h(x)$ on the restricted domain and $h^{-1}(x)$ on the same axes, we can notice symmetry: the graph of $h^{-1}(x)$ is the graph of $h(x)$ reflected over the line $y = x$.

We learned some important things about functions and their inverses from that last example. Let's summarize before we move on.

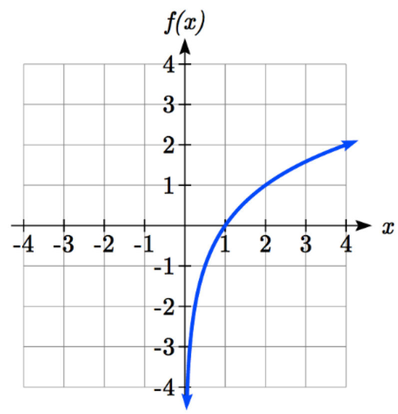
Features of functions and their inverses
Suppose $f(x)$ and $f^{-1}(x)$ are inverses. Then f and f^{-1} must both be one-to-one The graph of f^{-1} is the graph of f reflected over the line $y = x$ The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.

These features are a direct consequence of the main idea of creating inverses:

To make an inverse of a one-to-one function, swap all the input and output relationships!

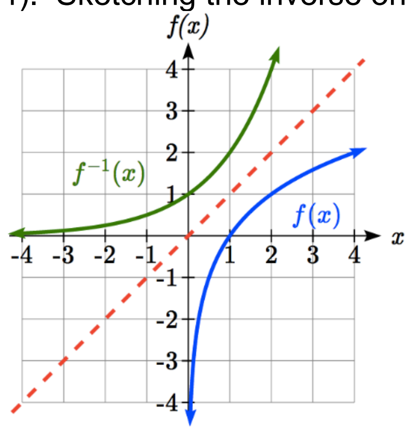
Example 7

Given the graph of $f(x)$ shown, sketch a graph of $f^{-1}(x)$.



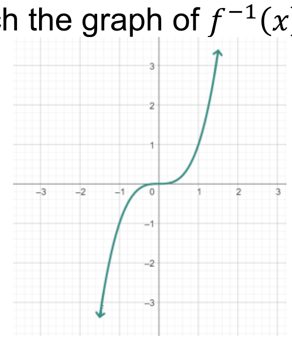
Solution:
This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of (0,∞) and range of (-∞,∞), so the inverse will have a domain of (-∞,∞) and range of (0,∞).

Reflecting this graph of the line $y = x$, the point (1, 0) reflects to (0, 1), and the point (4, 2) reflects to (2, 4). Sketching the inverse on the same axes as the original graph:



Try it Now

Given the graph of $f(x)$, sketch the graph of $f^{-1}(x)$.



Example 8

Given $f(x) = 8x - 4$, find a formula for $f^{-1}(x)$.

Solution:
First, we should check that this is a one-to-one function. If we sketch the graph, we see that it makes a diagonal line, which passes both the vertical and horizontal line tests, so this function is one-to-one.

Now, we know that an inverse reverses the x- and y-relationship, so we can think of making $y = 8x - 4$ into $x = 8y - 4$. If we isolate y, we'll have a formula for our inverse function.

$$x = 8y - 4$$
$$x + 4 = 8y$$
$$\frac{x+4}{8} = y$$
$$f^{-1}(x) = \frac{x+4}{8}$$

Try it Now:

Given $f(x) = \frac{3}{4}x + 1$, find a formula for $f^{-1}(x)$.

Important Topics of this Section
Definition of an inverse function
Composition of inverse functions yield the original input value
Not every function has an inverse function
To have an inverse a function must be one-to-one
Restricting the domain of functions that are not one-to-one.
Finding a formula for an inverse function.

Section 1.6 Exercises

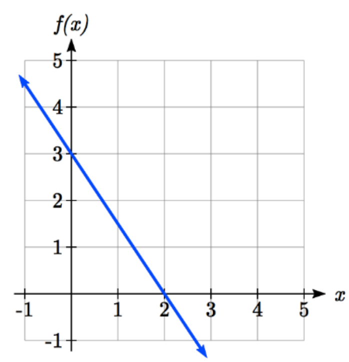
Conceptual Questions

1. What is the meaning of the notation $f^{-1}(x)$?
2. Why must a function f be one-to-one to find its inverse?
3. How are the functions $f(x)$ and $f^{-1}(x)$ related?
4. Are $f^{-1}(x)$ and $\frac{1}{f(x)}$ the same thing? Why or why not?

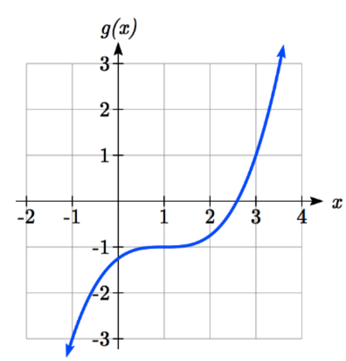
Practice Problems

Assume that the function f is a one-to-one function.

1. If $f(6) = 7$, find $f^{-1}(7)$
2. If $f(3) = 2$, find $f^{-1}(2)$
3. If $f^{-1}(-4) = -8$, find $f(-8)$
4. If $f^{-1}(-2) = -1$, find $f(-1)$
5. If $f(5) = 2$, find $\left(f(5)\right)^{-1}$
6. If $f(1) = 4$, find $\left(f(1)\right)^{-1}$
7. Using the graph of $f(x)$ shown
   a. Find $f(0)$
   b. Solve $f(x) = 0$
   c. Find $f^{-1}(0)$
   d. Solve $f^{-1}(x) = 0$



8. Using the graph shown
   a. Find $g(1)$
   b. Solve $g(x) = 1$
   c. Find $g^{-1}(1)$
   d. Solve $g^{-1}(x) = 1$



9. Use the table below to find the indicated quantities.

x	0	1	2	3	4	5	6	7	8	9
f(x)	8	0	7	4	2	6	5	3	9	1

   a. Find $f(1)$
   b. Solve $f(x) = 3$
   c. Find $f^{-1}(0)$
   d. Solve $f^{-1}(x) = 7$

10. Use the table below to fill in the missing values.

t	0	1	2	3	4	5	6	7	8
h(t)	6	0	1	7	2	3	5	4	9

   a. Find $h(6)$
   b. Solve $h(t) = 0$
   c. Find $h^{-1}(5)$
   d. Solve $h^{-1}(t) = 1$

For each table below, create a table for $\boldsymbol{f^{-1}(x)}$.

11.

x	3	6	9	13	14
f(x)	1	4	7	12	16

12.

x	3	5	7	13	15
f(x)	2	6	9	11	16

For each function below, find $\boldsymbol{f^{-1}(x)}$

13. $f(x) = x + 3$
14. $f(x) = x + 5$
15. $f(x) = 2 - x$
16. $f(x) = 3 - x$
17. $f(x) = 11x + 7$
18. $f(x) = 9 + 10x$

For each function, use a graphing calculator to find a domain on which $\boldsymbol{f}$ is one-to-one and non-decreasing, then find the inverse of $\boldsymbol{f}$ restricted to that domain.

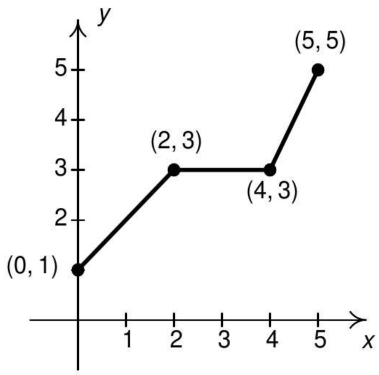
19. $f(x) = (x + 7)^2$
20. $f(x) = (x - 6)^2$
21. $f(x) = x^2 - 5$
22. $f(x) = x^2 + 1$

23. If $f(x) = x^3 - 5$ and $g(x) = \sqrt[3]{x+5}$, find
   a. $f(g(x))$
   b. $g(f(x))$
   c. What does this tell us about the relationship between $f(x)$ and $g(x)$?

24. If $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$, find
   a. $f(g(x))$
   b. $g(f(x))$
   c. What does this tell us about the relationship between $f(x)$ and $g(x)$?

Section 1.7: Graphical Transformations

Throughout this book, we're going to want to be able to create quick sketches of functions. Before we get to specific functions, we'll investigate in general how graphs of functions can be moved, stretched, and flipped around the coordinate plane. Our motivational example for the results in this section is the graph of $y = f(x)$ below. While we could formulate an expression for $f(x)$ as a piecewise-defined function consisting of linear and constant parts, we wish to focus more on the geometry here. That being said, we do record some of the function values - the 'key points', or 'parent values', if you will - to track through each transformation.



$y = f(x)$

x	$y = f(x)$
0	1
2	3
4	3
5	5

$(x, f(x))$
(0,1)
(2,3)
(4,3)
(5,5)

Vertical and Horizontal Shifts

Suppose we wished to graph $g(x) = f(x) + 2$. From a procedural point of view, we start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. Using the sample values for f from the table above we can create a table of values for g below, hence generating points on the graph of g.

x	$f(x) \rightarrow$	$g(x) = f(x) + 2$
0	1	$1 + 2 = 3$
2	3	$3 + 2 = 5$
4	3	$3 + 2 = 5$
5	5	$5 + 2 = 7$

$(x, g(x))$
(0,3)
(2,5)
(4,5)
(5,7)

For an interactive demonstration of what we're seeing with this table, you can check out the web activity linked here: https://www.geogebra.org/m/ubsnjbus.To get a feel for what's happening, you can adjust the slider labelled "d" in the interactive demonstration. Consider the following: as d becomes larger, how does the graph change? What about as d becomes smaller (more negative)?

Slide slider d to 2, and you'll see that geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* moves or 'shifts' the entire graph of f up 2 units. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four 'key points' we moved in the same manner in which they were connected before.

You'll note that the domain of f and the domain of g are the same, namely [0,5], but that the range of f is [1,5] while the range of g is [3,7]. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range.

You can perhaps imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1,3]$. In general, we have:

Vertical Shifts

Suppose f is a function and d is a real number.

To graph $F(x) = f(x) + d$, add d to each of the y-coordinates of the points on the graph of $y = f(x)$.

NOTE: This results in a vertical shift up d units if $d > 0$ or down d units if $d < 0$.

Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x+2)$. In other words, we are looking to see what happens when we add 2 to the input of the function. Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties. For instance, when we substitute $x = 4$ into the formula $g(x) = f(x+2)$, we are asked to find $f(4+2) = f(6)$ which doesn't exist because the domain of f is only $[0,5]$. The same thing happens when we attempt to find $g(5)$.

x	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	1	$g(0) = f(0+2) = f(2) = 3$	$(0,3)$

x	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
2	3	$g(2) = f(2+2) = f(4) = 3$	(2,3)
4	3	$g(4) = f(4+2) = f(6) = ?$	?
5	5	$g(5) = f(5+2) = f(7) = ?$	?

What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$so that the quantity $x + 2$, works out to be 0. Solving $x + 2 = 0$ gives $x = -2$, and $g(-2) = f\big((-2) + 2\big) = f(0) = 1$ so $(-2,1)$ is on the graph of g.

Similarly, to use the fact that $f(2) = 3$, we set $x + 2 = 2$ to get $x = 0$, which is the input we need for our g function. Substituting gives $g(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we produce the table below.

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(-2+2) = f(0) = 1$	$(-2,1)$
0	2	$g(0) = f(0+2) = f(2) = 3$	(0,3)
2	4	$g(2) = f(2+2) = f(4) = 3$	(2,3)
3	5	$g(3) = f(3+2) = f(5) = 5$	(3,5)

In summary, the points (0,1), (2,3), (4,3) and (5,5) on the graph of $y = f(x)$ give rise to the points (−2,1), (0,3), (2,3) and (3,5) on the graph of $y = g(x)$, respectively.

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x + 2 = a$ gives $x = a - 2$ so that $g(a-2) = f\big((a-2)+2\big) = f(a) = b$. As such, $(a-2, b)$ is on the graph of$y = g(x)$. The point$(a-2, b)$ is exactly 2 units to the *left* of the point (a, b)so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units. To achieve this shift left, we subtract c from our key x-values in the parent table. To get a feel for this, in the online demonstration (https://www.geogebra.org/m/ubsnjbus), adjust the slider labelled c. Investigate what happens when c is positive versus negative. It seems counterintuitive at first, but a positive c value shifts the graph *left* while a negative c value shifts the graph *right*!

Note that when we make $c = 2$, while the ranges of f and g are the same, the domain of g is $[-2,3]$ whereas the domain of f is $[0,5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph$j(x) = f(x-2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

Horizontal Shifts

Suppose f is a function and c is a real number.

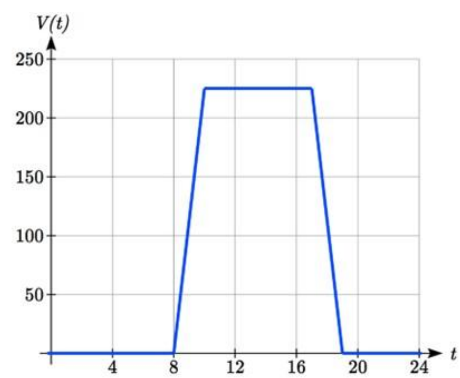
To graph $F(x) = f(x + c)$, subtract c from each of the x-coordinates of the points on the graph of $y = f(x)$.

Note: This results in a horizontal shift right c units if $c < 0$ or left c units if $c > 0$.

These observations about vertical and horizontal shifts present a theme which will run common throughout the section: changes to the *outputs* from a function result in some kind of *vertical change*; changes to the *inputs* to a function result in some kind of *horizontal* change.

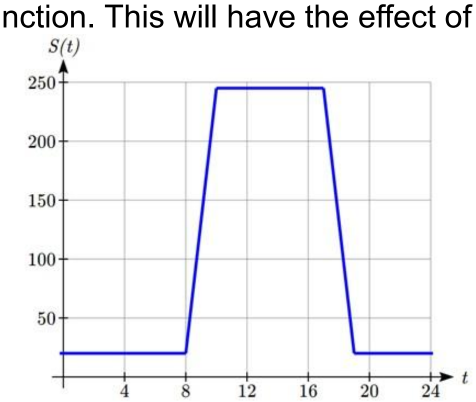
Example 1

To regulate temperature in a green building, air flow vents near the roof open and close throughout the day to allow warm air to escape. The graph below shows the open vents V (in square feet) throughout the day, t in hours after midnight. During the summer, the facilities staff decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day. Sketch a graph of this new function.



Solution:

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph up.



Notice that in the second graph, for each input value, the output value has increased by twenty, so if we call the new function S(t), we could write $S(t) = V(t) + 20$.

Note that this notation tells us that for any value of t, $S(t)$ can be found by evaluating the V function at the same input, then adding twenty to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units.

Notice that with a vertical shift the input values stay the same and only the output values change.

In our next example, we're going to set up an organizational method for managing transformations in tables. We'll put our beginning "parent" x and $y = f(x)$ values in the central column of the table. Then, we'll apply any x-transformations by adding columns onto the left side of the table (going out from the x column), and we'll apply any y-transformations by adding columns onto the right side of the table (going out from the $y = f(x)$ column). That way, when we're done with the transformations, the outermost columns will form the coordinates for our transformed function.

Example 2

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = f(x) - 3$.

x	f(x)
2	1
4	3
6	7
8	11

Solution:
The formula $g(x) = f(x) - 3$ tells us that we can find the output values of the g function by subtracting 3 from the output values of the f function. For example,

$f(2) = 1$ is found from the given table
$g(x) = f(x) - 3$ is our given transformation
$g(2) = f(2) - 3 = 1 - 3 = -2$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$:

x	$f(x) \rightarrow$	$g(x) = f(x) - 3$
2	1	$g(2) = 1 - 3 = -2$
4	3	$g(4) = 3 - 3 = 0$
6	7	$g(6) = 7 - 3 = 4$
8	11	$g(8) = 11 - 3 = 8$

$(x, g(x))$
$(2, -2)$
$(4,0)$
$(6,4)$
$(8,8)$

Try it Now

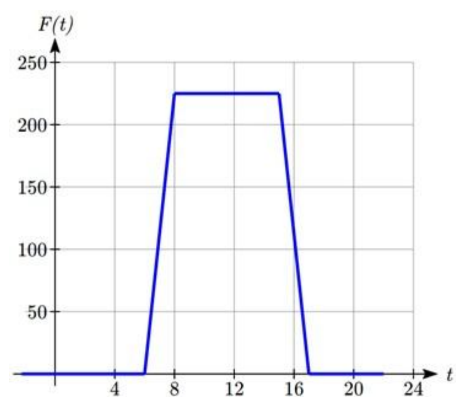
The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upwards from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10 meter building. Relate this new height function $b(t)$ to $h(t)$, then find a formula for $b(t)$.

Example 3

Returning to our building air flow example from the beginning of the section, suppose that in Fall, the facilities staff decides that the original venting plan starts too late, and they want to move the entire venting program to start two hours earlier. Sketch a graph of the new function.

Solution:

In the new graph, which we can call $F(t)$, at each time, the air flow is the same as the original function $V(t)$ was two hours later. For example, in the original function V, the air flow starts to change at 8am, while for the function $F(t)$ the air flow starts to change at 6am. The comparable function values are $V(8) = F(6)$.



Notice also that the vents first opened to 220 sq. ft. at 10 a.m. under the original plan, while under the new plan the vents reach 220 sq. ft. at 8 a.m., so $V(10) = F(8)$.

In both cases we see that since $F(t)$ starts 2 hours sooner, the same output values are reached when,$F(t) = V(t + 2)$.

Note that $V(t + 2)$ had the effect of shifting the graph to the left.

Example 4

A function f(x) is given as a table below. Create a table for the function $g(x) = f(x-3)$.

x	$f(x)$
2	1
4	3
6	7
8	11

Solution:
The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f with an input value three smaller. For example, we know that $f(2) = 1$. To get the same output from the g function, we will need an input value that is 3 larger: We input a value that is three larger for $g(x)$ because the function takes three away before evaluating the function f.

$$g(5)$$
$$= f(5-3)$$
$$= f(2)$$
$$= 1$$

So, to create our list of corresponding input values for the function g, we need to figure out what did we subtract that 3 from to get our list of inputs for $f(x)$? If we go in reverse, then we should *add* 3 to each of the input values for $f(x)$ to find what the inputs for g need to be.

x	$g(x)$
5	1
7	3
9	7
11	11

That makes our final set of coordinates the following:

$X = x + 3$	$\leftarrow\ x$	$f(x)$
$X = 2 + 3 = 5$	2	1
$X = 4 + 3 = 7$	4	3
$X = 6 + 3 = 9$	6	7
$X = 8 + 3 = 11$	8	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$ in the chart, but the corresponding input values, x, have shifted to the right by 3: 2 shifted to 5, 4 shifted to 7, 6 shifted to 9 and 8 shifted to 11.

When thinking about horizontal and vertical shifts, it is good to keep in mind that vertical shifts are affecting the output values of the function, while horizontal shifts are affecting the input values of the function.

Example 5

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)$ and $G(m+10)$.

Solution:
$G(m)+10$ is adding 10 to the output, gallons. This is 10 gallons of gas more than is required to drive m miles. So, this is the gas required to drive m miles, plus another 10 gallons of gas.
$G(m+10)$ is adding 10 to the input, miles. This is the number of gallons of gas required to drive 10 miles more than m miles.

Reflections about the Coordinate Axes

We now turn our attention to reflections. You may know from algebra courses that to reflect a point (x, y) across the x-axis, we replace y with $-y$. If (x, y) is on the graph of f, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis. To get a feel for this, in the online demonstration at https://www.geogebra.org/m/ubsnjbus, adjust the sliders labelled a and b.

Reflections

Suppose f is a function.

To graph $F(x) = -f(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x) by - 1$.

NOTE: This results in a reflection across the x-axis.

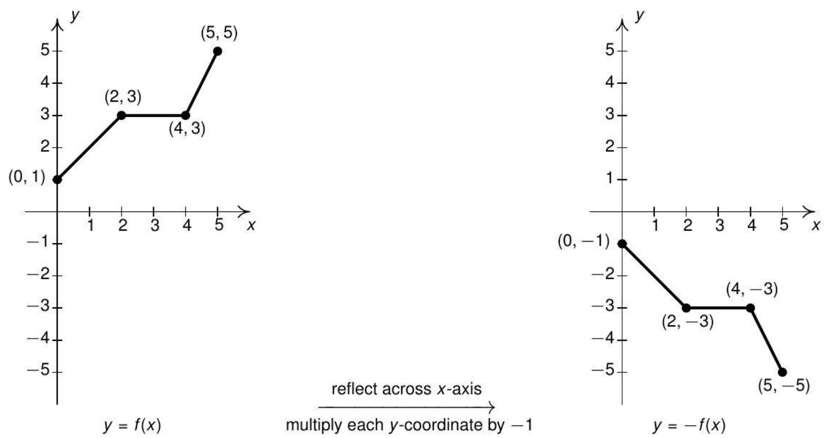
To graph $F(x) = f(-x)$, multiply each of the x-coordinates of the points on the graph of $y = f(x)$ by -1.

NOTE: This results in a reflection across the y-axis.

Using the language of inputs and outputs, we are saying that multiplying the *outputs* from a function by -1 reflects its graph across the *horizontal* axis, while multiplying the *inputs* to a function by -1 reflects the graph across the *vertical* axis.

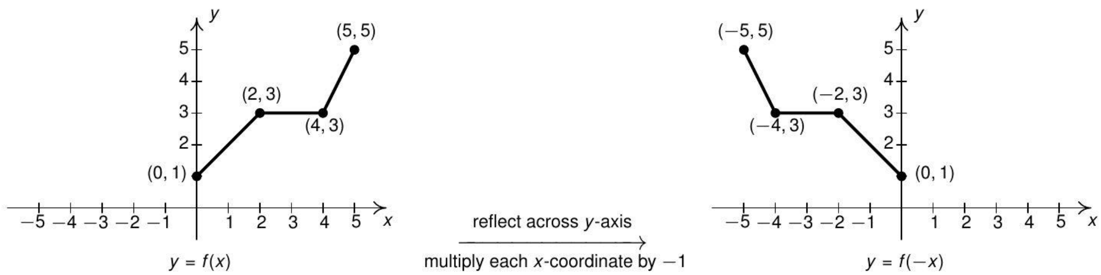
Applying these ideas to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis.

x	$y = f(x) \rightarrow$	$g(x) = -1 \cdot y$	$(x, g(x))$
0	1	-1	$(0, -1)$
2	3	-3	$(2, -3)$
4	3	-3	$(4, -3)$
5	5	-5	$(5, -5)$



By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.

$\boldsymbol{x}$	$\boldsymbol{-x}$	$\boldsymbol{g(x) = f(-x)}$	$\boldsymbol{(x, g(x))}$
0	0	$g(0) = f\big(-(-0)\big) = f(0) = 1$	$(0,1)$
-2	2	$g(-2) = f\big(-(-2)\big) = f(2) = 3$	$(-2,3)$
-4	4	$g(-4) = f\big(-(-4)\big) = f(4) = 3$	$(-4,3)$
-5	5	$g(-5) = f\big(-(-5)\big) = f(5) = 5$	$(-5,5)$



Example 6

A function $\mathrm{f(x)}$ *is given as a table below. Create a table for the function* $\mathrm{g(x)} = -\mathrm{f(x)}$ *and* $\mathrm{h(x)} = \mathrm{f(-x)}$.

x	f(x)
2	1
4	3
6	7
8	11

Solution:

For $\mathrm{g(x)}$, *this is a vertical reflection, so the* x *values stay the same and each output value will be the opposite of the original output value.*

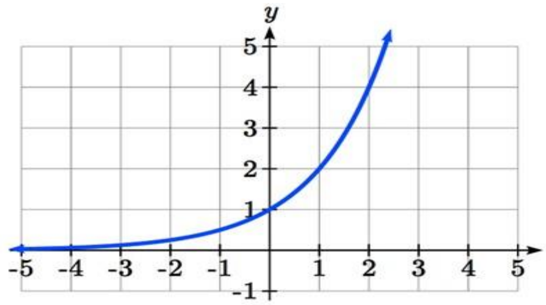
$\boldsymbol{x}$	$\boldsymbol{y = f(x)} \rightarrow$	$\boldsymbol{g(x) = -1 \cdot y}$	$\boldsymbol{(x, g(x))}$
2	1	-1	$(2, -1)$
4	3	-3	$(4, -3)$
6	7	-7	$(6, -7)$
8	11	-11	$(8, -11)$

For $\mathrm{h(x)}$, *this is a horizontal reflection, and each input value will be the opposite of the original input value and the* $\mathrm{h(x)}$ *values stay the same as the* $\mathrm{f(x)}$ *values:*

$X = x \div -1$	x	$f(x)$	$(X, g(X))$
-2	2	1	$(-2,1)$
-4	4	3	$(-4,3)$
-6	6	7	$(-6,7)$
-8	8	11	$(-8,11)$

Example 7

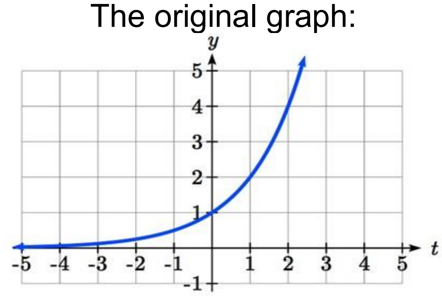
A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^{t}$ shown here. Sketch a graph of $k(t)$.



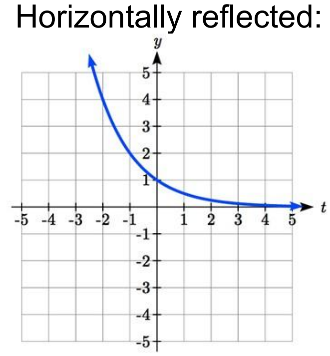
Solution:
This equation combines three transformations into one equation.
A horizontal reflection:
$f(-t) = 2^{-t}$
combined with a vertical reflection:
$-f(-t) = -2^{-t}$
combined with a vertical shift up 1:
$-f(-t) + 1 = 2^{-t} + 1$.

We can sketch a graph by applying these transformations one at a time to the original function:

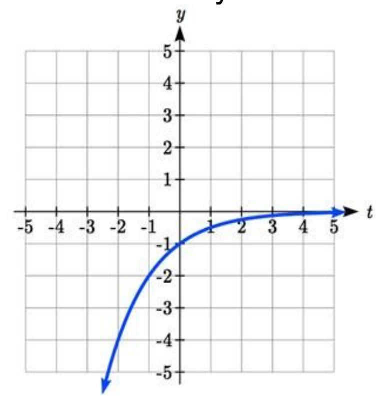
The original graph:



Horizontally reflected:

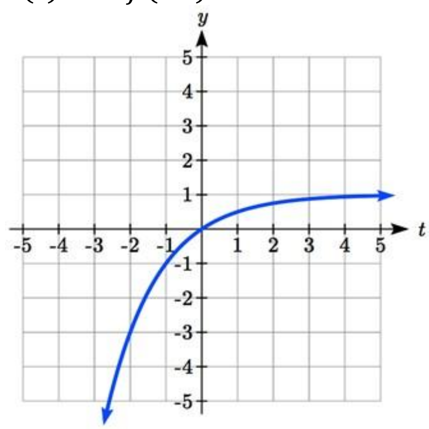


Then vertically reflected



Then, after shifting up 1, we get the final graph.

$$k(t) = -f(-t) + 1 = -2^{-t} + 1$$



Note: As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range [0,1).

Scalings

We now turn our attention to our last class of transformations: **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**.

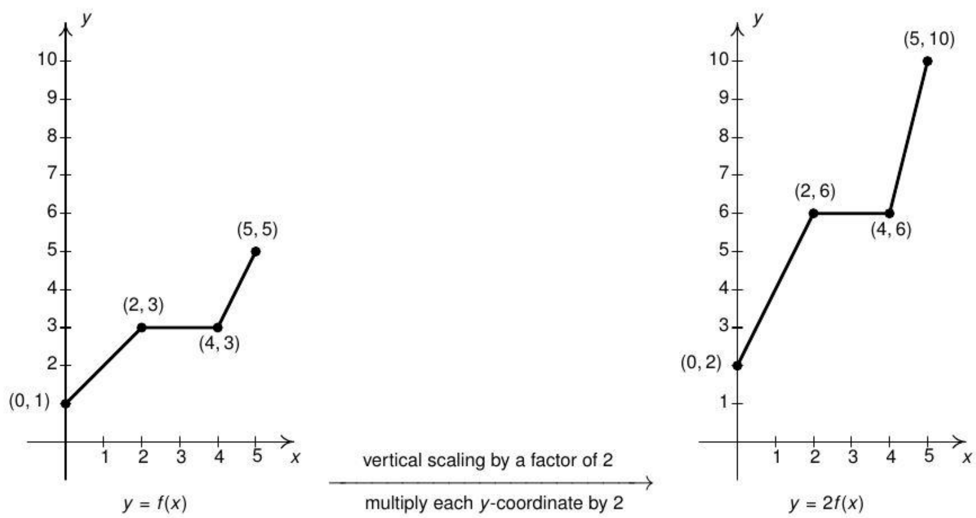
Simply put, rigid transformations preserve the distances between points on the graph - only their position and orientation in the plane change. If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be affecting the distance between points. These sorts of transformations are hence called **non-rigid**. As always, we motivate the general theory with an example.

Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph, we can build a table of values for g as before.

x	$f(x)$	$g(x) = 2f(x)$
0	1	2
2	3	6
4	3	6
5	5	10

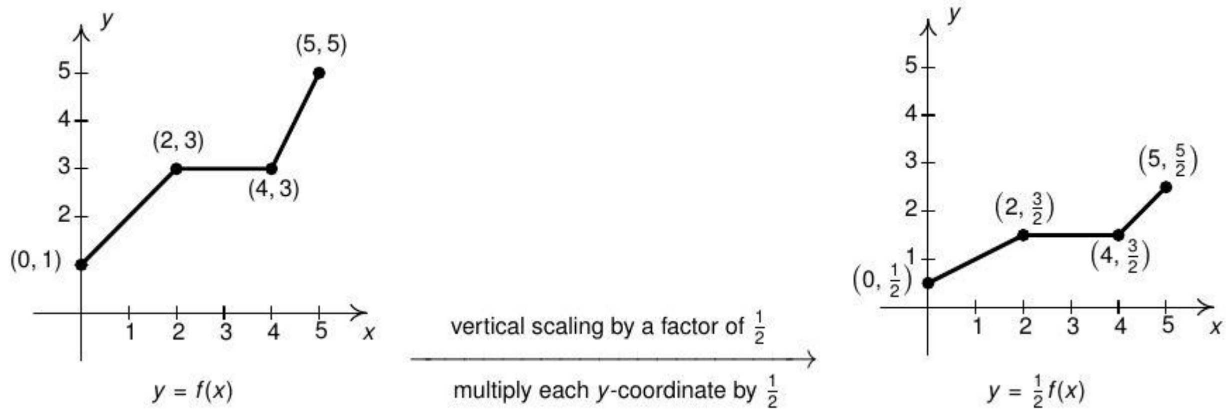
$(x, g(x))$
(0,2)
(2,6)
(4,6)
(5,10)

Graphing, we get:



In general, if (a, b) is on the graph of f, then $f(a) = b$so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling by a factor of 2', or a vertical stretch.

If we wish to graph $y = \frac{1}{2}f(x)$,we multiply the all of the y-coordinates of the points on the graph of f by ½ . This creates a 'vertical scaling by a factor of $1/2$', or a vertical compression, as seen below.



We generalize these results below.

Vertical Scalings.

Suppose f is a function and a>0 is a real number.

To graph $F(x) = af(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by a.

- If $a > 1$, we say the graph of f has undergone a vertical stretch by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrink by a factor of $\frac{1}{a}$.

Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to our horizontal shifts, as seen in the following table.

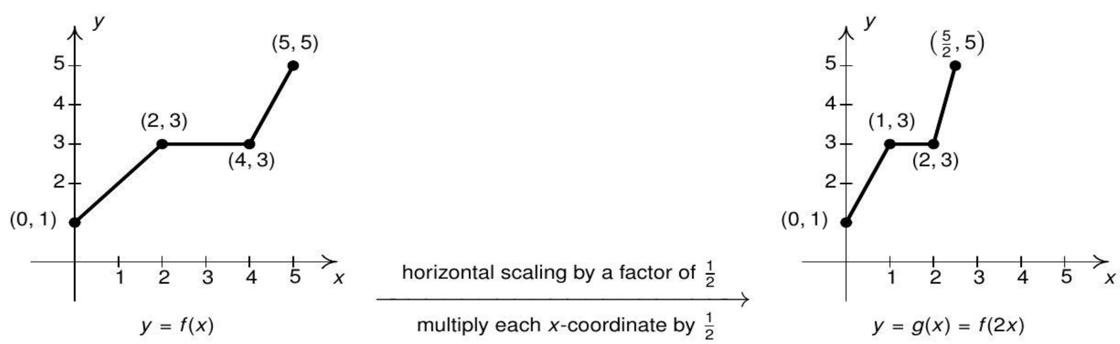
x	$f(x)$	$g(x) = f(2x)$
0	1	$f(2 \cdot 0) = f(0) = 1$
2	3	$f(2 \cdot 2) = f(4) = 3$
4	3	$f(2 \cdot 4) = f(8) = ?$
5	5	$f(2 \cdot 5) = f(10) = ?$

$(x, g(x))$
(0,1)
(2,3)

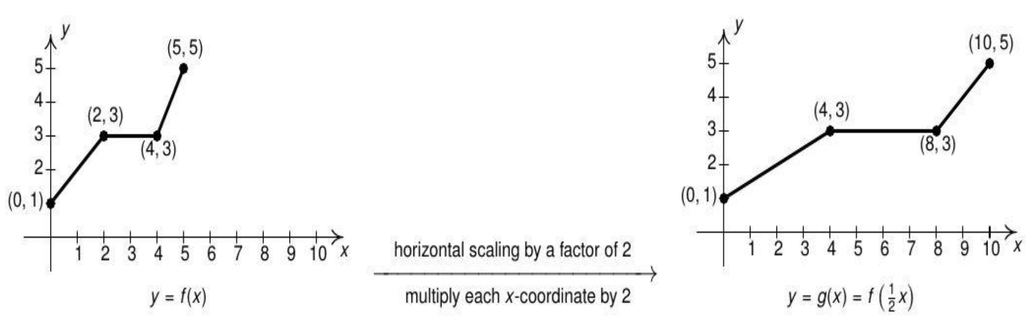
We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point (2,3) on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1,3)$ is on the graph of g.

In general, if (a,b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling by a factor of $1/2$.

$\boldsymbol{X = \frac{x}{2}}$	$\boldsymbol{x}$	$\boldsymbol{g(X) = f(2X) = f(x)}$	$\boldsymbol{(X, g(X))}$
0	0	$g(0) = f(2 \cdot 0) = f(0) = 1$	$(0, 1)$
1	2	$g(1) = f(2 \cdot 1) = f(2) = 3$	$(1,3)$
2	4	$g(2) = f(2 \cdot 2) = f(4) = 3$	$(2,3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f\left(2 \cdot \frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$



If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling by a factor of 2, as demonstrated below.



We generalize these results below.

Horizontal Scalings.

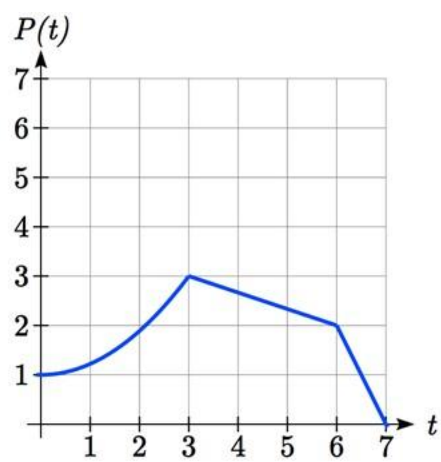
Suppose f is a function and $b > 0$ is a real number.

To graph $F(x) = f(bx)$, divide each of the x-coordinates of the points on the graph of $y = f(x)$ by b.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretch by a factor of $\frac{1}{b}$
- If $b > 1$, we say the graph of f has undergone a horizontal shrink by a factor of b.

Example 8

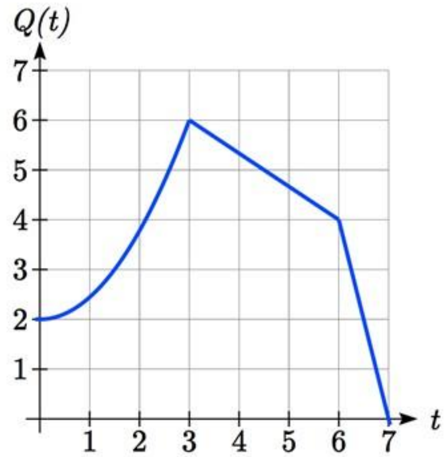
A function $P(t)$ models the growth of a population of fruit flies. The growth is shown in the graph. A scientist is comparing this to another population, Q, that grows the same way, but starts twice as large. Sketch a graph of this population.



Solution:
Since the population is always twice as large, the new population's output values are always twice the original function output values. Graphically, this would look like the second graph shown.

Symbolically, $Q(t) = 2P(t)$.

This means that for any input t, the value of the Q function is twice the value of the P function. Notice the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.



Example 9

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

x	f(x)
2	1
4	3
6	7
8	11

Solution:

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then $g(4) = \frac{1}{2}\ f(4) = \frac{1}{2}(3) = \frac{3}{2}$.

x	$y = f(x)$	$g(x) = \frac{1}{2} \cdot\ y$	$(x, g(x))$
2	1	$1 \cdot \frac{1}{2} = \frac{1}{2}$	$\left(2, \frac{1}{2}\right)$
4	3	$3 \cdot \frac{1}{2} = \frac{3}{2}$	$\left(4, \frac{3}{2}\right)$
6	7	$7 \cdot \frac{1}{2} = \frac{7}{2}$	$\left(6, \frac{7}{2}\right)$
8	11	$11 \cdot \frac{1}{2} = \frac{11}{2}$	$\left(8, \frac{11}{2}\right)$

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value has been cut in half, so the graph would now be half the original height.

Example 10

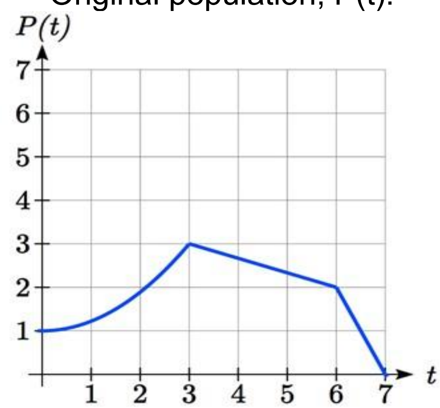
Returning to the fruit fly population we looked at earlier, suppose the scientist is now comparing it to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount the original population did in 2 hours, and in 2 hours, will progress as much as the original population did in 4 hours. Sketch a graph of this population.

Solution:
Symbolically, we could write $R(1) = P(2)$ $R(2) = P(4)$, and in general,

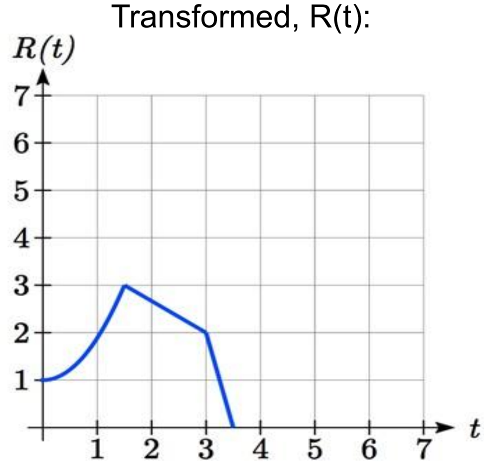
$$R(t) = P(2t).$$

Graphing this, we have

Original population, P(t):



Transformed, R(t):



Note the effect on the graph is a horizontal compression, where all input values are half their original distance from the vertical axis.

Example 11

A function f(x) is given as a table below. Create a table for $g(x) = f\left(\frac{1}{2}x\right)$.

x	f(x)
2	1
4	3
6	7
8	11

Solution:

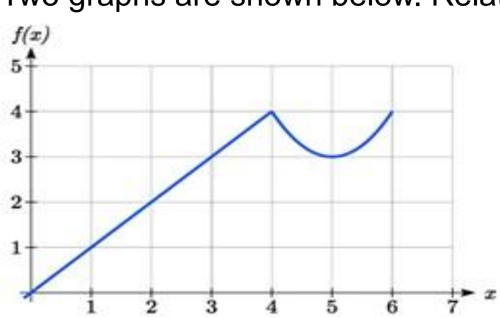
The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we don't have enough information to determine since $g(2) = f\left(\frac{1}{2} \cdot 2\right) = f(1)$, and we do not have a value for in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine since $g(4) = f\left(\frac{1}{2} \cdot 4\right) = f(2) = 1$. Simply put, we'll multiply all our x-values by 2.

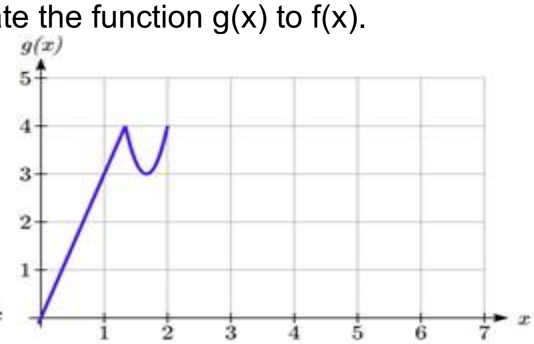
$X = 2 \cdot x$	$\leftarrow x$	$f(x)$	$(X, g(X))$
4	2	1	$(4, 1)$
8	4	3	$(8, 3)$
12	6	7	$(12, 7)$
16	8	11	$(16, 11)$

Since each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by 2.

Example 12

Two graphs are shown below. Relate the function g(x) to f(x).





Solution:
The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Since $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4) we can see that the x values have been compressed by 1/3, because $6\left(\frac{1}{3}\right) = 2$. We might also notice that $g(2) = f(6)$, and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by 1/3.

Transformations in Sequence

Now that we have studied three basic classes of transformations: shifts, reflections, and scalings, we present a result below which provides one algorithm to follow to transform the graph of $y = f(x)$ into the graph of $y = af(bx + c) + d$.

Transformations in Sequence.

Suppose f is a function. If a, $b \neq 0$, then to graph $g(x) = a\,f(bx + c) + d$ start with the graph of $y = f(x)$ and follow the steps below.

1. Subtract c from each of the x-coordinates of the points on the graph of f.

Note: This results in a horizontal shift to the left if $c < 0$ or right if $c > 0$.

2. Divide the x-coordinates of the points on the graph obtained in Step 1 by b.

Note: This results in a horizontal scaling of factor $\frac{1}{b}$, but includes a reflection about the y-axis if $b < 0$.

3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by a.

Note: This results in a vertical scaling of factor a, but includes a reflection about the x-axis if $a < 0$.

4. Add d to each of the y-coordinates of the points on the graph obtained in Step 3.

Note: This results in a vertical shift up if $d > 0$ or down if $d < 0$.

A convenient way to remember this order is "C-BAD".

We should note that technically, the horizontal (b and c) and vertical (a and d) transformations are independent of one another, so it doesn't matter whether we do the horizontal or the vertical first. For consistency's sake, though, we'll stick with the "C-BAD" order from here on out.

Example 13

Given the table of values for the function $f(x)$ below, create a table of values for the function $g(x) = 2f(3x) + 1$.

x	$y = f(x)$
6	10
12	14
18	15
24	17

Solution:

$Since\ g(x) = 2f(3x) + 1$, we have $a = 2, b = 3$, and $d = 1$.
Technically, $c = 0$, so we'll skip that transformation.
We'll put our original, parent values in the center of our table and work outward from there in C-BAD order.

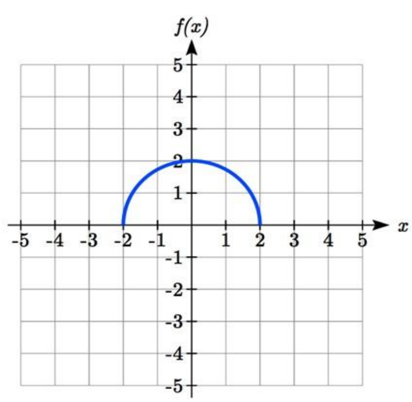
$X = \frac{1}{3}x$	$\leftarrow x$	$y = f(x) \rightarrow$	$2 \cdot y \rightarrow$	$2 \cdot y + 1$
2	6	10	20	21
4	12	14	28	29
6	18	15	30	31
8	24	17	34	35

The outer columns then form our final, transformed function:

$\boldsymbol{X}$	$\boldsymbol{g(X)}$
2	21
4	29
6	31
8	35

Example 14

Using the graph of $f(x)$ below, sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.



Solution:

We'll start by identifying a, b, c, and d. Here we have $b = \frac{1}{2}, c = 1$, and $d = -3$. Technically $a = 1$, but that won't reflect or stretch the graph, so we'll ignore it. We can follow our same C-BAD process if we pick out a few points on our graph to use as the parent function. We'll use (-2, 0), (0, 2), and (2, 0). We'll put those parent values in the center of our table, and work outwards from there in C-BAD order.

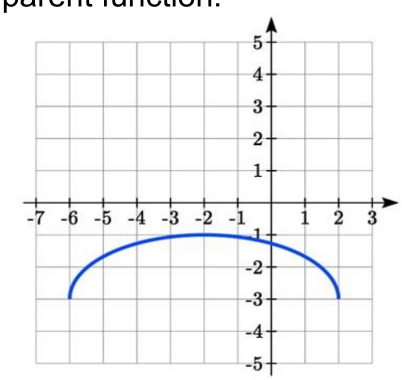
First, subtract 1 from the x-values (a shift to the left). Then divide the new x-values by $\frac{1}{2}$ (which is the same as multiplying by the reciprocal, 2, a horizontal stretch). Finally, we'll subtract 3 from the y-values (a vertical shift down).

$X = (x-1) \cdot 2$	$\leftarrow x - 1$	$\leftarrow x$	$y = f(x) \rightarrow$	$y - 3$
-6	-3	-2	0	-3
-2	-1	0	2	-1
2	1	2	0	-3

So, our final table for $k(x)$, taken from the outer columns of our transformation, is:

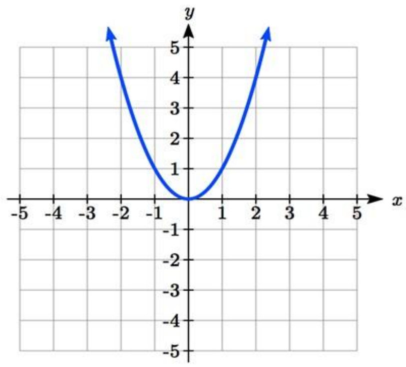
X	$k(X)$
-6	-3
-2	-1
2	-3

We can now plot those points and connect them with the same semicircular shape that we saw in the parent function:

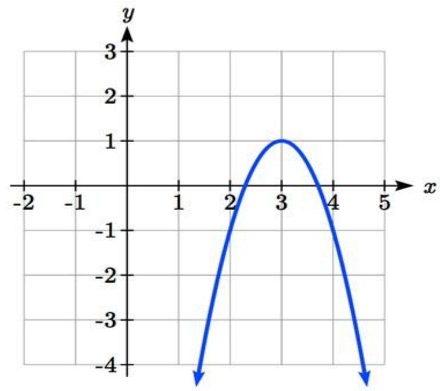


Example 15

Given the original function $f(x)$ and transformed function $g(x)$ pictured below, find a, b, c, and d so that $g(x) = a\,f(b\,x\ +\ c)\ +\ d$.



$f(x)$



$g(x)$

Solution:
The original function has been flipped over the x axis, some kind of stretch or compression has occurred, and we can see a shift to the right 3 units and a shift up 1 unit.

In total there are four operations:
- Vertical reflection, requiring a negative sign outside the function
- Vertical Stretch or Horizontal Compression*
- Horizontal Shift Right 3 units, which tells us to put x-3 on the inside of the function
- Vertical Shift up 1 unit, telling us to add 1 on the outside of the function

* It is unclear from the graph whether it is showing a vertical stretch or a horizontal compression. For this shape, it turns out we could represent it either way, so we'll use a vertical stretch. You may be able to determine the vertical stretch by observation.

By observation, the basic function has a vertex at (0, 0) and symmetrical points at (1, 1) and (-1, 1). These points are one unit up and one unit over from the vertex. The new points on the transformed graph are one unit away horizontally but 2 units away vertically. They have been stretched vertically by two.

So, a vertical stretch of 2 means $a = 2$. A reflection across the x-axis furthermore means this a value should be *negative*. A move *right* means that our c value needs to be *negative* 3. A move up 1 means $d = 1$. Therefore g$(x) = -2f(x - 3) + 1$.

Important Topics of this Section
Transformations
Vertical Shift (up & down)
Horizontal Shifts (left & right)
Reflections over the vertical & horizontal axis
Vertical Stretches & Compressions
Horizontal Stretches & Compressions
Combinations of Transformation

Section 1.7 Exercises

Conceptual Questions

1. In the structure $g(x) = a\,f(bx + c) + d$, what do each of the constants a, b, c, and d do?
2. In the structure $g(x) = af(bx + c) + d$, what is special about the way that b and c work, versus a and d?
3. In the structure $g(x) = af(bx + c) + d)$, what values affect the x's? What values affect the y's?

Practice Problems

Describe how each function is a transformation of the original function $\boldsymbol{f(x)}$.

1. $f(x - 49)$
2. $f(x + 43)$
3. $f(x + 3)$
4. $f(x - 4)$
5. $f(x) + 5$
6. $-f(x)$
7. $f(5x)$
8. $3f(-x)$
9. $6f(x)$
10. $f\left(\frac{1}{5}x\right)$
11. $f(x) + 8$
12. $f(x) - 2$
13. $f(x) - 7$
14. $f(x - 2) + 3$
15. $f(x + 4) - 1$
16. $4f(x)$
17. $f\left(\frac{1}{3}x\right)$
18. $f(-x)$
19. $f(2x)$
20. $-f(3x)$

21. Write a formula for f(x) shifted up 1 unit and left 2 units.
22. Write a formula for f(x) shifted down 3 units and right 1 unit.
23. Write a formula for f(x) shifted down 4 units and right 3 units.
24. Write a formula for f(x) shifted up 2 units and left 4 units.
25. Write a formula for f(x) reflected about the x-axis.
26. Write a formula for f(x) reflected about the y-axis.
27. Write a formula for f(x) reflected over the y axis and horizontally compressed by a factor of $\frac{1}{4}$.
28. Write a formula for f(x) reflected over the x axis and horizontally stretched by a factor of 2.

29. Write a formula for f(x) vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

30. Write a formula for f(x) vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

31. Write a formula for f(x), shifted to the right 5 units, horizontally compressed by a factor of 1/2, and shifted up 1 unit.

32. Write a formula for f(x) shifted to the left 4 units, horizontally stretched by a factor of 3, then shifted down 3 units.

33. Tables of values for f(x), g(x), and h(x) are given below. Write g(x) and h(x) as transformations of f(x).

x	$f(x)$
-2	-2
-1	-1
0	-3
1	1
2	2

x	$g(x)$
-1	-2
0	-1
1	-3
2	1
3	2

$\boldsymbol{x}$	$\boldsymbol{h(x)}$
-2	-1
-1	0
0	-2
1	2
2	3

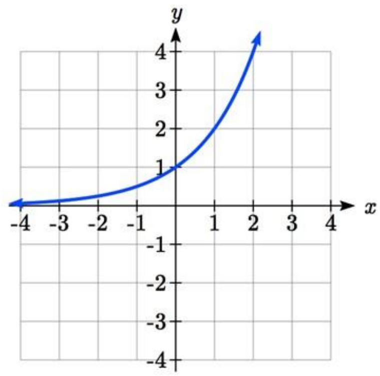
34. Tables of values for f(x), g(x), and h(x) are given below. Write g(x) and h(x) as transformations of f(x).

x	$f(x)$
-2	-1
-1	-3
0	4
1	2
2	1

x	$g(x)$
-3	-1
-2	-3
-1	4
0	2
1	1

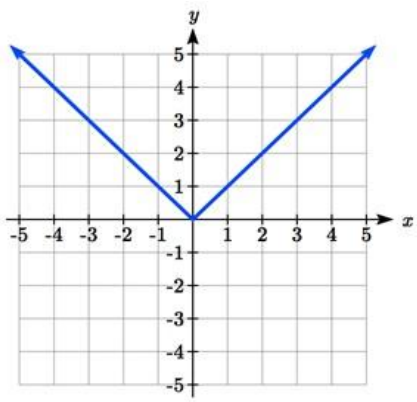
x	$h(x)$
-2	-2
-1	-4
0	3
1	1
2	0

35. The graph of $f(x) = 2^x$ is shown. Sketch a graph of each transformation of f(x). Then describe the domain and range of each transformed graph.



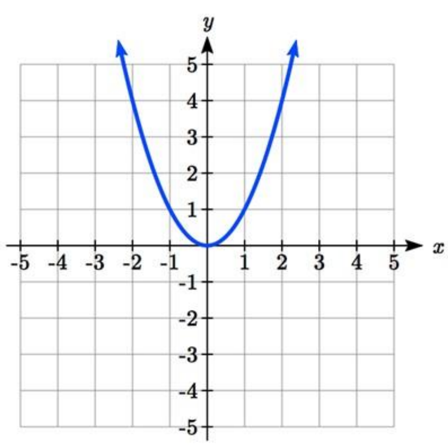
a. $g(x) = f(x) + 1$
b. $h(x) = f(x) - 3$
c. $w(x) = f(x-1)$
d. $q(x) = f(x+3)$
e. $r(x) = -f(x) + 1$
f. $j(x) = f(-x)$

36. The graph of $f(x) = |x|$ is shown. Sketch a graph of each transformation of f(x). Then, describe the interval(s) on which the function is increasing or decreasing.



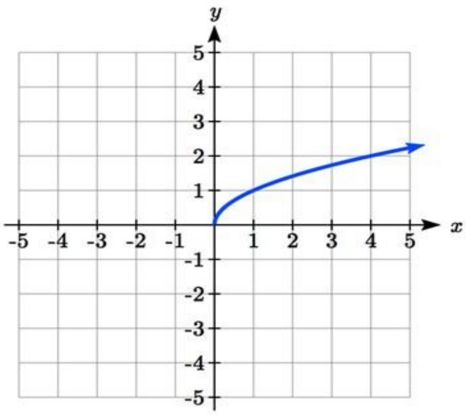
a. $g(x) = f(x+1) - 3$
b. $h(x) = f(x-1) + 4$
c. $k(x) = f(x-2) - 1$
d. $m(x) = 3 + f(x+2)$
e. $p(x) = -2f(x-4) + 3$
f. $n(x) = \frac{1}{3} f(x-2)$

37. The graph of $f(x) = x^2$ is shown. Sketch a graph of each transformation of f(x). Then, describe the interval(s) on which the function is increasing or decreasing. Finally, determine the interval(s) on which the function is concave up and concave down.



a. $g(x) = 4f(x+1) - 5$
b. $h(x) = 5f(x+3) - 2$
c. $p(x) = f\left(\frac{1}{3}x\right) - 3$

38. The graph of $f(x) = \sqrt{x}$ is shown. Sketch a grph of each transformation of f(x). Then, describe the interval(s) on which the function is increasing or decreasing. Finally, determine the interval(s) on which the function is concave up and concave down.

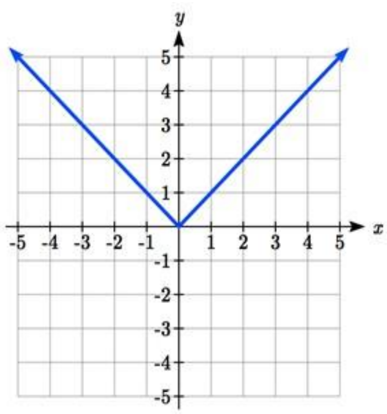


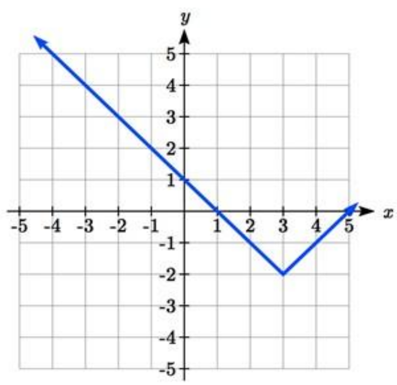
a. $k(x) = -3f(x) - 1$

b. $a(x) = f(-x + 4)$

Given each graph of f(x) and g(x) below, find a, b, c, and d such that $\boldsymbol{g(x) = af(bx + c) + d}$.

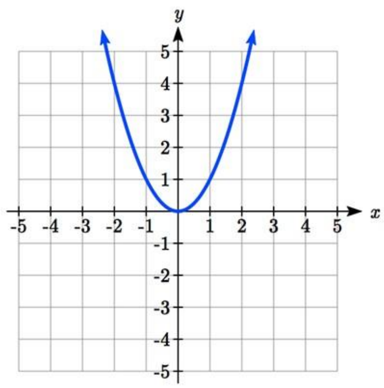
39. f(x) g(x)

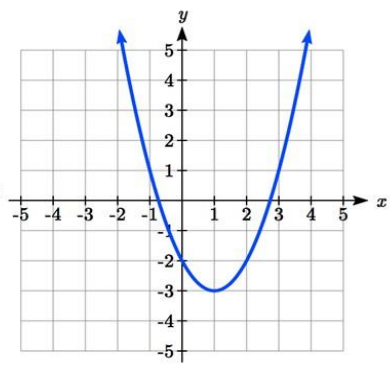




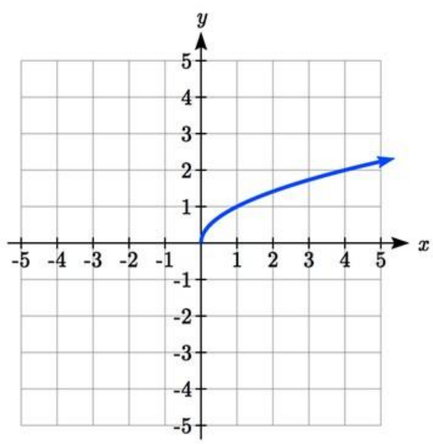
Given each graph of f(x) and g(x) below, find a, b, c, and d such that $g(x) = af(bx + c) + d$.

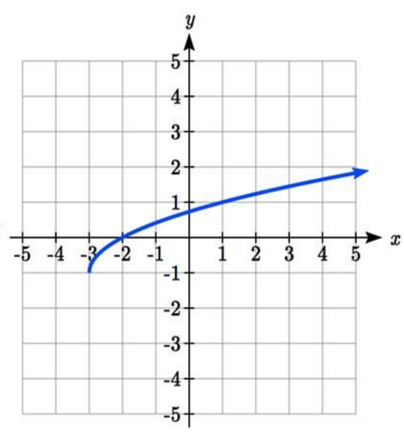
40. f(x) g(x)



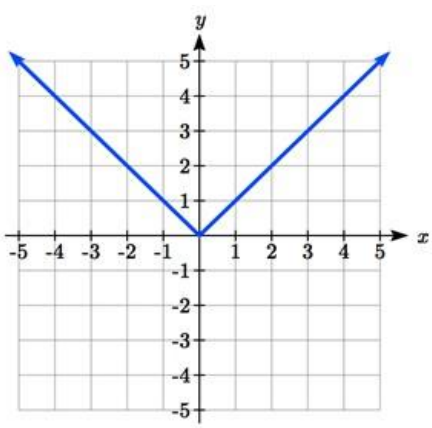


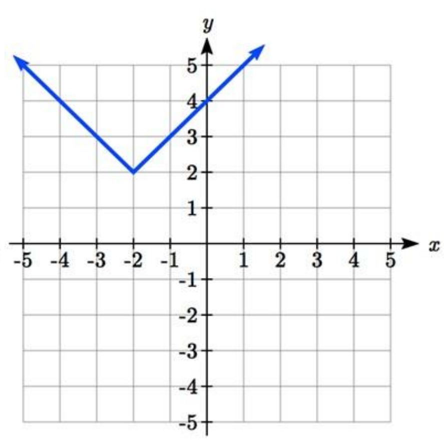
41. f(x) g(x)





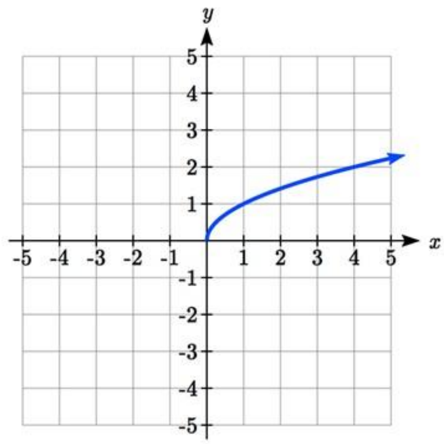
42. f(x) g(x)

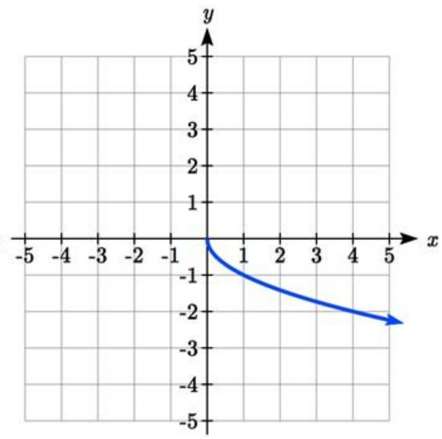




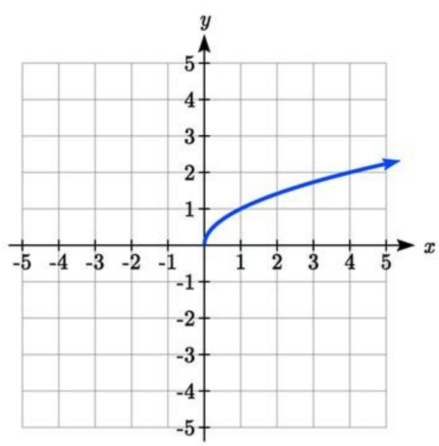
Given each graph of f(x) and g(x) below, find a, b, c, and d such that $\boldsymbol{g(x) = af(bx + c) + d}$.

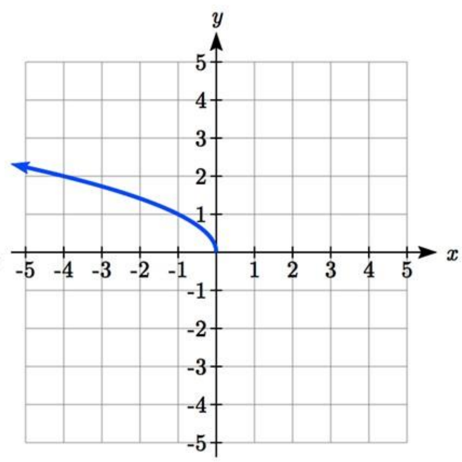
43. f(x) g(x)



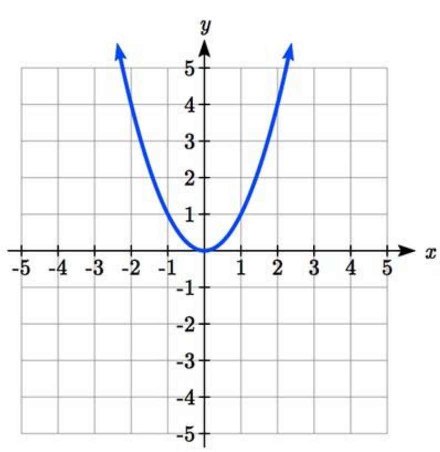


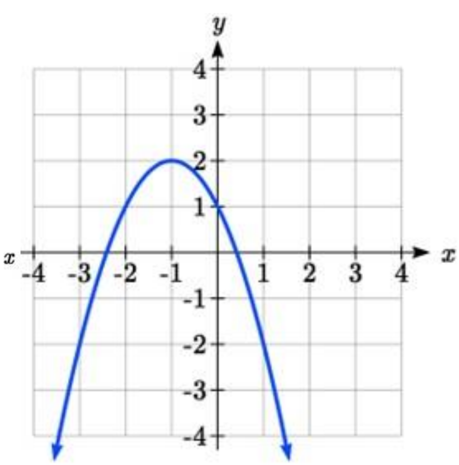
44. f(x) g(x)





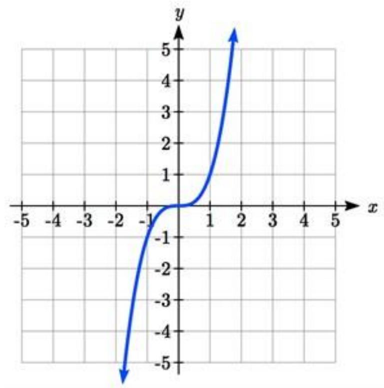
45. f(x) g(x)

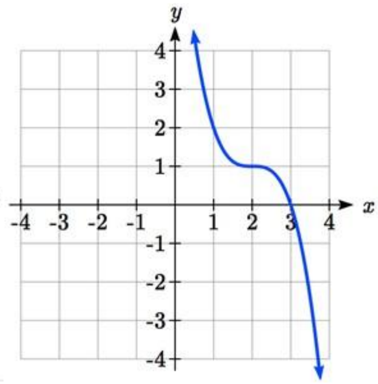




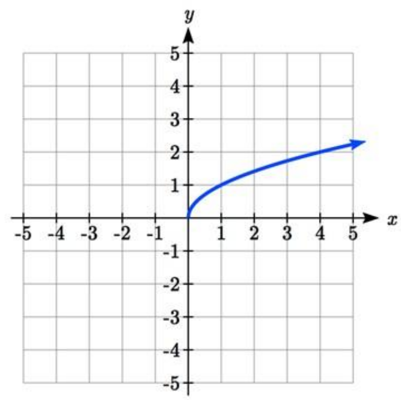
Given each graph of f(x) and g(x) below, find a, b, c, and d such that $\boldsymbol{g(x) = af(bx + c) + d}$.

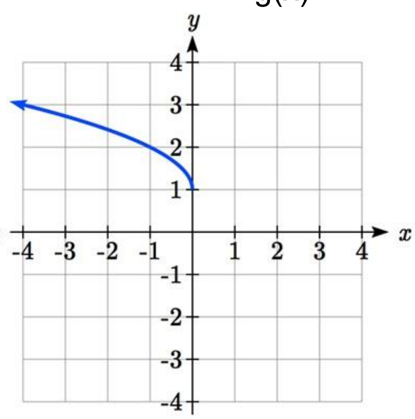
46. f(x) g(x)



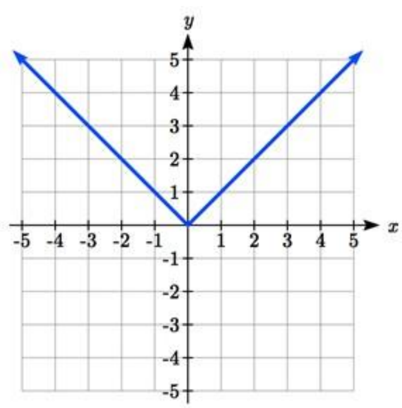


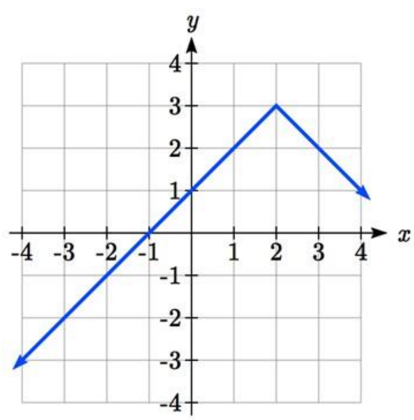
47. f(x) g(x)



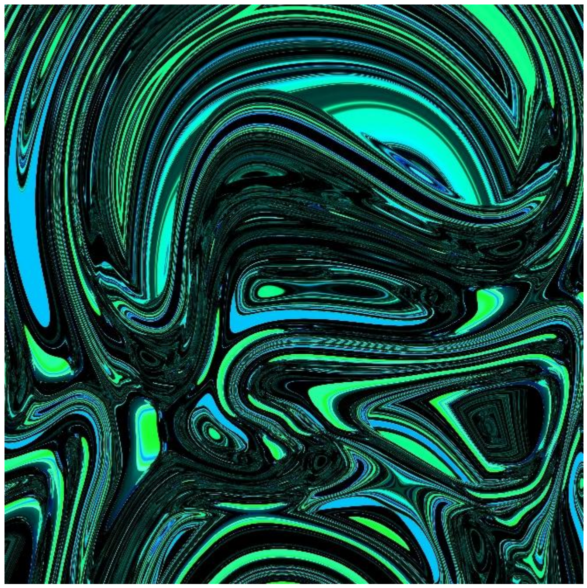


48. f(x) g(x)





Chapter 2: Linear and Polynomial Functions



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Chapter one was a window that gave us a peek into the entire course. Our goal was to understand the basic structure of functions and function notation, the toolkit functions, domain and range, how to recognize and understand composition and transformations of functions and how to understand and utilize inverse functions. With these basic components in hand we will further research the specific details and intricacies of each type of function in our toolkit and use them to model the world around us.

Mathematical Modeling
As we approach day to day life we often need to quantify the things around us, giving structure and numeric value to various situations. This ability to add structure enables us to make choices based on patterns we see that are weighted and systematic. With this structure in place we can model and even predict behavior to make decisions. Adding a numerical structure to a real world situation is called **Mathematical Modeling.**

When modeling real world scenarios, there are some common growth patterns that are regularly observed. We will devote this chapter and the rest of the book to the study of the functions used to model these growth patterns.

Section 2.1 Linear Functions

As you hop into a taxicab in Las Vegas, the meter will immediately read \$3.50; this is the "drop" charge made when the taximeter is activated. After that initial fee, the taximeter will add \$2.76 for each mile the taxi drives[6]. In this scenario, the total taxi fare depends upon the number of miles ridden in the taxi, and we can ask whether it is possible to model this type of scenario with a function. Using descriptive variables, we choose *m* for miles and *C* for Cost in dollars as a function of miles: *C(m)*.

We know for certain that $C(0) = 3.50$, since the \$3.50 drop charge is assessed regardless of how many miles are driven. Since \$2.67 is added for each mile driven, then

$$C(1) = 3.50 + 2.67 = 6.17.$$

If we then drove a second mile, another \$2.67 would be added to the cost:

$$C(2) = 3.50 + 2.67 + 2.67 = 3.50 + 2.67(2) = 8.84$$

If we drove a third mile, another \$2.67 would be added to the cost:

$$C(3) = 3.50 + 2.67 + 2.67 + 2.67 = 3.50 + 2.67(3) = 11.51$$

From this we might observe the pattern, and conclude that if *m* miles are driven,$C(m) = 3.50 + 2.67m$ because we start with a \$3.50 drop fee and then for each mile increase we add \$2.67.

It is good to verify that the units make sense in this equation. The \$3.50 drop charge is measured in dollars; the \$2.67 charge is measured in dollars per mile.

$$C(m) = 3.50 \text{ dollars} + \left(2.67 \frac{\text{dollars}}{\text{mile}}\right)(m \text{ miles})$$

When dollars per mile are multiplied by a number of miles, the result is a number of dollars, matching the units on the 3.50, and matching the desired units for the *C* function.

Notice this equation $C(m) = 3.50 + 2.67m$ consisted of two quantities. The first is the fixed \$3.50 charge which does not change based on the value of the input. The

[6] Nevada Taxicab Authority, retrieved Aug 4, 2020. There is also a waiting fee assessed when the taxi is waiting at red lights, but we'll ignore that in this discussion.

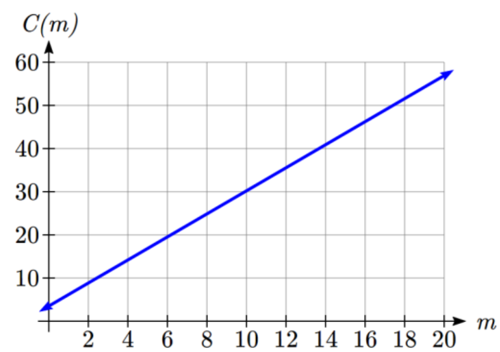
second is the \$2.67 dollars per mile value, which is a **rate of change**. In the equation, this rate of change is multiplied by the input value.

Looking at this same problem in table format we can also see the cost changes by \$2.67 for every 1 mile increase.

m	0	1	2	3
C(m)	3.50	6.17	8.84	11.51

It is important here to note that in this equation, the **rate of change is constant**; over any interval, the rate of change is the same.

Graphing this equation, $C(m) = 3.50 + 2.67m$ we see the shape is a line, which is how these functions get their name: **linear functions**.



When the number of miles is zero the cost is \$3.50, giving the point (0, 3.50) on the graph. This is the vertical or *C(m)* intercept. The graph is increasing in a straight line from left to right because for each mile the cost goes up by \$2.67; this rate remains consistent.

In this example, you have seen the taxicab cost modeled in words, an equation, a table and in graphical form. Whenever possible, ensure that you can link these four representations together to continually build your skills. It is important to note that you will not always be able to find all 4 representations for a problem and so being able to work with all 4 forms is very important.

Slope

That first example taught us some important facts about linear functions. One crucial idea is that the rate of change is constant. In fact, we give the rate of change a special name: the slope of the function.

Slope: Rate of Change in Linear Functions
Linear functions have a **constant** rate of change. That is, no matter what pair of points we use to calculate an average rate of change for a linear function, we will get the same average rate of change every time. This constant rate of change of a linear function is called the **slope** and is often described with the variable m.

Slope and Increasing/Decreasing

m is the constant rate of change of the function (also called **slope**). The slope determines if the function is an increasing function or a decreasing function. For a linear function, $f(x)$,

$f(x)$ is an **increasing** function if $m > 0$
$f(x)$ is a **decreasing** function if $m < 0$

If $m = 0$, the rate of change is zero, and the function $f(x)$ is just a horizontal line, neither increasing nor decreasing.

Calculating Slope (rate of change)

Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2, or a set of points, (x_1, y_1) and (x_2, y_2), we can calculate the slope (the constant rate of change), m:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

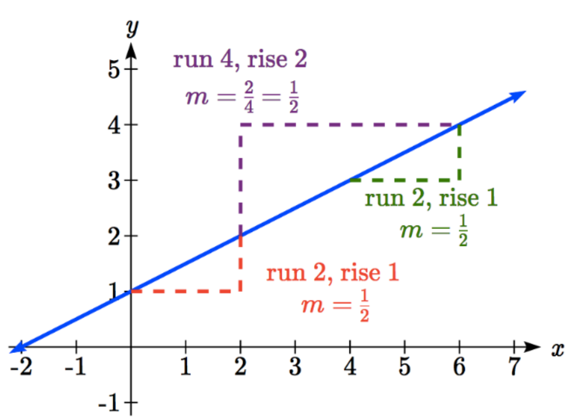
Note in function notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

One of the ways of thinking about the slope is

$$m = \frac{\text{change of output}}{\text{change of input}} = \frac{rise}{run}$$

We can see this on a graph by stair-stepping our way from one point to another. The vertical distance is the “rise” and the horizontal distance is the “run.”



Example 1

The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

Solution:

The rate of change will relate the change in population to the change in time. The population increased by $27800 - 23400 = 4400$ people over the 4 year time interval. To find the rate of change, the number of people per year the population changed by:

$$\frac{4400\text{ people}}{4\text{ years}} = 1100\frac{\text{people}}{\text{year}} = 1100\text{ people per year}$$

Notice that we knew the population was increasing, so we would expect our value for *m* to be positive. This is a quick way to check to see if your value is reasonable.

Example 2

The pressure, *P*, in pounds per square inch (PSI) on a diver depends upon their depth below the water surface, *d*, in feet, following the equation $P(d) = 14.696 + 0.434d$. Interpret the components of this function.

Solution:

The rate of change, or slope, 0.434 would have units $\frac{\text{output}}{\text{input}} = \frac{\text{pressure}}{\text{depth}} = \frac{\text{PSI}}{\text{ft}}$. This tells us the pressure on the diver increases by 0.434 PSI for each foot their depth increases.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 PSI.

Example 3

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find the rate of change.

Solution:

$f(3) = -2$ tells us that the input 3 corresponds with the output -2, and $f(8) = 1$ tells us that the input 8 corresponds with the output 1. To find the rate of change, we divide the change in output by the change in input:

$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1-(-2)}{8-3} = \frac{3}{5}$. If desired we could also write this as *m* = 0.6

Note that it is not important which pair of values comes first in the subtractions so long as the first output value used corresponds with the first input value used.

Try it Now

Given the two points (2, 3) and (0, 4), find the rate of change. Is this function increasing or decreasing?

Slope-Intercept Form

There are different ways that we can think of a linear function's equation. Different formats of the equation can reveal different features of the function.

As we study different functions, let's focus on the idea that

Algebra isn't really only about "solving for x." Algebra allows us to rearrange functions into different forms so we can understand

We'll focus first on a special form of a linear function: the slope-intercept form.

Slope-Intercept Form of a Linear Function

A **linear function** is a function whose graph produces a line. Linear functions can always be written in the form

$$f(x) = mx + b$$

where

b is the vertical **intercept**, i.e. the initial or starting value of the function, and

m is the **slope**, i.e. the constant rate of change of the function

Since this format of a linear function readily reveals the slope, *m*, and the vertical intercept, *b,* the equation $f(x) = mx + b$ is called the **slope-intercept** form of a line.

Many people like to write linear functions in the form $f(x) = b + mx$ because it corresponds to the way we tend to speak: "The output starts at *b* and increases at a rate of *m*."

If we think of transformations then the form $f(x) = mx + b$ tells us the vertical stretch is a factor of m and the vertical shift is b.

Example 4

Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, *N*, in his iTunes collection as a function of the number of months, *m*. How many songs will he own in a year?

Solution:

The initial value for this function is 200, since he currently owns 200 songs, so $N(0) = 200$. The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. With this information, we can write the formula:

$$N(m) = 15m + 200.$$

N(m) is an increasing linear function.

With this formula we can predict how many songs he will have in 1 year (12 months):

$$N(12) = 15(12) + 200 = 180 + 200 = 380.$$

Marcus will have 380 songs in 12 months.

Try it Now

If you earn $30,000 per year and you spend $29,000 per year write an equation for the amount of money you save after *y* years, if you start with nothing.
"The most important thing, spend less than you earn![7]"

Example 5

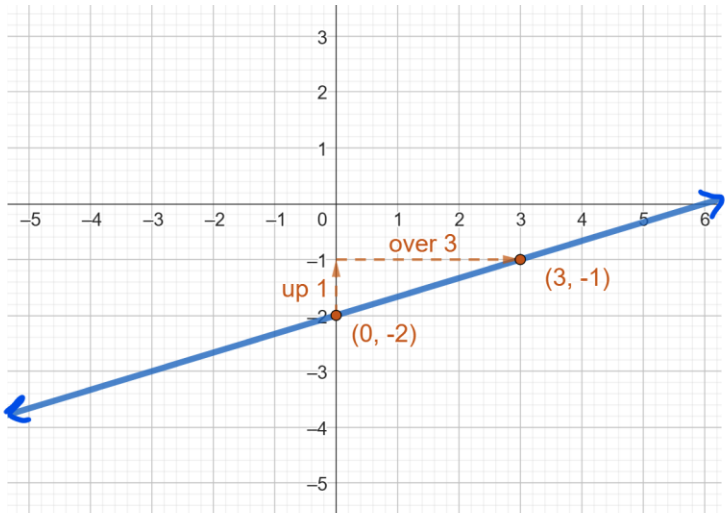
Graph the equation of the line $f(x) = \frac{1}{3}x - 2$

Solution:

Let's use what we know about the $f(x) = mx + b$ form of a line.

Here, we know the y-intercept is -2, and the slope is $\frac{1}{3}$. So, we could start at the point -2 on the y-axis, and then find another point by going up 1 and over 3, since the slope describes the rise over the run. Connect those points, and we have the graph of our line!

[7] http://www.thesimpledollar.com/2009/06/19/rule-1-spend-less-than-you-earn/



We can now find the rate of change given two input-output pairs, and can write an equation for a linear function once we have the rate of change and initial value. There is another format of the equation of a line that can also be helpful if we know the slope of the line and a point which lies on the line.

Point-Slope Form and the Toolkit Linear Function

Point-Slope form of a Linear Function
If a line passes through the point (c, d) and has slope m, then an equation for the line is $$f(x) = m(x - c) + d$$

The point-slope form of a linear equation is directly connected to transformations. We can think of this as a linear equation where the "starting point" (0,0) has be shifted c units to the right or left, and d units up or down. Then the graph is vertically stretched by the slope, m.

In fact, let's take a moment to create a toolkit linear function that we can use to think about these transformations.

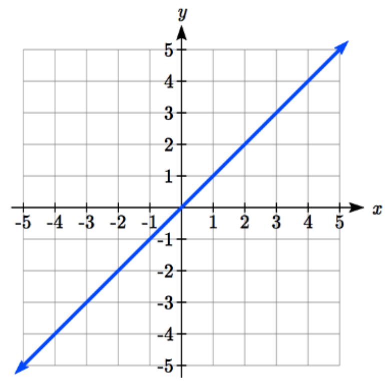
Toolkit Linear Function

Toolkit Linear Function
The "toolkit" linear function is $$f(x) = x.$$

It is sometimes called the "identity" function.

This function passes through the points, and its graph is a straight, constantly increasing line.

x	$y = f(x)$
-2	2
-1	1
0	0
1	1
2	2



Features of the toolkit linear function

- Intercepts: The x and y intercept is (0,0)
- Domain: $(-\infty, \infty)$
- Range: $(-\infty, \infty)$
- End behavior: As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to \ -\infty$
- Increasing on $(-\infty, \infty)$
- Origin symmetry (odd)

Example 6

Graph the equation of the line $g(x) = -3(x + 1) - 2$.

Solution:

We'll take this from two points of view: transformations of the toolkit, and the point-slope form of the equation.

First, if we want to use transformation techniques, we want to identify the parent function. This means, we want to decide, what structure is the input a part of? Here, our input x isn't really a part of anything special (not in a power, or a root, or anything like that). Our parent function is the linear function, $f(x) = x$.

Our transformed function, g, can then be written as

$$\begin{aligned} g(x) &= a\, f(bx + c) + d \\ &= -3(x + 1) - 2. \end{aligned}$$

From this, we see that $a = -3$, $c = 1$, and $d = 2$. This means our toolkit function will be shifted left 1, flipped around the x-axis and vertically stretched by 3, and shifted down 2.

$x - 1$	$\leftarrow x$	$f(x) = x \rightarrow$	$-3 \cdot y \rightarrow$	$y - 2$
$-2 - 1 = -3$	-2	-2	$-3 \cdot -2 = 6$	$6 - 2 = 4$
$-1 - 1 = -2$	-1	-1	$-3 \cdot -1 = 3$	$3 - 2 = 1$
$0 - 1 = -1$	0	0	$-3 \cdot 0 = 0$	$0 - 2 = -2$
$1 - 1 = 0$	1	1	$-3 \cdot 1 = -3$	$-3 - 2 = -5$
$2 - 1 = 1$	2	2	$-3 \cdot 2 = -6$	$-6 - 2 = -8$

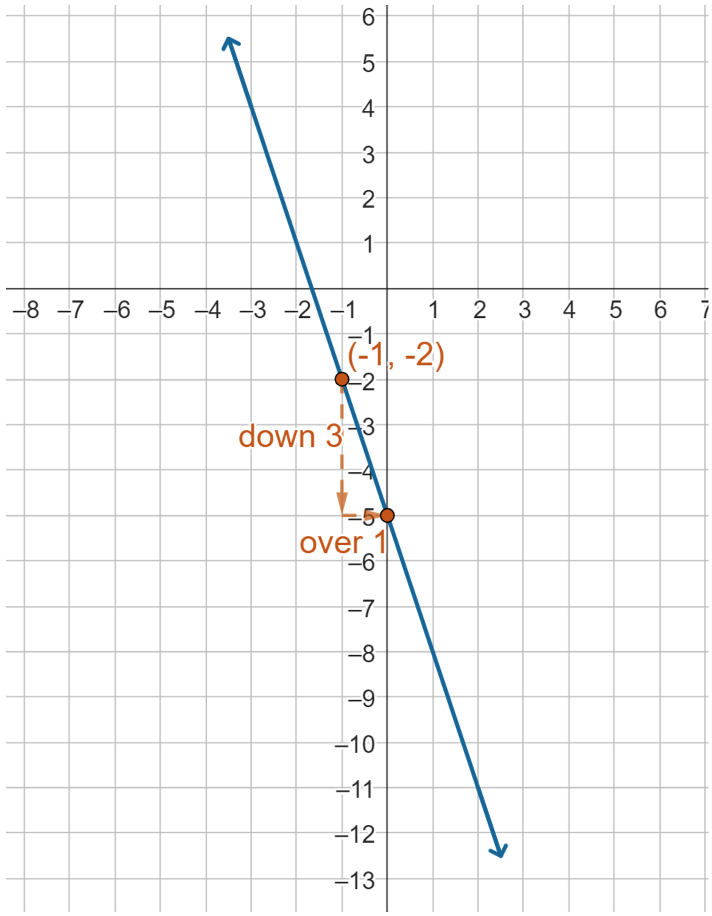
Our final table of values to graph then are

x	$g(x)$
-3	4
-2	1
-1	-2
0	-5
1	-8

Perhaps, though, things would be more simple if we considered the format of the equation and what it tells us about the line.

The equation $g(x) = -3(x+1) - 2$ looks most similar to the format $f(x) = m(x - c) + d$, which is a line through the point (c, d) with slope m. So, we know our line should pass through the point $(-1, -2)$ and have a slope $-\frac{3}{1}$. That means, we could graph the point $(-1, -2)$, then go down three and right one to find the next point.

Following these directions, we produce this line:



Note that this line passes through all the points indicated in the table we made using transformations!

In a moment, we'll describe the features of the function from the previous example. First, let's remind ourselves about how to find x- and y-intercepts.

Reminder: Intercepts
For a function $f(x)$...
To find the y-intercept, evaluate $f(0)$.
To find x-intercepts, solve $f(x) = 0$.

Example 7

Describe the features of the function $g(x) = -3(x+1) - 2$.

Solution:

We'll focus on the rate of change, whether the function is increasing or decreasing, the intercepts, the domain and range, and the end behavior.

First, the rate of change (aka the slope) is a constant -3. That means the function is always decreasing.

Now, for the intercepts. Our graph in the previous example clearly shows the y-intercept is at -5. We could also find this by evaluating $g(0) = -3(0+1) - 2 = -5$.

For the x-intercept, we'll need to solve $g(x) = 0$:

$$-3(x+1) - 2 = 0$$
$$-3x - 3 - 2 = 0$$
$$-3x - 5 = 0$$
$$-3x = 5$$
$$x = -\frac{5}{3}$$

That is consistent with the x-intercept we saw on the graph in previous example.

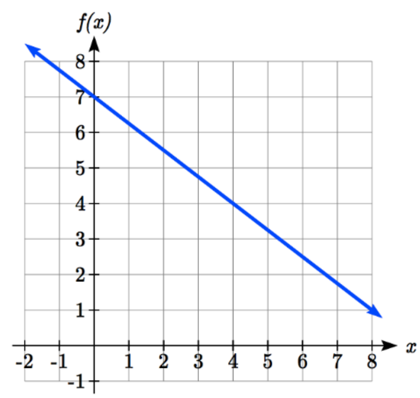
The domain and range are both $(-\infty, \infty)$.
Since the function is decreasing, for end behavior we see that as $x \to \infty, g(x) \to -\infty$ and as $x \to -\infty, g(x) \to \infty$. That is, the graph falls on the right, and rises on the left.

Try it Now

Graph the line $f(x) = \frac{2}{3}(x+2) - 1$. Then describe all its features.

Example 8

Write an equation for the linear function graphed to the right.



Solution:
Looking at the graph, we might notice that it passes through the points (0, 7) and (4, 4). From the first value, we know the initial value (y-intercept) of the function is *b* = 7, so in this case we will only need to calculate the rate of change (slope):

$$m = \frac{4-7}{4-0} = \frac{-3}{4}$$

Now we know the slope and the y-intercept; this allows us to write the slope-intercept equation:

$$f(x) = mx + b$$
$$f(x) = -\frac{3}{4}x + 7$$

We could, equivalently, use our calculated slope $m = -\frac{3}{4}$ and any point on the line to write an equation of the line. Let's use the point (4,4) and see what happens. If we have point $(4,4)$ and slope $-\frac{3}{4}$, then our equation would be

$$f(x) = m(x - c) + d$$
$$f(x) = -\frac{3}{4}(x - 4) + 4$$

But, is this really the same as our other equation, $f(x) = -\frac{3}{4}x + 7$? Let's simplify to find out.

$$f(x) = -\frac{3}{4}x + 3 + 4$$
$$= -\frac{3}{4}x + 7$$

Yes, they're the same!

The previous example is a good illustration of our big theory of algebra. Different forms of the equation of the line allowed us to understand different features of the line!

Example 9

If $f(x)$ is a linear function where $f(3) = -2$ and $f(8) = 1$, find an equation for the function. Then describe the features of the function.

Solution:

In example 3, we computed the rate of change to be $m = \frac{3}{5}$. In this case, we do not know the initial value $f(0)$, so the slope-intercept form of the equation isn't our best bet. Let's use the point-slope form of the equation. Once we have that, we can simplify to make the slope-intercept form if we really want it.

The slope is $\frac{3}{5}$ and a point on the line is $(3, -2)$, so

$$f(x) = m(x - c) + d$$
$$f(x) = \frac{3}{5}(x - 3) - 2$$

This is an equation for the function, but we can simplify it if we want.

$$f(x) = \frac{3}{5}x - \frac{9}{5} - 2$$
$$f(x) = \frac{3}{5}x - \frac{9}{5} - \frac{10}{5}$$
$$f(x) = \frac{3}{5}x - \frac{19}{5}$$

Now we know the slope is $\frac{3}{5}$ and the initial value (y-intercept) is $-\frac{19}{5}$.

This is enough information to describe most of the features of the function.

- The rate of change is $\frac{3}{5}$ (the slope).
- Since the slope is positive, the function is increasing on $(-\infty, \infty)$.
- The domain and range are both $(-\infty, \infty)$.
- Since the function is increasing, we know that as $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to -\infty$.

We also readily know that the y-intercept is $-\frac{19}{5}$, but we'll have to do a little work to find the x-intercept. Let's solve $f(x) = 0$ to find it.

$$\frac{3}{5}x - \frac{19}{5} = 0$$
$$\frac{3}{5}x = \frac{19}{5}$$
$$x = \frac{19}{3}$$

So, the x-intercept is at $\frac{19}{3}$.

Example 10

Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so Ilya's weekly income, *I*, depends on the number of new policies, *n*, he sells during the week. Last week he sold 3 new policies, and earned $760 for the week. The week before, he sold 5 new policies, and earned $920. Find an equation for *I(n)*, and interpret the meaning of the components of the equation.

Solution:

The given information gives us two input-output pairs: (3,760) and (5,920). We start by finding the rate of change.

$$m = \frac{920 - 760}{5 - 3} = \frac{160}{2} = 80$$

Keeping track of units can help us interpret this quantity. Income increased by $160 when the number of policies increased by 2, so the rate of change is $80 per policy; Ilya earns a commission of $80 for each policy sold during the week.

Now we know the slope is 80 and a point on the line is $(3, 760)$, giving us the equation

$$I(n) = 80(n - 3) + 760$$

This equation hides some important information, so let's simplify it

$$I(n) = 80n - 240 + 760$$
$$I(n) = 80n + 520$$

Now we can see the y-intercept is 520. This value is the starting value for the function. This is Ilya's income when *n* = 0, which means no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

Our final interpretation is: Ilya's base salary is $520 per week and he earns an additional $80 commission for each policy sold each week.

Try it Now

The balance in your college payment account, *C*, is a function of the number of quarters, *q*, you attend. Interpret the function $C(a) = \ -4000q + 2000$ in words. How many quarters of college can you pay for until this account is empty?

Example 11

Given the table below write a linear equation that represents the table values

w, number of weeks	0	2	4	6
$P(w)$, number of rats	1000	1080	1160	1240

Solution:

We can see from the table that the initial value of rats is 1000 so in the linear format $P(w) = mw + b$, b = 1000.

Rather than solving for m, we can notice from the table that the population goes up by 80 for every 2 weeks that pass. This rate is consistent from week 0, to week 2, 4, and 6. The rate of change is 80 rats per 2 weeks. This can be simplified to 40 rats per week and we can write

$P(w) = mw + b$ as $P(w) = 40w + 1000$.

If you didn't notice this from the table you could still solve for the slope using any two points from the table. For example, using (2, 1080) and (6, 1240),

$m = \frac{1240-1080}{6-2} = \frac{160}{4} = 40$ rats per week

Important Topics of this Section
Definition of Modeling
Definition of a linear function
Increasing & Decreasing functions
Finding the slope/rate of change, m
Structure of a linear function
Slope-intercept form of a line
Point-slope (transformation) form of a line

Section 2.1 Exercises

Conceptual Questions

1. What are the different forms of the equation of a line? When are each of the forms useful?
2. How can we identify the slope of a line from an equation? From a graph? From two points?
3. Consider a line in slope-intercept form. How is this form related to graphical transformations?
4. Suppose we use a slope-intercept linear function to describe a real-world situation. What would each part of the function represent in that context?
5. True or false: the slope of a linear function is always increasing or always decreasing.
6. True or false: A linear function is neither concave up nor concave down.
7. True or false: The slope is the only piece of information we need in order to write an equation for a linear function.

Practice Problems

8. A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1700 people each year. Write an equation, $P(t)$, for the population *t* years after 2003.

9. A town's population has been growing linearly. In 2005, the population was 69,000, and the population has been growing by 2500 people each year. Write an equation, $P(t)$, for the population *t* years after 2005.

10. Sonya is currently 10 miles from home, and is walking further away at 2 miles per hour. Write an equation for her distance from home *t* hours from now.

11. A boat is 100 miles away from the marina, sailing directly towards it at 10 miles per hour. Write an equation for the distance of the boat from the marina after *t* hours.

12. Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding *n* rides?

13. At noon, a barista notices she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves *n* more customers during her shift?

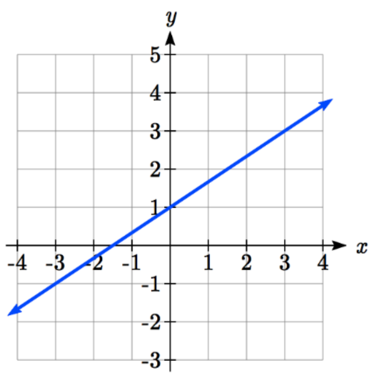
Determine if each function is increasing or decreasing

14. $f(x) = 4x + 3$
15. $g(x) = 5x + 6$
16. $a(x) = 5 - 2x$
17. $b(x) = 8 - 3x$
18. $h(x) = -2x + 4$
19. $k(x) = -4x + 1$
20. $j(x) = \frac{1}{2}x - 3$
21. $p(x) = \frac{1}{4}x - 5$
22. $n(x) = -\frac{1}{3}x - 2$
23. $m(x) = -\frac{3}{8}x + 3$

Find the slope of the line that passes through the two given points

24. (2, 4) and (4, 10)
25. (1, 5) and (4, 11)
26. (-1,4) and (5, 2)
27. (-2, 8) and (4, 6)
28. (6,11) and (-4,3)
29. . (9,10) and (-6,-12)

Find the slope of the lines graphed

30.


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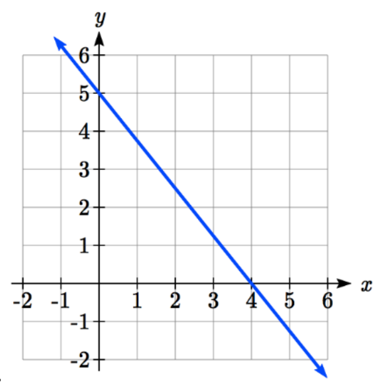


25. Sonya is walking home from a friend’s house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate?

26. A gym membership with two personal training sessions costs \$125, while gym membership with 5 personal training sessions costs \$260. What is the rate for personal training sessions?

27. A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the slope of the population growth (or decline) and make a statement about the population rate of change in people per year.

28. A city's population in the year 1958 was 2,113,000. In 1991 the population was 2,099,800. Compute the slope of the population growth (or decline) and make a statement about the population rate of change in people per year.

29. A phone company charges for service according to the formula: $C(n) = 24 + 0.1n$, where *n* is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars.
    Find and interpret the rate of change and initial value.

30. A phone company charges for service according to the formula: $C(n) = 26 + 0.04n$, where *n* is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars.
    Find and interpret the rate of change and initial value.

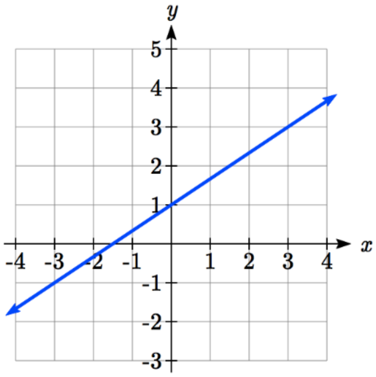
31. Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after *t* seconds is given by $E(t) = 3000 - 70t$. Write a complete sentence describing Terry's starting elevation and how it is changing over time.

32. Maria is climbing a mountain. Maria's elevation, $E(t)$, in feet after *t* minutes is given by $E(t) = 1200 + 40t$. Write a complete sentence describing Maria's starting elevation and how it is changing over time.

Given each set of information, find a linear equation satisfying the conditions, if possible. Then describe the features of the function (rate of change, intervals of increasing/decreasing, domain and range, end behavior, and intercepts).

33. $f(-5) = -4$, and $f(5) = 2$
34. $f(-1) = 4$, and $f(5) = 1$
35. Passes through (2, 4) and (4, 10)
36. Passes through (1, 5) and (4, 11)
37. Passes through (-1,4) and (5, 2)
38. Passes through (-2, 8) and (4, 6)
39. *x* intercept at (-2, 0) and *y* intercept at (0, -3)
40. *x* intercept at (-5, 0) and *y* intercept at (0, 4)

Find an equation for the function graphed



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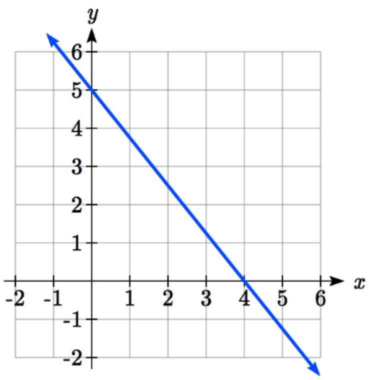

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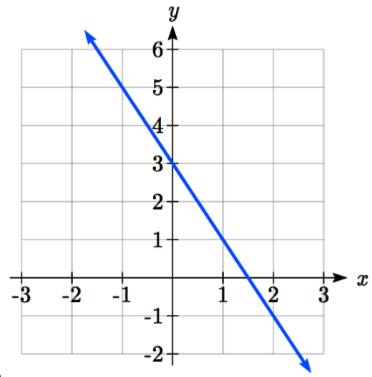

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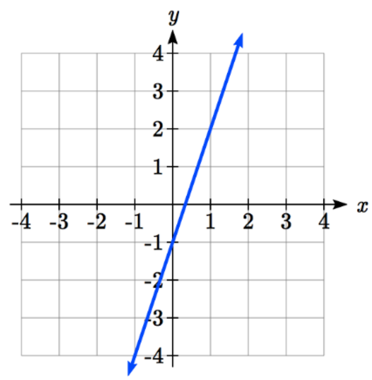


44.

Graph each of the lines described by the following equations. Then describe the features of the function (rate of change, intervals of increasing/decreasing, domain and range, end behavior, and intercepts).

45. $f(x) = \frac{4}{5}x - 2$
46. $f(x) = 3 - \frac{5}{4}x$
47. $f(x) = 2(x - 1) + 4$
48. $g(x) = -\frac{1}{3}(x + 2) - 5$
49. $f(x) = -2x - 1$
50. $g(x) = -3x + 2$
51. $h(x) = \frac{1}{3}x + 2$
52. $k(x) = \frac{2}{3}x - 3$
53. $k(t) = 3 + 2t$
54. $p(t) = -2 + 3t$

55. A clothing business finds there is a linear relationship between the number of shirts, *n*, it can sell and the price, *p*, it can charge per shirt. In particular, historical data shows that 1000 shirts can be sold at a price of $30, while 3000 shirts can be sold at a price of $22. Find a linear equation in the form $p = mn + b$ that gives the price *p* they can charge for *n* shirts.

56. A farmer finds there is a linear relationship between the number of bean stalks, *n*, she plants and the yield, *y*, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when *n* stalks are planted.

57. Which of the following tables could represent a linear function? For each that could be linear, find a linear equation models the data.

x	***g(x)***
0	5
5	-10
10	-25
15	-40

x	***h(x)***
0	5
5	30
10	105
15	230

x	***f(x)***
0	-5
5	20
10	45
15	70

x	***k(x)***
5	13
10	28
20	58
25	73

58. Which of the following tables could represent a linear function? For each that could be linear, find a linear equation models the data.

x	***g(x)***
0	6
2	-19
4	-44
6	-69

x	***h(x)***
2	13
4	23
8	43
10	53

x	***f(x)***
2	-4
4	16
6	36
8	56

x	***k(x)***
0	6
2	31
6	106
8	231

59. While speaking on the phone to a friend in Oslo, Norway, you learned that the current temperature there was -23 Celsius (-23°C). After the phone conversation, you wanted to convert this temperature to Fahrenheit degrees, °F, but you could not find a reference with the correct formulas. You then remembered that the relationship between °F and °C is linear. [UW]
    a. Using this and the knowledge that 32°F = 0 °C and 212 °F = 100 °C, find an equation that computes Celsius temperature in terms of Fahrenheit; i.e. an equation of the form C = "an expression involving only the variable F."
    b. Likewise, find an equation that computes Fahrenheit temperature in terms of Celsius temperature; i.e. an equation of the form F = "an expression involving only the variable C."
    c. How cold was it in Oslo in °F?

Sketch a line with the given features.

60. An *x*-intercept of (-4, 0) and *y*-intercept of (0, -2)

61. An *x*-intercept of (-2, 0) and *y*-intercept of (0, 4)

62. A vertical intercept of (0, 7) and slope $-\frac{3}{2}$

Sketch a line with the given features.

63. A vertical intercept of (0, 3) and slope $\frac{2}{5}$

64. Passing through the points (-6,-2) and (6,-6)

65. Passing through the points (-3,-4) and (3,0)

66. If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $3/4$, a shift left by 2, and a shift down by 4
    a. Write an equation for $g(x)$
    b. What is the slope of this line?
    c. Find the vertical intercept of this line.

67. If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $1/3$, a shift right by 1, and a shift up by 3
    a. Write an equation for $g(x)$
    b. What is the slope of this line?
    c. Find the vertical intercept of this line.

Find the horizontal and vertical intercepts of each equation

68. $f(x) = -x + 2$

69. $g(x) = 2x + 4$

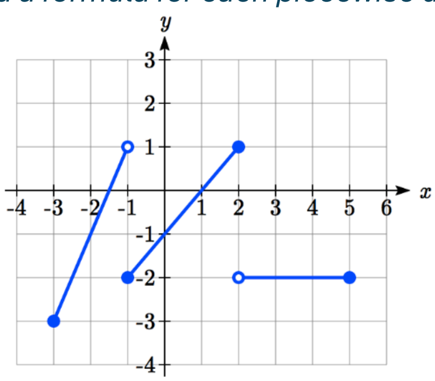
70. $h(x) = 3x - 5$

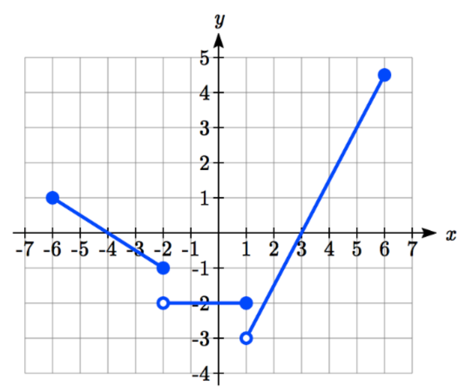
71. $k(x) = -5x + 1$

72. $-2x + 5y = 20$

73. $7x + 2y = 5$

Find a formula for each piecewise defined function.

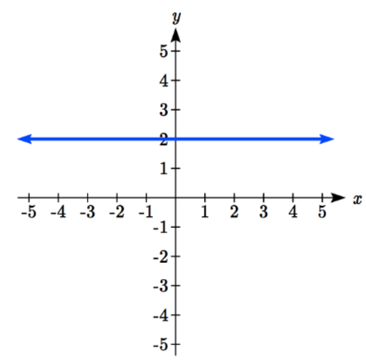
74. 

75. 

Section 2.2: More Features of Linear Functions

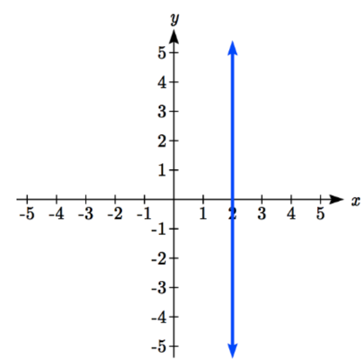
In this section, we'll continue our study of linear functions. We'll look at some special cases of linear functions,

Horizontal and Vertical Lines



There are two special cases of lines: a horizontal line and a vertical line. In a horizontal line like the one graphed to the right, notice that between any two points, the change in the outputs is 0. In the slope equation, the numerator will be 0, resulting in a slope of 0. Using a slope of 0 in the $f(x) = b + mx$, the equation simplifies to $f(x) = b$.

Notice a horizontal line has a vertical intercept, but no horizontal intercept (unless it's the line *f(x)* = 0).



In the case of a vertical line, notice that between any two points, the change in the inputs is zero. In the slope equation, the denominator will be zero, and you may recall that we cannot divide by the zero; the slope of a vertical line is undefined. You might also notice that a vertical line is not a function. To write the equation of vertical line, we simply write input=value, like $x = b$.
Notice a vertical line has a horizontal intercept, but no vertical intercept (unless it's the line *x* = 0).

Horizontal and Vertical Lines
Horizontal lines have equations of the form $f(x) = b$. The slope of a horizontal line is 0. **Vertical lines** have equations of the form $x = a$. The slope of a vertical line is undefined.

Example 1

Write an equation for the horizontal line graphed above.

Solution:
This line would have equation $f(x) = 2$

Example 2

Write an equation for the vertical line graphed above.

Solution:
This line would have equation $x = 2$

Example 3

Write an equation for the line through each pair of points:
a. $(5, 3)$ and $(5, -1)$
b. $(5, 3)$ and $(-1, 3)$

Solutions:
For any equation of a line, we need to find the slope. We'll start there and see where we go from there.

a.

$$m = \frac{-1-3}{5-5}$$
$$= -\frac{4}{0}$$
$$= \text{undefined}$$

This line has an undefined slope, so it is a vertical line. The equation of this line will describe the x-value that all the points on the line have in common: $x = 5$.

b.

$$m = \frac{3-3}{-1-5}$$
$$= \frac{0}{-6}$$
$$= 0$$

The slope of this line is 0, so it is a horizontal line. Its equation describes the constant height of the line, $f(x) = 3$.

Parallel and Perpendicular Lines

When two lines are graphed together, the lines will be **parallel** if they are increasing at the same rate – if the rates of change are the same. In this case, the graphs will never cross (unless they're the same line).

Parallel Lines
Two lines are **parallel** if the slopes are equal (or, if both lines are vertical).

Example 4

Find a line parallel to $f(x) = 3x + 6$ that passes through the point (3, 0).

Solution:
We know the line we're looking for will have the same slope as the given line, m = 3. We readily know the slope and a point on the line, so we can use point-slope form:

$$g(x) = m(x - c) + d$$
$$g(x) = 3(x - 3) + 0$$
$$g(x) = 3x - 9$$

If two lines are not parallel, one other interesting possibility is that the lines are perpendicular, which means the lines form a right angle (90 degree angle – a square corner) where they meet. In this case, the slopes when multiplied together will equal -1. Solving for one slope leads us to the definition:

Perpendicular Lines

Given two linear equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$,

the lines will be **perpendicular** if $m_1m_2 = -1$, and so $m_2 = \frac{-1}{m_1}$.

We often say the slope of a perpendicular line is the "opposite reciprocal" of the other line's slope.

Example 5

Find the slope of a line perpendicular to a line with:

a) a slope of 2. b) a slope of -4. c) a slope of $\frac{2}{3}$.

Solution:

a) If the original line had slope 2, the perpendicular line's slope would be $m_2 = \frac{-1}{2}$
b) If the original line had slope -4, the perpendicular line's slope would be $m_2 = \frac{-1}{-4} = \frac{1}{4}$
c) If the original line had slope $\frac{2}{3}$, the perpendicular line's slope would be $m_2 = \frac{-1}{2/3} = \frac{-3}{2}$

Example 6

Find the equation of a line perpendicular to $f(x) = 6 + 3x$ and passing through the point (3, 0)

Solution:

The original line has slope $m = 3$. The perpendicular line will have slope $m = \frac{-1}{3}$. Using this and the given point, we can find the equation for the line in point-slope form.

$$g(x) = -\frac{1}{3}(x - 3) + 0$$
$$g(x) = -\frac{1}{3}x + 1$$

Try it Now

Given the line $h(t) = -4 + 2t$, find an equation for the line passing through (0, 0) that is: a) parallel to *h(t)*. b) perpendicular to *h(t)*.

Example 7

A line passes through the points (-2, 6) and (4, 5). Find the equation of a perpendicular line that passes through the point (4, 5).

Solution:

From the two given points on the reference line, we can calculate the slope of that line:

$$m_1 = \frac{5 - 6}{4 - (-2)} = \frac{-1}{6}$$

The perpendicular line will have slope

$$m_2 = \frac{-1}{-1/6} = 6$$

Now we know a point (4, 5) and the slope $m = 6$, so we can write the point-slope form of the line:

$$f(x) = 6(x - 4) + 5$$

We can simplify this if we want the slope-intercept form:

$$f(x) = 6x - 24 + 5$$
$$f(x) = 6x - 19$$

Intersections of Lines

The graphs of two lines will intersect if they are not parallel. They will intersect at the point that satisfies both equations. To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal, and solve for the input that satisfies the equation.

Example 8

Find the intersection of the lines $h(t) = 3t - 4$ and $j(t) = 5 - t$

Solution:

Setting $h(t) = j(t)$,

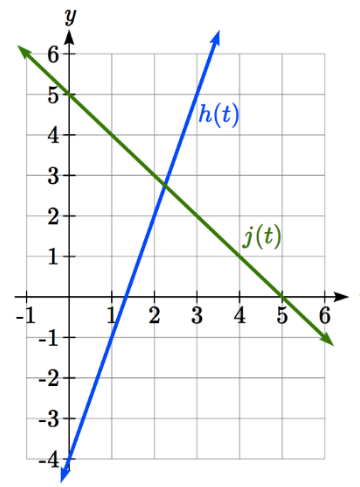
$$3t - 4 = 5 - t$$
$$4t = 9$$
$$t = \frac{9}{4}$$

This tells us the lines intersect when the input is $\frac{9}{4}$.

We can then find the output value of the intersection point by evaluating either function at this input

$$j\left(\frac{9}{4}\right) = 5 - \frac{9}{4} = \frac{11}{4}$$

These lines intersect at the point $\left(\frac{9}{4}, \frac{11}{4}\right)$. Looking at the graph, this result seems reasonable.



Two parallel lines can also intersect if they happen to be the same line. In that case, they intersect at every point on the lines.

Finding the intersection allows us to answer other questions as well, such as discovering when one function is larger than another.

Example 9

Using the functions from the previous example, for what values of *t* is $h(t) > j(t)$

Solution:

To answer this question, it is helpful first to know where the functions are equal, since that is the point where *h(t)* could switch from being greater to smaller than *j(t)* or vice-versa. From the previous example, we know the functions are equal at $t = \frac{9}{4}$.

By examining the graph, we can see that *h(t)*, the function with positive slope, is going to be larger than the other function to the right of the intersection. So $h(t) > j(t)$ when $t > \frac{9}{4}$.

We could also set the formulas of the function so that $h(t) > j(t)$ and solve similarly to how we solve equations.

$3t - 4$	$> 5 - t$	
$4t - 4$	> 5	Add t to both sides
$4t$	> 9	Add 4 to both sides
t	$> \frac{9}{4}$	Divide by 4 on both sides

Again we see that the solution set is all t such that $t > \frac{9}{4}$. In interval notation, we say $\left(\frac{9}{4}, \infty\right)$.

If we are going to take the algebraic approach to solving inequalities, we must be careful about one particular situation: multiplying and dividing by negative numbers.

Let's take the numbers 2 and 4 for example. We know that $2 < 4$, since 2 is to the left of 4 on the number line. If we multiply both sides of this inequality by -1, now we have the numbers -2 and -4. It is no longer correct to say that $-2 < -4$, though, because -4 is to the left of -2 on the number line! The inequality symbol changes directions, and we write that $-2 > -4$.

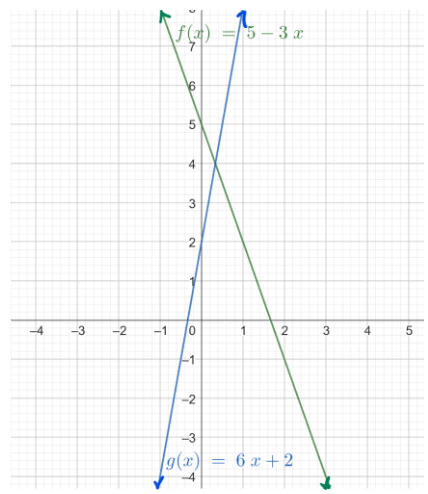
Solving Linear Inequalities
To solve an inequality in the form $f(x) < g(x)$ or $f(x) > g(x)$ where f and g are both linear functions 1. Simplify each side 2. Move all variable terms to same side 3. Isolate the variable CAUTION: If we multiply or divide by a negative number, we must reverse the direction of the inequality symbol!

Example 10

Given the functions $f(x) = 5 - 3x$ and $g(x) = 6x + 2$, find for what values of x is $f(x) \le g(x)$.

Solution:

$5 - 3x$	$\le$	$6x + 2$	
$5 - 9x$	$\le$	2	Subtract 6x from both sides
$-9x$	$\le$	-3	Subtract 5 from both sides
x	$\ge$	$\frac{1}{3}$	Divide by-9 on both sides and reverse the inequality!



We see that $f(x) \le g(x)$ when $x \ge \frac{1}{3}$; that is, on the interval $\left[\frac{1}{3}, \infty\right)$.

When we observe the graphs of f and g, we see that on the left side, f is above g. They meet when $x = \frac{1}{3}$, and after that f is below g.

Try it Now

Given the functions $f(x) = 3(x - 1) + 4$ and $g(x) = 7 - 2x$, find

a. The intersection of f and g.
b. The values of x for which $g(x) \ge f(x)$.

Inverses

Linear functions are great for learning how to algebraically find an inverse of a function. Remember, the heart of the inverse relationship is that we're swapping inputs and outputs.

Example 11

Find the inverse of $f(x) = 3x - 4$.

Solution:

Before we proceed, we need to make sure our function is one-to-one. We know that $f(x) = 3x - 4$ will make a line with vertical intercept -4 and slope 3; that's a picture that passes both the horizontal and vertical line tests, so $f(x)$ is one-to-one.

To find the inverse function, we need to swap inputs (x) with outputs ($y = f(x)$). To help make this easier, we'll start by replacing $f(x)$ with y.

$$y = 3x - 4$$

Then replace all y's with x's and all x's with y's.

$$x = 3y - 4$$

Now, we want to rearrange the equation so that the y is isolated. This will make the new input/output relationship more clear.

$$\begin{aligned} x + 4 &= 3y && \text{Add 4 to both sides} \\ \frac{x+4}{3} &= y && \text{Divide by 3 on both sides} \\ \frac{1}{3}(x+4) &= y && \text{Simplify} \end{aligned}$$

Now we have an inverse! We can rename this function $f^{-1}(x)$.

$$y = f^{-1}(x) = \frac{1}{3}(x + 4)$$

Let's use that example as an opportunity to dig deeper into the function/inverse relationship

Example 12

Compare the features of $f(x) = 3x - 4$ with its inverse $f^{-1}(x) = \frac{1}{3}(x + 4)$

Solution:

For both functions, the domain and range remains $(-\infty, \infty)$.

For $f(x)$, we see an x-intercept at $(\frac{4}{3}, 0)$; in the inverse, that becomes a y-intercept at $\left(0, \frac{4}{3}\right)$. Similarly, in $f(x)$ the y-intercept is $(0, -4)$; in the inverse, the x-intercept is $(-4, 0)$. This is a great illustration of the fact that the x/y relationship is swapped!

In $f(x)$, since the slope is positive, the function is increasing and as $x \to \infty, f(x) \to \infty$, and as $x \to -\infty, f(x) \to -\infty$. In the inverse function, the function is decreasing, so as $x \to \infty, f(x) \to -\infty$ and as $x \to -\infty, f(x) \to \infty$.

Describing an inverse function

Given a function $f(x)$, do the following to find its inverse, $f^{-1}(x)$.

1. Determine whether $f(x)$ is one-to-one. If necessary and possible, limit the domain of $f(x)$ to a one-to-one portion that contains the entire range.
2. Replace $f(x)$ with y.
3. Swap all x's with y's and all y's with x's.
4. Isolate y to describe the new input/output relationship
5. Rename the function $f^{-1}(x)$.

Once complete, we've not only described the new input/output relationship, but we should also be able to compare f^{-1} to f to see that all the input/output relationships and features have been reversed.

Try it Now

Given $f(x) = 5(x - 2)$, find the inverse function. Then compare the features of f and its inverse.

Important Topics

Equations of horizontal and vertical lines.

Equations of parallel and perpendicular lines

Calculating intersections of lines.

Solving linear inequalities.

Finding the inverse of a linear function.

Section 2.2 Exercises

Conceptual Questions

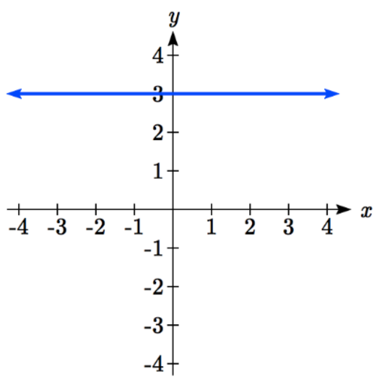
1. Why is the equation of a vertical line in the form $x = n$?
2. If $f(x)$ and $g(x)$ are linear functions, how can we identify the solution to $f(x) < g(x)$ using a graph?
3. How do we find a formula for the inverse of a linear function?
4. True or false: if f is increasing, then f^{-1} must be decreasing.
5. True or false: if f and g are perpendicular lines, then one must be increasing and the other must be decreasing.
6. True or false: if f is a linear function with slope $-\frac{1}{2}$, a line perpendicular to f must have slope -2.

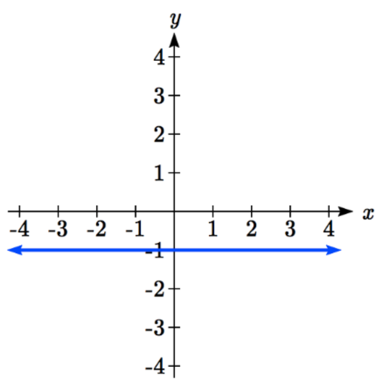
Practice Problems

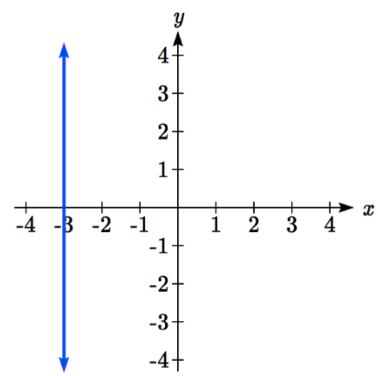
Sketch the graph of each equation.

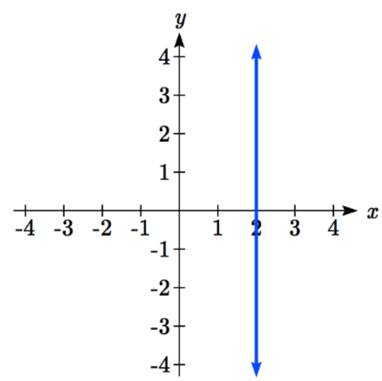
1. $x = 3$
2. $x = -2$
3. $r(x) = 4$
4. $q(x) = 3$

Write the equation of each line shown

5. 

6. 

7. 

8. 

Given below are descriptions of two lines. Find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular or neither?

9. Line 1: Passes through $(0,6)$ and $(3,-24)$
   Line 2: Passes through $(-1,19)$ and $(8,-71)$

10. Line 1: Passes through $(-8,\ -55)$ and $(10,\ 89)$
    Line 2: Passes through $(9,\ -44)$ and $(4,-14)$

11. Line 1: Passes through $(2,3)$ and $(4,-1)$
    Line 2: Passes through $(6,3)$ and $(8,5)$

12. Line 1: Passes through $(1,\ 7)$ and $(5,5)$
    Line 2: Passes through $(-1,-3)$ and $(1,1)$

13. Line 1: Passes through $(0,\ 5)$ and $(3,3)$
    Line 2: Passes through $(1,-5)$ and $(3,-2)$

14. Line 1: Passes through $(2,5)$ and $(5,-1)$
    Line 2: Passes through $(-3,7)$ and $(3,-5)$

15. Write an equation for a line parallel to $f(x) = -5x - 3$ and passing through the point (2,-12)

16. Write an equation for a line parallel to $g(x) = 3x - 1$ and passing through the point (4,9)

17. Write an equation for a line perpendicular to $h(t) = -2t + 4$ and passing through the point (-4,-1)

18. Write an equation for a line perpendicular to $p(t) = 3t + 4$ and passing through the point (3,1)

19. Find the point at which the line $f(x) = -2x - 1$ intersects the line $g(x) = -x$. Then find the x-values for which $f(x) > g(x)$.

20. Find the point at which the line $f(x) = 2x + 5$ intersects the line $g(x) = -3x - 5$. Then find the x-values for which $f(x) \leq g(x)$.

21. Use algebra to find the point at which the line $f(x) = -\frac{4}{5}x + \frac{274}{25}$ intersects the line $h(x) = \frac{9}{4}x + \frac{73}{10}$

22. Use algebra to find the point at which the line $f(x) = \frac{7}{4}x + \frac{457}{60}$ intersects the line $g(x) = \frac{4}{3}x + \frac{31}{5}$

Solve each inequality. Write your final answer in inequality notation, on a number line, and in interval notation.

23. $4x - 7 \le 9$
24. $3x + 2 \ge 7x - 1$
25. $-2x + 3 > x - 5$
26. $4(x + 3) \ge 2x - 1$
27. $-\frac{1}{2}x \le -\frac{5}{4} + \frac{2}{5}x$
28. $-5(x - 1) + 3 > 3x - 4 - 4x$
29. $-3(2x + 1) > -2(x + 4)$
30. $\frac{x+3}{8} - \frac{x+5}{5} \le \frac{3}{5}$
31. $\frac{x+1}{3} + \frac{x+2}{5} \le \frac{3}{5}$

Find the inverse of each linear function. Then compare the intercepts and end behavior of each function and its inverse.

32. $f(x) = 5x - 1$
33. $f(x) = \frac{1}{5}(x + 2)$
34. $f(x) = -3(x - 1) + 2$
35. $f(x) = 2(x + 3)$

36. A car rental company offers two plans for renting a car.
    Plan A: 30 dollars per day and 18 cents per mile
    Plan B: 50 dollars per day with free unlimited mileage
    How many miles would you need to drive for plan B to save you money?
37. You're comparing two cell phone companies.
    Company A: \$20/month for unlimited talk and text, and \$10/GB for data.
    Company B: \$65/month for unlimited talk, text, and data.
    Under what circumstances will company A save you money?
38. Sketch an accurate picture of the equation $f(x) = 2 - \frac{1}{2}x$. Let *c* be an unknown constant. [UW]
    a. Find the point of intersection between the line you have graphed and the line $g(x) = 1 + cx$; your answer will be a point in the *xy* plane whose coordinates involve the unknown *c*.
    b. Find *c* so that the intersection point in (a) has *x*-coordinate 10.
    Find *c* so that the intersection point in (a) lies on the x-axis.

Section 2.3: Modeling with Linear Functions

Now that we know a great deal about how to describe and graph linear functions, we can start to use these functions to solve problems. We'll begin with some typical algebraic problems, and then move on to modeling problems.

Modeling with Linear Functions

When modeling scenarios with a linear function and solving problems involving quantities changing linearly, we typically follow the same problem solving strategies that we would use for any type of function:

Problem solving strategy

1) Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2) Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
3) Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
4) Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
5) When needed, find a formula for the function.
6) Solve or evaluate using the formula you found for the desired quantities.
7) Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8) Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

Example 1

Emily saved up \$3500 for her summer visit to Seattle. She anticipates spending \$400 each week on rent, food, and fun. Find and interpret the horizontal intercept and determine a reasonable domain and range for this function.

Solution:

In the problem, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can define our variables, including units.

Output: M, money remaining, in dollars

Input: t, time, in weeks

Reading the problem, we identify two important values. The first, \$3500, is the initial value for M. The other value appears to be a rate of change – the units of dollars per week match the units of our output variable divided by our input variable. She is spending money each week, so you should recognize that the amount of money remaining is decreasing each week and the slope is negative.

To answer the first question, looking for the horizontal intercept, it would be helpful to have an equation modeling this scenario. Using the intercept and slope provided in the problem, we can write the equation: $M(t) = 3500 - 400t$.

To find the horizontal intercept, we set the output to zero, and solve for the input:

$$0 = 3500 - 400t$$

$$t = \frac{3500}{400} = 8.75$$

The horizontal intercept is 8.75 weeks. Since this represents the input value where the output will be zero, interpreting this, we could say: Emily will have no money left after 8.75 weeks.

When modeling any real life scenario with functions, there is typically a limited domain over which that model will be valid – almost no trend continues indefinitely. In this case, it certainly doesn't make sense to talk about input values less than zero. It is also likely that this model is not valid after the horizontal intercept (unless Emily's going to start using a credit card and go into debt).

The domain represents the set of input values and so the reasonable domain for this function is $0 \leq t \leq 8.75$.

However, in a real world scenario, the rental might be weekly or nightly. She may not be able to stay a partial week and so all options should be considered. Emily could stay in Seattle for 0 to 8 full weeks (and a couple of days), but would have to go into debt to stay 9 full weeks, so restricted to whole weeks, a reasonable domain without going in to debt would be $0 \leq t \leq 8$, or $0 \leq t \leq 9$if she went into debt to finish out the last week.

The range represents the set of output values and she starts with $3500 and ends with $0 after 8.75 weeks so the corresponding range is $0 \leq M(t) \leq 3500$. If we limit the rental to whole weeks, however, the range would change. If she left after 8 weeks because she didn't have enough to stay for a full 9 weeks, she would have $M(8) = 3500-400(8) =$ $300 dollars left after 8 weeks, giving a range of $300 \leq M(t) \leq 3500$. If she wanted to stay the full 9 weeks she would be $100 in debt giving a range of $-100 \leq M(t) \leq 3500$.

Most importantly remember that domain and range are tied together, and what ever you decide is most appropriate for the domain (the independent variable) will dictate the requirements for the range (the dependent variable).

Try it Now

A database manager is loading a large table from backups. Getting impatient, she notices 1.2 million rows had been loaded. Ten minutes later, 2.5 million rows had been loaded. How much longer will she have to wait for all 80 million rows to load?

Example 2

Jamal is choosing between two moving companies. The first, U-Haul, charges an up-front fee of $20, then 59 cents a mile. The second, Budget, charges an up-front fee of $16, then 63 cents a mile[8]. When will U-Haul be the better choice for Jamal?

Solution:

The two important quantities in this problem are the cost, and the number of miles that are driven. Since we have two companies to consider, we will define two functions:

Input: *m*, miles driven
Outputs:
Y(*m*): cost, in dollars, for renting from U-Haul
B(*m*): cost, in dollars, for renting from Budget

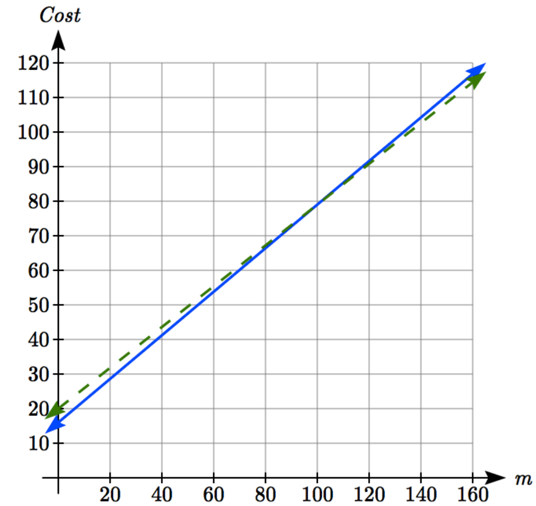
Reading the problem carefully, it appears that we were given an initial cost and a rate of change for each company. Since our outputs are measured in dollars but the costs per mile given in the problem are in cents, we will need to convert these

[8] Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/

quantities to match our desired units: \$0.59 a mile for U-Haul, and \$0.63 a mile for Budget.

Looking to what we're trying to find, we want to know when U-Haul will be the better choice. Since all we have to make that decision from is the costs, we are looking for when U-Haul will cost less, or when $Y(m) < B(m)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, then look to see where the *Y(m)* function is smaller. Using the rates of change and initial charges, we can write the equations:

$$Y(m) = 20 + 0.59m$$
$$B(m) = 16 + 0.63m$$



These graphs are sketched to the right, with *Y(m)* drawn dashed.

To find when U-Haul will cost less, we'll solve $20 + 0.59m < 16 + 0.63m$.

$$20 + 0.59m < 16 + 0.63m$$
$$20 < 16 + 0.04m$$
$$4 < 0.04m$$
$$100 < m$$
$$m > 100$$

We can conclude that U-Haul will be the cheaper price when more than 100 miles are driven.

Example 3

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. If this trend continues,

a. Predict the population in 2013

b. When will the population reach 15000?

Solution:

The two changing quantities are the population and time. While we could use the actual year value as the input quantity, doing so tends to lead to very ugly equations, since the vertical intercept would correspond to the year 0, more than 2000 years ago!

To make things a little nicer, and to make our lives easier too, we will define our input as years since 2004:

Input: *t*, years since 2004

Output: *P(t)*, the town's population

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point (0, 6200). Notice that through our clever choice of variable definition, we have "given" ourselves the vertical intercept of the function. The year 2009 would correspond to $t = 5$, giving the point (5, 8100).

To predict the population in 2013 ($t = 9$), we would need an equation for the population. Likewise, to find when the population would reach 15000, we would need to solve for the input that would provide an output of 15000. Either way, we need an equation. To find it, we start by calculating the rate of change:
$m = \frac{8100-6200}{5-0} = \frac{1900}{5} = 380$people per year

Since we already know the vertical intercept of the line, we can immediately write the equation:

$$P(t) = 6200 + 380t$$

To predict the population in 2013, we evaluate our function at $t = 9$

$$P(9) = 6200 + 380(9) = 9620$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$15000 = 6200 + 380t$$
$$8800 = 380t$$
$$t \approx 23.158$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Before our next example, we need to recall an important fact about movement.

Try it Now

The amount of students at a school is decreasing linearly. In 2020, there are 2 thousand students at the school. In 2022, there are 1.5 thousand students at the school.

a. Predict how many students will be at the school in 2024.
b. If this decrease continues, when will there be 800 students left?

Distance rate and time

Distance = rate · time

To find the distance that a moving object has traveled, multiply the rate (speed), by the time in motion.

Example 3

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio with a range of 2 miles. How long after they start walking will they fall out of radio contact?

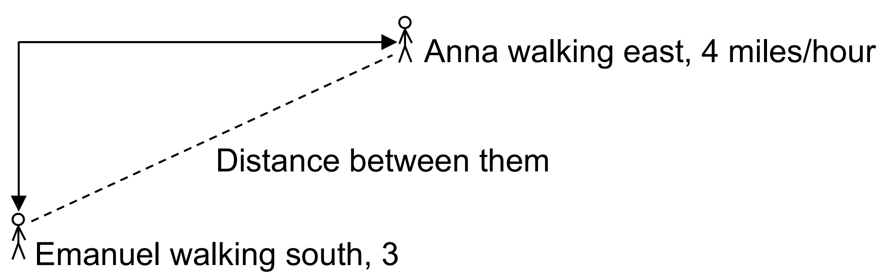
Solution:

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question: how long will it take them to be 2 miles apart?

In this problem, our changing quantities are time and the two peoples' positions, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define

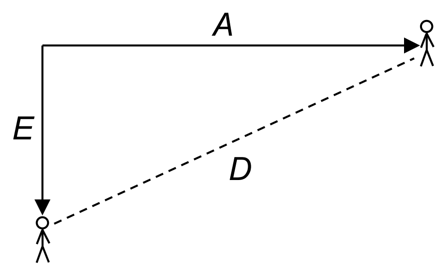
Input: t, time in hours.

Since it is not obvious how to define our output variables, we'll start by drawing a picture.

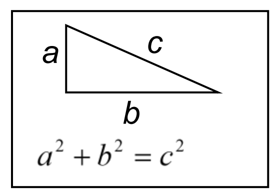


Because of the complexity of this question, it may be helpful to introduce some intermediary variables. These are quantities that we aren't directly interested in, but seem important to the problem. For this problem, Anna's and Emanuel's distances from the starting point seem important. To notate these, we are going to define a coordinate system, putting the "starting point" at the intersection where they both started, then we're going to introduce a variable, A, to represent Anna's position, and define it to be a measurement from the starting point, in the eastward direction. Likewise, we'll introduce a variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system we specified both the origin, or starting point, of the measurement, as well as the direction of measure.

While we're at it, we'll define a third variable, *D*, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the picture is often helpful.
Looking at the variables on the picture, we remember we need to know how long it takes for D, the distance between them, to equal 2 miles.



Seeing this picture we remember that in order to find the distance between the two, we can use the Pythagorean Theorem, a property of right triangles.



From here, we can now look back at the problem for relevant information. Anna is walking 4 miles per hour, and Emanuel is walking 3 miles per hour, which are rates of change. Using those, we can write formulas for the distance each has walked.

They both start at the same intersection and so when *t* = 0, the distance travelled by each person should also be 0, so given the rate for each, and the initial value for each, we get:

$$A(t) = 4t$$
$$E(t) = 3t$$

Using the Pythagorean theorem we get:

$D(t)^2 = A(t)^2 + E(t)^2$ substitute in the function formulas
$D(t)^2 = (4t)^2 + (3t)^2 = 16t^2 + 9t^2 = 25t^2$ solve for *D(t)* using the square root
$D(t) = \pm\sqrt{25t^2} = \pm 5|t|$

Since in this scenario we are only considering positive values of *t* and our distance *D(t)* will always be positive, we can simplify this answer to $D(t) = 5t$

Interestingly, the distance between them is also a linear function. Using it, we can now answer the question of when the distance between them will reach 2 miles:

$$D(t) = 2$$
$$5t = 2$$
$$t = \frac{2}{5} = 0.4$$

They will fall out of radio contact in 0.4 hours, or 24 minutes

Example 4

Sofia and Surya are at the park together. Sofia leaves first, traveling in a straight line at 5 ft/s. Surya leaves 90 seconds later, jogging at a rate of 8 ft/s in the same direction. How long after she leaves will it take for Surya to catch up with Sofia?

Solution:
First, we gather up our information. For Sofia, we know that she travels at a rate of 5ft/s, but we don't know how long she's been traveling. We need a variable for that; let's use t_1 to represent Sofia's time. Then for Sofia's distance, we have $d_S = 5t_1$.

For Surya, we know that she travels at a rate of 8 ft/s, and we also don't know her time. We can use t_2 to represent her time. That makes Surya's distance $d_U = 8t_2$.

We also know that Sofia has been traveling for 90 seconds longer than Surya. That makes $t_1 = t_2 + 90$. Substituting that into Sofia's distance formula, we have $d_S = 5(t_2 + 90)$.

Now, how do we put all this information together? We want to find how long it will take for Surya to catch up; at that point, the two girls will have traveled the same distance, so $d_S = d_U$. We'll use this to set up our equation and solve.

$$d_S = d_U$$
$$5(t_2 + 90) = 8t_2$$
$$5t_2 + 450 = 8t_2$$
$$450 = 3t_2$$
$$t_2 = 150$$

Remember, t_2 represents Surya's time, so this answer tells us that Surya will catch up 150 seconds after she leaves the park.

Try It Now

Kai and Lalit leave the pickleball court at the same time, heading in opposite directions. Kai walks at a speed of 3 yards per minute. After 5 minutes, Kai and Lalit are 25 yards apart. How fast did Lalit walk?

The last, very common type of linear problem that we will examine is mixture problems. These types of situations come up everywhere from chemistry to food science to engineering. The main idea is this:

Mixture problems

If x units of a mixture is made up of $p\%$ of a substance, then there are

$$\frac{p}{100}\cdot x$$

units of the substance in that mixture.

Example 5

An artisan lotion maker wants to create 5 quarts of a lotion that is 40% cocoa butter. He has two base mixtures; one that is 60% cocoa butter, and one that is 20% cocoa butter. How much of each of these base mixtures should he use?

Solution:
We'll start again by gathering up our information.

In the end, we want 5 quarts of a 40% cocoa butter lotion. That means that the final mixture will contain $\frac{40}{100}\cdot 5$ quarts of cocoa butter.

We'll use some amount of the 60% cocoa butter lotion. Let's call that x quarts, so we'll have $\frac{60}{100}\cdot x$ quarts of cocoa butter.
We'll also use some amount of the 20% cocoa butter lotion. We can call that y quarts, so we'll have $\frac{20}{100}\cdot y$ quarts of cocoa butter.

We also know that our 60% and 20% mixtures should add up to a total of 5 quarts, so $5 = x + y$. We can rearrange that to see that $x = 5 - y$.

Now to form our equation, we'll focus purely on the amount of cocoa butter involved. The idea is that the amount of cocoa butter in the 60% lotion plus the amount in the 20% lotion should add up to the amount of cocoa butter in the final 40% mixture. So

$$\frac{60}{100}x + \frac{20}{100}y = \frac{40}{100}\cdot 5$$

If we replace x with $5 - y$, we have

$$\frac{60}{100}(5 - y) + \frac{20}{100}y = \frac{40}{100}\cdot 5$$

This is a linear equation we can solve!

$$0.6(5 - y) + 0.2y = 0.4\cdot 5$$

$$3 - 0.6y + 0.2y = 2$$
$$3 - 0.4y = 2$$
$$-0.5y = -1$$
$$y = 2$$

The variable y represents the amount of the 20% cocoa butter lotion. We also know that $x = 5 - y = 5 - 2 = 3$ is the amount of 60% lotion. So our final answer is that we should mix 2 qt of the 20% and 3 qt of the 60% lotion to make 5 quarts of lotion which is 40% cocoa butter.

Try it Now

A chemist needs to make a 30% hydrochloric acid solution. The lab has a 23% solution and a 42% solution. How much of each must be mixed together to make 10 liters of a 30% solution?

Important Topics of this Section

The problem solving process

1) Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2) Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
3) Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
4) Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
5) When needed, find a formula for the function.
6) Solve or evaluate using the formula you found for the desired quantities.
7) Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8) Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

Section 2.3 Exercises

Conceptual Questions

1. Create a situation that could be modeled by the function $f(x) = -3x + 100$.
2. Create a situation that could b modeled by the function $f(x) = 0.2(x - 3) + 4$.
3. Give an example of an equation that might help us to solve for when two people have traveled the same distance.
4. Given an example of an equation that might help us to solve for when two people have traveled in opposite directions and end up 100 m apart.

Practice Problems

5. In 2004, a school population was 1001. By 2008 the population had grown to 1697. Assume the population is changing linearly.
    a. How much did the population grow between the year 2004 and 2008?
    b. How long did it take the population to grow from 1001 students to 1697 students?
    c. What is the average population growth per year?
    d. What was the population in the year 2000?
    e. Find an equation for the population, *P*, of the school *t* years after 2000.
    f. Using your equation, predict the population of the school in 2011.

6. In 2003, a town's population was 1431. By 2007 the population had grown to 2134. Assume the population is changing linearly.
    a. How much did the population grow between the year 2003 and 2007?
    b. How long did it take the population to grow from 1431 people to 2134?
    c. What is the average population growth per year?
    d. What was the population in the year 2000?
    e. Find an equation for the population, *P*, of the town *t* years after 2000.
    f. Using your equation, predict the population of the town in 2014.

7. A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used on the phone. If a customer uses 410 minutes, the monthly cost will be $71.50. If the customer uses 720 minutes, the monthly cost will be $118.
    a. Find a linear equation for the monthly cost of the cell plan as a function of *x*, the number of monthly minutes used.
    b. Interpret the slope and vertical intercept of the equation.
    c. Use your equation to find the total monthly cost if 687 minutes are used.

8. A phone company has a monthly cellular data plan where a customer pays a flat monthly fee and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be \$11.20. If the customer uses 130 MB, the monthly cost will be \$17.80.
    a. Find a linear equation for the monthly cost of the data plan as a function of x, the number of MB used.
    b. Interpret the slope and vertical intercept of the equation.
    c. Use your equation to find the total monthly cost if 250 MB are used.

9. In 1991, the moose population in a park was measured to be 4360. By 1999, the population was measured again to be 5880. If the population continues to change linearly,
    a. Find a formula for the moose population, P.
    b. What does your model predict the moose population to be in 2003?

10. In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. If the population continues to change linearly,
    a. Find a formula for the owl population, P.
    b. What does your model predict the owl population to be in 2012?

11. The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010, and is being depleted by about 2.1 billion cubic feet each year.
    a. Give a linear equation for the remaining federal helium reserves, R, in terms of t, the number of years since 2010.
    b. In 2015, what will the helium reserves be?
    c. If the rate of depletion doesn't change, when will the Federal Helium Reserve be depleted?

12. Suppose the world's current oil reserves are 1820 billion barrels. If, on average, the total reserves is decreasing by 25 billion barrels of oil each year:
    a. Give a linear equation for the remaining oil reserves, R, in terms of t, the number of years since now.
    b. Seven years from now, what will the oil reserves be?
    c. If the rate of depletion isn't change, when will the world's oil reserves be depleted?

13. You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of \$19.95 *plus* 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

14. You are choosing between two different window washing companies. The first charges \$5 per window. The second charges a base fee of \$40 plus \$3 per

window. How many windows would you need to have for the second company to be preferable?

15. When hired at a new job selling jewelry, you are given two pay options:
Option A: Base salary of \$17,000 a year, with a commission of 12% of your sales
Option B: Base salary of \$20,000 a year, with a commission of 5% of your sales
How much jewelry would you need to sell for option A to produce a larger income?

16. When hired at a new job selling electronics, you are given two pay options:
Option A: Base salary of \$14,000 a year, with a commission of 10% of your sales
Option B: Base salary of \$19,000 a year, with a commission of 4% of your sales
How much electronics would you need to sell for option A to produce a larger income?

17. Find the area of a triangle bounded by the *y* axis, the line $f(x) = 9 - \frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.

18. Find the area of a triangle bounded by the *x* axis, the line $f(x) = 12 - \frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.

19. Find the area of a parallelogram bounded by the *y* axis, the line $x = 3$, the line $f(x) = 1 + 2x$, and the line parallel to $f(x)$ passing through (2, 7)

20. Find the area of a parallelogram bounded by the *x* axis, the line $g(x) = 2$, the line $f(x) = 3x$, and the line parallel to $f(x)$ passing through (6, 1)

21. If $b > 0$ and $m < 0$, then the line $f(x) = b + mx$ cuts off a triangle from the first quadrant. Express the area of that triangle in terms of *m* and *b*. [UW]

22. Find the value of *m* so the lines $f(x) = mx + 5$ and $g(x) = x$ and the *y*-axis form a triangle with an area of 10. [UW]

23. The median home values in Mississippi and Hawaii (adjusted for inflation) are shown below. If we assume that the house values are changing linearly,

Year	Mississippi	Hawaii
1950	25200	74400
2000	71400	272700

a. In which state have home values increased at a higher rate?
b. If these trends were to continue, what would be the median home value in Mississippi in 2010?

c. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd)

24. The median home value ins Indiana and Alabama (adjusted for inflation) are shown below. If we assume that the house values are changing linearly,

Year	Indiana	Alabama
1950	37700	27100
2000	94300	85100

a. In which state have home values increased at a higher rate?
b. If these trends were to continue, what would be the median home value in Indiana in 2010?
c. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd)

25. Pam is taking a train from the town of Rome to the town of Florence. Rome is located 30 miles due West of the town of Paris. Florence is 25 miles East, and 45 miles North of Rome. On her trip, how close does Pam get to Paris? [UW]

26. You're flying from Joint Base Lewis-McChord (JBLM) to an undisclosed location 226 km south and 230 km east. Mt. Rainier is located approximately 56 km east and 40 km south of JBLM. If you are flying at a constant speed of 800 km/hr, how long after you depart JBLM will you be the closest to Mt. Rainier?

27. Zahira and Joel leave their college algebra PASS meeting at the same time. Zahira walks due north to her dorm. Joel walks due south towards the dining hall. After 20 minutes, they are 2600 feet apart. If Joel walks at a rate of 140 feet per minute, how fast did Zahira walk?

28. Two sailboats leave from the same port. The first boat travels in a straight line at a speed of 3 knots (nautical miles per hour). One quarter hour later, the second boat leaves, traveling in the same direction as the first at a rate of 4 knots. When will the second boat catch up to the first?

29. Lill and Nesta are hanging out in Pikes Peak National forest when Bigfoot appears. Lill runs directly towards Bigfoot at a speed of 7 meters per second.

Nesta hesitates for 2 seconds, then runs in the opposite direction at a speed of 6.5 meters per second. When will Lill and Nesta be 50 meters apart?

30. Demetrios wants to make brownies, and needs 70% dark chocolate. He has 50% and 100% chocolate. How much of each of these does he need to mix together to get 80 grams of a 70% dark chocolate?

31. Ivan is a jeweler. He needs to make an 80% silver alloy. He has a 68% and a 92% alloy. How much of each of these does he need to blend together to make 2 pounds of the 80% silver alloy?

Section 2.4 Absolute Value Functions

So far in this chapter we have been studying the behavior of linear functions. The Absolute Value Function is a piecewise-defined function made up of two linear functions.

Absolute Value Function

The absolute value function can be defined as

$$f(x) = |x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

The absolute value function is commonly used to determine the distance between two numbers on the number line. Given two values *a* and *b*, then $|a - b|$ will give the distance, a positive quantity, between these values, regardless of which value is larger.

Example 1

Describe all values, *x*, within a distance of 4 from the number 5.

Solution:

We want the distance between *x* and 5 to be less than or equal to 4. The distance can be represented using the absolute value, giving the expression

$$|x - 5| \leq 4$$

Example 2

A 2010 poll reported 78% of Americans believe that people who are gay should be able to serve in the US military, with a reported margin of error of 3%[9]. The margin of error tells us how far off the actual value could be from the survey value[10]. Express the set of possible values using absolute values.

Solution:

Since we want the size of the difference between the actual percentage, *p*, and the reported percentage to be less than 3%,

$$|p - 78| \leq 3$$

[9] http://www.pollingreport.com/civil.htm, retrieved August 4, 2010

[10] Technically, margin of error usually means that the surveyors are 95% confident that actual value falls within this range.

Try it Now

Students who score within 20 points of 80 will pass the test. Write this as a distance from 80 using the absolute value notation.

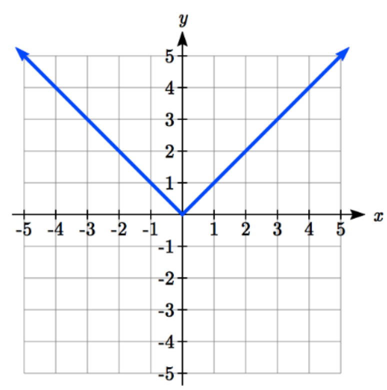
Important Features

The most significant feature of the absolute value graph is the corner point where the graph changes direction. When finding the equation for a transformed absolute value function, this point is very helpful for determining the horizontal and vertical shifts.

Absolute Value Toolkit Function

The absolute value toolkit function, $f(x) = |x|$, can be represented with the following table values and graph:

x	$f(x) = \|x\|$
-2	2
-1	1
0	0
1	1
2	2



Features of the Absolute Value Toolkit Function

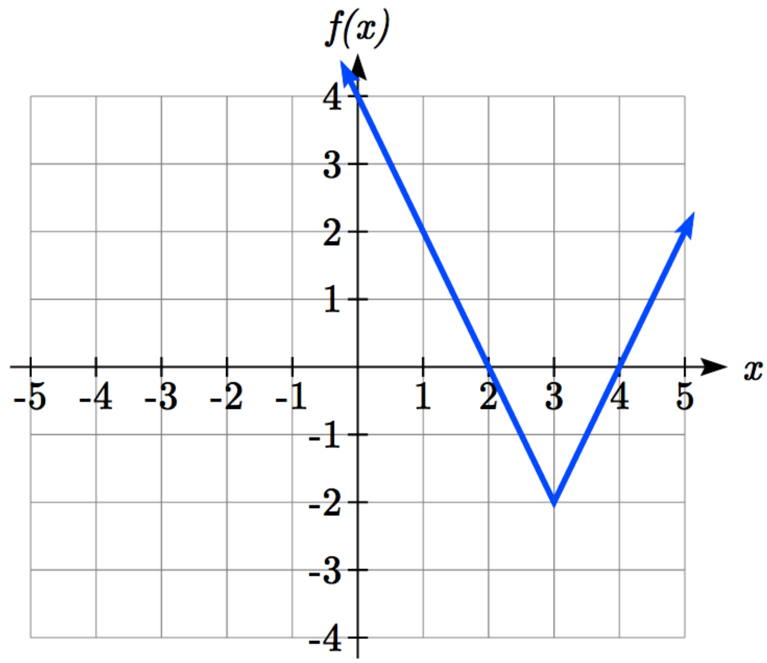
The Absolute Value Toolkit Function has the following features:

- Domain: $(-\infty, \infty)$
- Range: $[0, \infty)$
- Intercepts: x and y intercept of $(0,0)$
- End behavior: As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to \infty$
- Increasing on $(0, \infty)$
- Decreasing on $(-\infty, 0)$
- Absolute minimum at $(0,0)$.

- Origin symmetry (odd)

Example 3

Write an equation for the function graphed.



Solution:
First, we must recognize the sharp V-shape of the graph as an absolute value function, $g(x) = |x|$. We'll need to find a, b, c, and d, so we can write our function in the form

$$f(x) = ag(bx + c) + d,$$

that is,

$$f(x) = a|bx + c| + d.$$

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 and down 2 from the basic toolkit function. This tells us $c = -3$ and $d = 2$.

We might also notice that the graph appears stretched, since the linear portions have slopes of 2 and -2. From this information we could write the write the equation in two ways:

$f(x) = 2|x - 3| - 2,$ treating the stretch as a vertical stretch
$f(x) = |2(x - 3)| - 2,$ treating the stretch as a horizontal compression

Note that these equations are algebraically equivalent – the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch/compression.

If you had not been able to determine the stretch based on the slopes of the lines, you can solve for the stretch factor by putting in a known pair of values for *x* and $f(x)$.

$f(x) = a|x - 3| - 2$ Now substituting in the point (1, 2)
$2 = a|1 - 3| - 2$
$4 = 2a$
$a = 2$

Try it Now

Given the description of the transformed absolute value function write the equation. The absolute value function is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

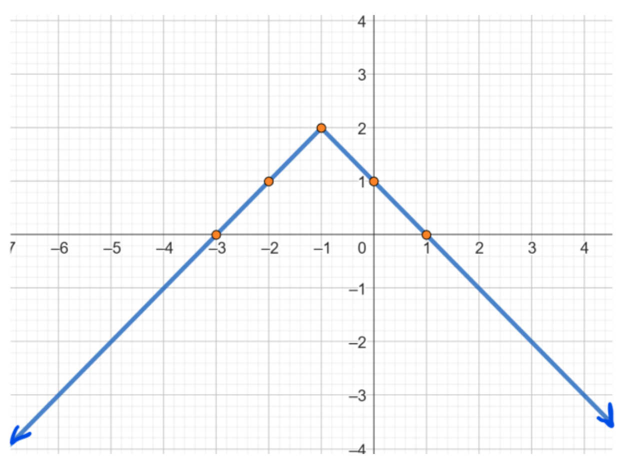
Example 4

Graph the function $g(x) = -|x + 1| + 2$. Then describe its domain and range, intercepts, end behavior, intervals of increasing and decreasing, and any maxima or minima.

Solution:

Taking a transformations perspective, we first notice the absolute value bars. That tells us our toolkit function is $f(x) = |x|$. If we match up our transformations to the form $g(x) = af(bx + c) + d = a|bx + c| + d$, we have $a = -1$, $c = 1$, and $d = 2$. We'll apply c first to create a shift left 1 unit, then apply a to create a vertical reflection. We'll end with d, a shift up 2. This process creates the following table and graph.

$X = x - 1$	$\leftarrow x$	$f(x) \rightarrow$	$y = -1 \cdot f(x) \rightarrow$	$g(X) = y + 2$
-3	-2	2	-2	0
-2	-1	1	-1	1
-1	0	0	0	2
0	1	1	-1	1
1	2	2	-2	0

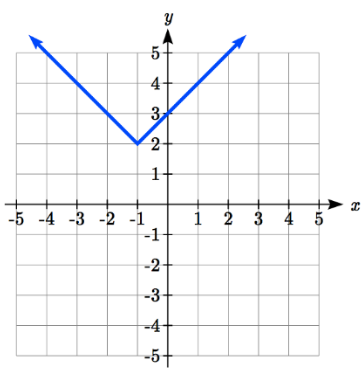


This function has domain $(-\infty, \infty)$ and range $(-\infty, 2]$.
The x-intercepts are $(-3,0)$ and $(1,0)$. The y-intercept is $(0,1)$.
The end behavior is as $x \to \infty, g(x) \to -\infty$ and as $x \to -\infty, g(x) \to -\infty$.
The graph is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$.
The function has an absolute maximum value of 2 at $x = -1$.

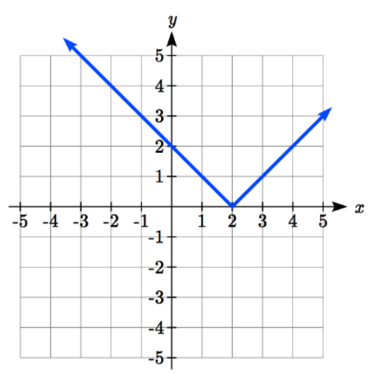
Try it Now
Graph the function $h(x) = |-2x - 3|$. Then describe its domain and range, intercepts, end behavior, intervals of increasing and decreasing, and any maxima or minima.

The graph of an absolute value function will have a vertical intercept when the input is zero. The graph may or may not have horizontal intercepts, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to have zero, one, or two horizontal intercepts.

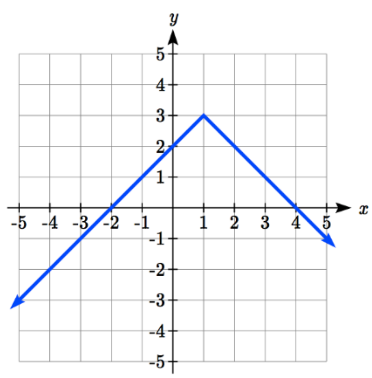
Zero horizontal intercepts



One



Two



To find the horizontal intercepts, we will need to solve an equation involving an absolute value.

Notice that the absolute value function is not one-to-one, so typically inverses of absolute value functions are not discussed.

Solving Absolute Value Equations

To solve an equation like $8 = |2x - 6|$, we can notice that the absolute value will be equal to eight if the quantity *inside* the absolute value were 8 or -8. This leads to two different equations we can solve independently:

$2x - 6 = 8$	or	$2x - 6 = -8$
$2x = 14$		$2x = -2$
$x = 7$		$x = -1$

Solutions to Absolute Value Equations

An equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$

Solving Absolute Value Equations

To solve an absolute value equation:

1. Isolate the absolute value so we have $|\text{stuff}| = B$
2. Rewrite so that stuff $= B$ or stuff $= -B$
3. Solve each resulting equation

Example 5

Find the horizontal intercepts of the graph of $f(x) = |4x + 1| - 7$

Solution:
The horizontal intercepts will occur when $f(x) = 0$. Solving,

$$0 = |4x + 1| - 7 \quad \text{Isolate the absolute value on one side of the equation}$$
$$7 = |4x + 1| \quad \text{Now we can break this into two separate equations:}$$
$$7 = 4x + 1 \qquad -7 = 4x + 1$$
$$6 = 4x \qquad -8 = 4x$$
$$x = \frac{6}{4} \qquad x = -\frac{8}{4}$$
$$x = \frac{3}{2} \qquad x = -2$$

The graph has two horizontal intercepts, at $x = \frac{3}{2}$ and $x = -2$.

Example 6

Solve $1 = 4|x - 2| + 2$

Solution:
Isolating the absolute value on one side the equation,

$$1 = 4|x - 2| + 2$$
$$-1 = 4|x - 2|$$
$$-\frac{1}{4} = |x - 2|$$

At this point, we notice that this equation has no solutions – the absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value.

Try it Now

Find the horizontal & vertical intercepts for the function $f(x) = -|x + 2| + 3$

Solving Absolute Value Inequalities

When absolute value inequalities are written to describe a set of values, like the inequality $|x - 5| \le 4$ we wrote earlier, it is sometimes desirable to express this set of values without the absolute value, either using inequalities, or using interval notation.

We will explore two approaches to solving absolute value inequalities:

1) Using the graph
2) Using test values

Example 7

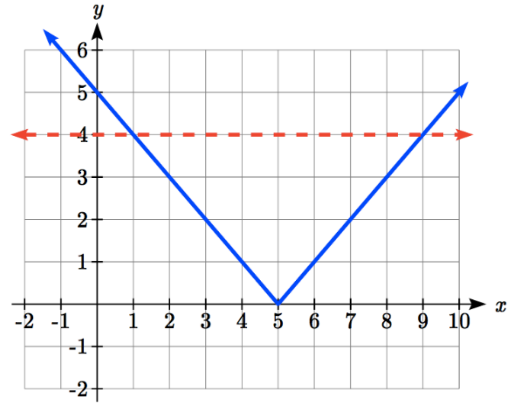
Solve $|x - 5| \le 4$

Solution:

With both approaches, we will need to know first where the corresponding *equality* is true. In this case, we first will find where $|x - 5| = 4$. We do this because the absolute value is a nice friendly function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x - 5| = 4$,

$$\begin{array}{ccc} x - 5 = 4 & \text{or} & x - 5 = -4 \\ x = 9 & & x = 1 \end{array}$$

To use a graph, we can sketch the function $f(x) = |x - 5|$. To help us see where the outputs are 4, the line $g(x) = 4$ could also be sketched.



On the graph, we can see that indeed the output values of the absolute value are equal to 4 at *x* = 1 and *x* = 9. Based on the shape of the graph, we can determine the absolute value is less than or equal to 4 between these two points, when $1 \leq x \leq 9$. In interval notation, this would be the interval [1,9].

As an alternative to graphing, after determining that the absolute value is equal to 4 at *x* = 1 and *x* = 9, we know the graph can only change from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals: *x*<1, 1<*x*<9, and *x*>9. To determine when the function is less than 4, we could pick a value in each interval and see if the output is less than or greater than 4.

Interval	Test *x*	*f(x)*	<4 or >4?
x<1	0	$\lvert 0-5\rvert = 5$	greater
1<*x*<9	6	$\lvert 6-5\rvert = 1$	less
x>9	11	$\lvert 11-5\rvert = 6$	greater

Since $1 \leq x \leq 9$ is the only interval in which the output at the test value is less than 4, we can conclude the solution to $|x-5| \leq 4$ is $1 \leq x \leq 9$.

Example 8

Given the function $f(x) = -\frac{1}{2}|4x-5|+3$, determine for what *x* values the function values are negative.

Solution:

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3 < 0$.
We begin by isolating the absolute value:

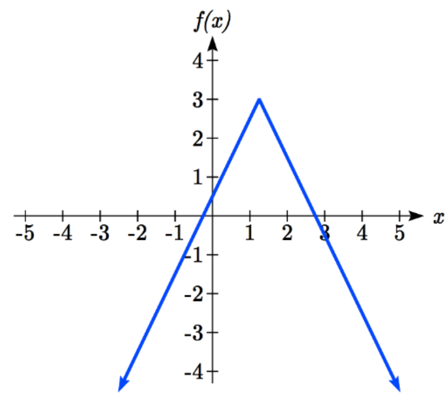
$-\frac{1}{2}|4x-5| < -3$ when we multiply both sides by -2, it reverses the inequality

$|4x-5| > 6$

Next we solve for the equality $|4x-5| = 6$

$$\begin{aligned} 4x-5 &= 6 \\ 4x &= 11 \\ x &= \frac{11}{4} \end{aligned} \quad \text{or} \quad \begin{aligned} 4x-5 &= -6 \\ 4x &= -1 \\ x &= -\frac{1}{4} \end{aligned}$$

We can now either pick test values or sketch a graph of the function to determine on which intervals the original function value are negative. Notice that it is not even really important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = \frac{-1}{4}$ and $x = \frac{11}{4}$, and that the graph has been reflected vertically due to the negative a-value.



From the graph of the function, we can see the function values are negative to the left of the first horizontal intercept at $x = \frac{-1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality:

$$x < \frac{-1}{4} \quad or \quad x > \frac{11}{4}$$

In interval notation, this would be $\left(-\infty, \frac{-1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$.

Try it Now

Solve $-2|k-4| \leq -6$

Important Topics of this Section
The properties of the absolute value function
Solving absolute value equations
Finding intercepts
Solving absolute value inequalities

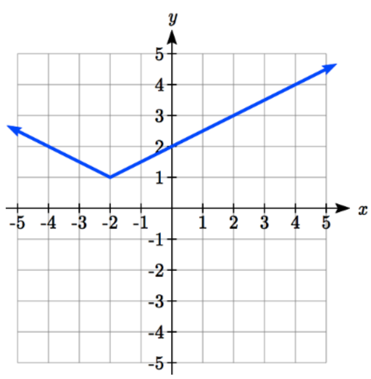
Section 2.4 Exercises

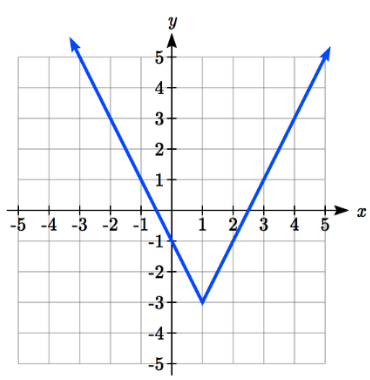
Conceptual Questions

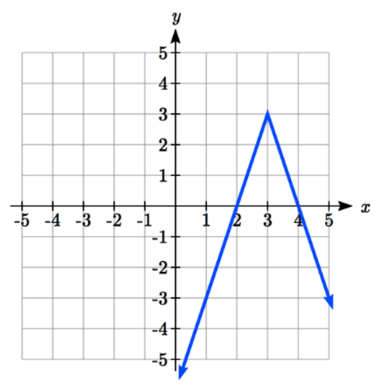
1. True or false: an absolute value function *must* have two x-intercepts.
2. True or false: an absolute value function *must* have only one y-intercept.
3. True or false: an absolute value function will always have y-axis symmetry.
4. True or false: the rate of change of an absolute value function is always constant.

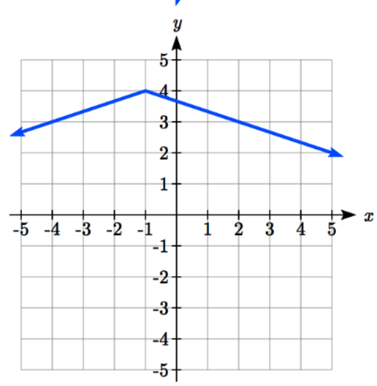
Practice Problems

Write an equation for each transformation of $\boldsymbol{f(x) = |x|}$

1. 

2. 

3. 

4. 

Sketch a graph of each function. Then describe the domain, range, intercepts, intervals of increasing and decreasing, and any maximum or minimum.

5. $f(x) = -|x-1|-1$
6. $f(x) = -|x+3|+4$
7. $f(x) = 2|x+3|+1$
8. $f(x) = 3|x-2|-3$
9. $f(x) = |2x-4|-3$
10. $f(x) = |3x+9|+2$

Solve each equation

11. $|5x - 2| = 11$
12. $|4x + 2| = 15$
13. $2|4 - x| = 7$
14. $3|5 - x| = 5$
15. $3|x + 1| - 4 = -2$
16. $5|x - 4| - 7$

Find the horizontal and vertical intercepts of each function

17. $f(x) = 2|x + 1| - 10$
18. $f(x) = 4|x - 3| + 4$
19. $f(x) = -3|x - 2| - 1$
20. $f(x) = -2|x + 1| + 6$

Solve each inequality

21. $|x + 5| < 6$
22. $|x - 3| < 7$
23. $|x - 2| \geq 3$
24. $|x + 4| \geq 2$
25. $|3x + 9| < 4$
26. $|2x - 9| \leq 8$

Section 2.5: Completing the Square

Essential Review

Example 1

Expand $(x + 9)^2$.

Solution:

The **really** important thing to remember here is that we can't "distribute" a square over addition. When we see that little exponent of 2, we need to think of that as telling us to multiply $x + 9$ by itself: $(x + 9)^2 = (x + 9)(x + 9)$. From there we use distribution (aka FOIL) to expand:

$$\begin{gathered}(x + 9)(x + 9) \\ = x \cdot x + 9x + 9x + 9 \cdot 9 \\ = x^2 + 18x + 81\end{gathered}$$

Example 2

Factor $x^2 - 14x + 49$.

Solution:

We could use our "ac" method here, or we could maybe recognize a special pattern. Notice that 14 is $7 \cdot 2$ and 49 is 7^2 ; this is a perfect square trinomial! Therefore, we can factor it as follows:

$$\begin{gathered}x^2 - 14x + 49 \\ = (x - 7)(x - 7) \\ = (x - 7)^2.\end{gathered}$$

Example 3

Factor $5x^2 + 40x + 80$.

Solution:

Here, we first notice that 5, 40, and 80 have a common factor of 5. Let's factor that out first and see where we can go from there.

$$\begin{gathered}5x^2 + 40x + 80 \\ = 5(x^2 + 8x + 16)\end{gathered}$$

Now, we see again that 8 is $4 \cdot 2$ and 16 is 4^2, so $x^2 + 8x + 16$ is another perfect square trinomial. Therefore, our final factorization is $5(x + 8)^2$.

In this section, we're going to make use of the patterns we see above in a new way. This will allow us to solve quadratics in a new way, and eventually it will help us to reveal hidden information about equations.

Completing The Square

Before we get into this technique, we need to work on some pattern recognition. Let's square some binomials.

$$(x+9)^2$$
$$= (x+9)(x+9)$$
$$= x^2 + 9x + 9x + 81$$
$$= x^2 + 9x + 81$$

What we see is that the middle bx term is twice the original 9, and the ending c term is the square of 9. We're going to make use of that pattern to **create** perfect square trinomials when we don't already have one.

Binomial Squares Pattern
If n is a real number, $$(x+n)^2$$ $$= x^2 + \mathbf{2n}x + \boldsymbol{n^2}$$

Now, our new goal is to "make" a perfect square trinomial.

Example 4

Rewrite $x^2 + 6x$ in the form $(x+n)^2 + m$.

Solution:
Our first goal is to identify what our n should be. We only have two terms, x^2 and $6x$. When we match this up to our binomial squares pattern, we see

$$\begin{array}{lll} x^2 & +\mathbf{6}x & \\ x^2 & +\mathbf{2n}x & +n^2 \end{array}$$

When we line up our terms, we see that our $2n$ and 6 match up. If we let $2n = 6$, we see that n needs to be 3. That makes the missing n^2 term $3^2 = 9$.

However, we can't just add a 9 onto the end of our expression, because that would completely change it! The expression $x^2 + 6x$ is **different** from $x^2 + 6x + 9$. What we're **always** allowed to add, though, is a fancy form of zero. So, we'll add +9-9 to the end of our expression.

$$x^2 + 6x = x^2 + 6x + 9 - 9$$

Now, we'll factor the perfect square portion of our expression, and leave the extra -9 at the end!

$$x^2 + 6x = (x+3)^2 - 9$$

Let's recap what just happened in that example before moving on.

Completing the square for $x^2 + bx$

Goal: Write in the form $(x+n)^2 + m$

1. To identify n, let $b = 2n$ and solve.
2. Add $n^2 - n^2$ to the end of the expression (a fancy form of zero!).
3. Factor the perfect square portion of the expression, leaving the $-n^2$ alone at the end.

Example 5

Write in the form $(x+n)^2 + m$ by completing the square.

$$x^2 - 26x$$

Solution:

Let's line things up again. We want to make use this time of the pattern $(x-n)^2 = x^2 - 2n\,x + n^2$.

$$\begin{array}{lll} x^2 & \mathbf{-26}x & \\ x^2 & +\mathbf{2n}x & +n^2 \end{array}$$

Here we see $-2n = -26$, so $n = 13$. That makes $n^2 = (-13)^2 = 169$, so we'll add 169-169 (our fancy form of zero) to the end of our expression, then factor the perfect square part, leaving the -169 at the end.

$$\begin{aligned} x^2 - 26x + 169 - 169 \\ = (x-13)^2 - 169 \end{aligned}$$

Example 6

Write in the form $(x+n)^2 + m$ by completing the square.

$$x^2 - 9x$$

Solution:

Let's line things up again.

$$\begin{array}{lll} x^2 & \mathbf{-9}x & \\ x^2 & +\mathbf{2n}x & +n^2 \end{array}$$

Here we see $2n = -9$, so $n = \frac{9}{2}$. It's not pretty, but that makes $n^2 = \left(-\frac{9}{2}\right)^2 = \frac{81}{4}$, so we'll add $\frac{81}{4} - \frac{81}{4}$ (our fancy form of zero) to the end of our expression, then factor the perfect square part, leaving the $-\frac{81}{4}$ at the end.

$$x^2 - 9x + \frac{81}{4} - \frac{81}{4}$$

Now, the factoring seems especially tricky, but remember the pattern we're using! We want to end up with $(x + n)^2$ for the factored portion, and we already figured out that $n = -\frac{9}{2}$. So, our final factored answer should be

$$\left(x - \frac{9}{2}\right)^2 - \frac{81}{4}.$$

Try it Now:
Given the expression $x^2 + 8x$, rewrite in the form $(x + n)^2 + m$ by completing the square.

Solving Quadratics by Completing the Square

One reason we might want to complete the square is to solve a quadratic. This strategy is actually **necessary** to derive the handy dandy quadratic formula! Let's see it in action.

Example 7
Solve $x^2 + 6x = 40$ by completing the square.

Solution:
First, we'll work on completing the square for the $x^2 + 6x$ terms. Using previous examples, we'll see that $n = 3$, so $n^2 = 9$, and we'll have:

$$x^2 + 6x = 40$$
$$x^2 + 6x + 9 - 9 = 40$$
$$(x + 3)^2 - 9 = 40$$

Now this equation is nicely set up to use the square root strategy, if we move the 9 to the other side.

$$(x + 3)^2 = 49$$
$$\sqrt{(x + 3)^2} = \sqrt{49}$$
$$x + 3 = \pm 7$$

We finish this up by solving $x + 3 = 7$ and $x + 3 = -7$, for solutions of $x = 4$ and $x = -10$, respectively.

Example 8

Solve $x^2 + 8x = 48$ by completing the square

Solution:
First, we'll work on completing the square for the $x^2 + 8x$ terms. Using previous examples, we'll see that $n = 4$, so $n^2 = 16$, and we'll have:

$$x^2 + 8x = 48$$
$$x^2 + 8x + 16 - 16 = 48$$
$$(x+4)^2 - 16 = 48$$

Now this equation is nicely set up to use the square root strategy, if we move the 16 to the other side.

$$(x+4)^2 = 64$$
$$\sqrt{(x+4)^2} = \sqrt{64}$$
$$x + 4 = \pm 8$$

We finish this up by solving $x + 4 = 8$ and $x + 4 = -8$, for solutions of $x = 4$ and $x = -12$, respectively.

Try it Now

Solve each equation by completing the square.

1. $x^2 + 2x + 5 = 0$
2. $2x^2 + 4x + 16 = 0$

Completing the Square when $a \neq 1$

So far, we've been looking at the best case scenarios. What happens when there's a leading coefficient of something other than 1, like this?

$$3x^2 - 12x$$

In this case, we'll need to do a little extra factoring before we can proceed (carefully).

Example 9

Write $3x^2 - 12x$ in the form $a(x+n)^2 + m$ by completing the square.

Solution:

Before we can make use of our $(x+n)^2$ pattern, we need to factor out the leading 3:

$$\begin{aligned} &3x^2 - 12x \\ &= 3(x^2 - 4x) \end{aligned}$$

Now, we're going to match the $x^2 - 4x$ portion of our expression up with the $x^2 + 2nx + n^2$ pattern, but we're going to keep that factored 3 in mind as we do.

$$\begin{array}{llll} 3(x^2 & -4x & &) \\ (x^2 & +2nx & +n^2) & \end{array}$$

Now we see that $2n = -4$, so $n = -2$. Here is where we need to be careful. Our n^2 is still $(-2)^2 = 4$, but when we add our fancy zero, $4 - 4$, that's all **inside** our parentheses, being multiplied by a 3 on the outside:

$$3(x^2 - 4x + 4 - 4).$$

Now, we'll factor the perfect square portion, and see what we have left.

$$3((x-2)^2 - 4)$$

If we distribute the 3, we have $3(x-2)^2 - 3 \cdot 4 = 3(x-2)^2 - 12$ for our final answer.

That's a little more tricky. Let's try another example.

Example 10

Write $4x^2 - 24x$ in the form $a(x+n)^2 + m$ by completing the square.

Solution:
Before we can make use of our $(x+n)^2$ pattern, we need to factor out the leading 4:

$$\begin{aligned} &4x^2 - 24x \\ &= 4(x^2 - 6x) \end{aligned}$$

Now, we're going to match the $x^2 - 6x$ portion of our expression up with the $x^2 + 2nx + n^2$ pattern, but we're going to keep that factored 4 in mind as we do.

$$\begin{array}{llll} 4(x^2 & -6x & &) \\ (x^2 & +2nx & +n^2) & \end{array}$$

Now we see that $2n = -6$, so $n = -3$. Here is where we need to be careful. Our n^2 is still $(-3)^2 = 9$, but when we add our fancy zero, $9 - 9$, that's all **inside** our parentheses, being multiplied by a 4 on the outside:

$$4(x^2 - 6x + 9 - 9).$$

Now, we'll factor the perfect square portion, and see what we have left.

$$4((x - 3)^2 - 9)$$

If we distribute the 4, we have $4(x - 3)^2 - 4 \cdot 9 = 4(x - 3)\text{^}2$ -36 for our final answer.

Try it Now

Rewrite $2x^2 + 16x$ in the form $a(x + n)^2 + m$ by completing the square.

Example 11

Solve by completing the square.

$$2x^2 + 16x + 14 = 0$$

Solution:
Start by factoring out the 2 from only the first terms. Remember, the 14 isn't really involved in completing the square; we're only concerned about the ax^2 and bx terms.

$$2x^2 + 16x + 14 = 0$$
$$2(x\text{^}2 + 8x + \quad) + 14 = 0$$

Now, we want to complete the square for $x\text{^}2 + 8x$. We see that $2n = 8$, so $n = 4$ and $n^2 = 16$. Then we add $16 - 16$ **inside** our parentheses.

$$2(x^2 + 8x + 16 - 16) + 14 = 0$$

Factor, then simplify, and we have

$$2((x + 4)^2 - 16) + 14 = 0$$
$$2(x + 4)^2 - 2 \cdot 16 + 14 = 0$$
$$2(x + 4)^2 - 18 = 0$$

This equation is ready for solving via square roots. We'll move the 18 to the other side of the equals sign, then divide by 2, then root both sides to undo the square.

$$2(x + 4)^2 = 18$$
$$(x + 4)^2 = 9$$
$$\sqrt{(x + 4)^2} = \sqrt{9}$$
$$x + 4 = \pm 3$$
$$x = -1, -7$$

Example 12

Solve by completing the square.

$$2x^2 - 3x = 20$$

Solution:

Start by factoring out the 2 from the terms on the left hand side. This feels a little weird, since 2 isn't a factor of 3. Keep in mind that factoring out a common factor is kind of like dividing each term by that factor.

$$2x^2 - 3x = 20$$
$$2(x^2 - \frac{3}{2}x + \qquad) = 20$$

Now, we want to complete the square for $x^2 - \frac{3}{2}x$. We see that $2n = -\frac{3}{2}, so\ n =$ and $n^2 = \frac{9}{16}$. Then we add $\frac{9}{16} - \frac{9}{16}$ **inside** our parentheses.

$$2\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right) = 20$$

Factor (remembering that our $n = \frac{3}{4}$, then simplify, and we have

$$2\left(\left(x - \frac{3}{4}\right)^2 - \frac{9}{16}\right) = 20$$
$$2\left(x - \frac{3}{4}\right)^2 - 2 \cdot \frac{9}{16} = 20$$
$$2\left(x - \frac{3}{4}\right)^2 - \frac{9}{8} = 20$$

This equation is ready for solving via square roots. We'll move the $\frac{9}{8}$ to the other side of the equals sign, then divide by 2, then root both sides to undo the square.

$$2\left(x - \frac{3}{4}\right)^2 = \frac{169}{8}$$
$$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$$
$$\sqrt{\left(x - \frac{3}{4}\right)^2} = \sqrt{\frac{169}{16}}$$
$$x - \frac{3}{4} = \frac{13}{4} \quad or \quad x - \frac{3}{4} = -\frac{13}{4}$$
$$x = \frac{16}{4}, -\frac{10}{4}$$
$$x = 4, -\frac{5}{2}$$

Section 2.5 Exercises

Conceptual Questions

1. Describe the process of completing the square when $a = 1$.
2. Describe the process of completing the square when $a \neq 1$.

Practice Problems

In the following exercises, complete the square to make a perfect square trinomial. Then write the result in the form $(\boldsymbol{x}+\boldsymbol{n})^2 + \boldsymbol{m}$.

1. $x^2 - 24x$
2. $x^2 - 11x$
3. $x^2 - \frac{1}{3}x$
4. $x^2 - 16x$
5. $x^2 + 15x$
6. $x^2 + 3/4\, x$
7. $x^2 - 22x$
8. $2x^2 + 10x$
9. $5x^2 + 2x$
10. $3x^2 - 18x$
11. $4x^2 - 28x$
12. $3x^2 - 2x$

Solve the following equations using the strategy of completing the square.

13. $x^2 + 2x = 3$
14. $x^2 + 12x = -11$
15. $x^2 - 20x - 21 = 0$
16. $x^2 - 8 = 2x$
17. $x^2 + 4x = -44$
18. $x^2 = 2x - 3$
19. $x^2 + 5x = 2$
20. $x^2 - 3x - 2 = 0$
21. $x^2 - 14x + 12 = -1$
22. $x^2 = 9x + 2$
23. $x^2 = 5x - 1$
24. $x^2 - 5 = 10x$
25. $x^2 - 14 = 6x$
26. $3x^2 + 30x - 27 = 6$
27. $2x^2 - 14x + 12 = 0$
28. $2x^2 + 4x = 26$
29. $5x^2 + 20x = 15$
30. $2x^2 + x = 6$
31. $3x^2 - 4x = 15$
32. $2x^2 + 7x - 15 = 0$
33. $3x^2 - 14x + 8 = 0$
34. $2x^2 + 7x = 14$
35. $3x^2 - 5x = 9$
36. $5x^2 - 3x = -10$
37. $7x^2 + 4x = -3$

38. Solve the equation $x^2 + 10x = -25$…
    a. by factoring
    b. by completing the square
    c. Which method do you prefer? Why?

Section 2.6 Power Functions & Polynomial Functions

A square is cut out of cardboard, with each side having length *L*. If we wanted to write a function for the area of the square, with *L* as the input and the area as output, you may recall that the area of a rectangle can be found by multiplying the length times the width. Since our shape is a square, the length & the width are the same, giving the formula:

$$A(L) = L \cdot L = L^2$$

Likewise, if we wanted a function for the volume of a cube with each side having some length *L*, you may recall volume of a rectangular box can be found by multiplying length by width by height, which are all equal for a cube, giving the formula:

$$V(L) = L \cdot L \cdot L = L^3$$

These two functions are examples of **power functions**, functions that are some power of the variable.

Power Function
A **power function** is a function that can be represented in the form $$f(x) = x^p$$ Where the base is a variable and the exponent, p, is a number.

Technically, the toolkit linear function $f(x) = x$ is a power function, since it can be written as $f(x) = x^1$. We'll be studying some power functions with more interesting exponents in this chapter.

Toolkit Power Functions (Quadratic)

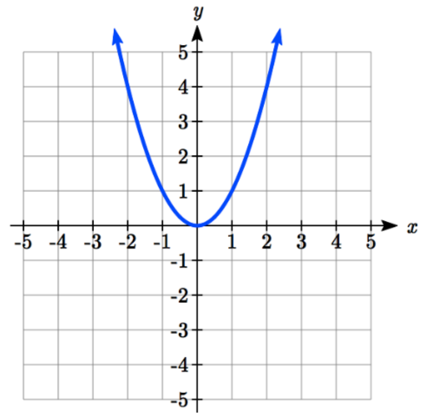
The toolkit power function with an exponent of 2 is

$$f(x) = x^2.$$

It is called the quadratic function.

This function is described by the following table and graph:

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4



Features of the toolkit quadratic function

The quadratic function has the following features:

- Domain: $(-\infty, \infty)$
- Range: $[0, \infty)$
- Intercepts: x and y intercepts at $(0,0)$.
- End behavior: As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to \infty$.
- Increasing on $(0, \infty)$
- Decreasing on $(-\infty, 0)$
- Concave up on (∞, ∞)
- Symmetric about the y-axis (even)

Toolkit Power Functions (Cubic)

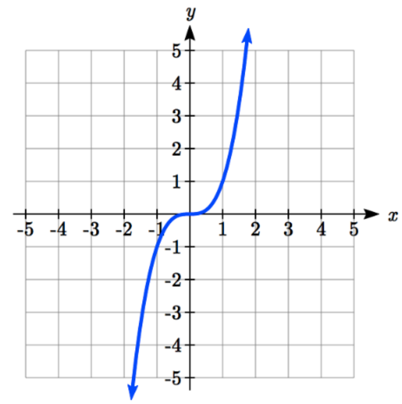
The toolkit power function with an exponent of 3 is

$$f(x) = x^3.$$

It is called the cubic function.

This function is described by the following table and graph:

x	$f(x)$
-2	-8
-1	-1
0	0
1	1
2	8



Features of the toolkit cubic function

The cubic function has the following features:

- Domain: $(-\infty, \infty)$
- Range: $(-\infty, \infty)$
- Intercepts: x and y intercepts at $(0,0)$.
- End behavior: As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to -\infty$.
- Increasing on $(-\infty, 0) \cup (0, \infty)$
- Concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.
- Symmetric about the origin (odd)

Let's spend a little time playing around with transformations of these functions.

Example 1

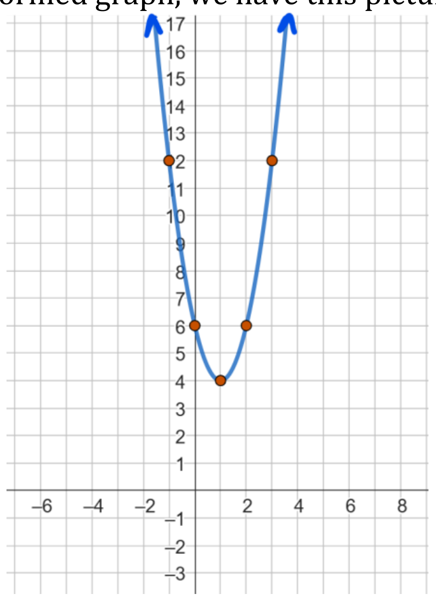
Graph $g(x) = 2(x-1)^2 + 4$ using transformation techniques. Then identify the domain, range, intercepts, end behavior, and intervals of increasing or decreasing. Finally, identify any symmetry present.

Solution:
First, we look at the structure to determine our toolkit function. Here, we see that exponent of 2; that tells us we're dealing with $f(x) = x^2$ as a parent. We want to identify a, b, c, and d for $g(x) = af(bx+c) + d = a(bx+c)^2 + d$.

In this case, we have $a = 2$, $c = -1$, and $d = 4$. We'll use c to shift the graph right 1, then use a to vertically stretch by a factor of 2, and finally use d to shift the graph up 4. In table form, that looks as follows.

$X = x + 1$	$\leftarrow x$	$f(x) = x^2 \rightarrow$	$y = 2 \cdot f(x) \rightarrow$	$g(X) = y + 4$
-1	-2	4	8	12
0	-1	1	2	6
1	0	0	0	4
2	1	1	2	6
3	2	4	8	12

When we graph our transformed graph, we have this picture:



The function has the following features:

- Domain: $(-\infty, \infty)$
- Range: $[4, \infty)$
- Intercepts: y intercept at (0,6). No x-intercepts.
- End behavior: As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to \infty$.
- Increasing on $(-\infty, 1) \cup (1, \infty)$
- Symmetry: symmetric about the line $x = 1$. Neither even nor odd.

In that example, we were using the graph of the function to identify our features. We could have used algebra to figure out things like the intercepts and whether the function was even or odd. Depending on what we want to do, sometimes graphing will be quicker, and sometimes algebra will be easier for determining these features!

Example 2

Graph $g(x) = \left(\frac{1}{2}x + 1\right)^3 - 1$ using transformation techniques. Then identify the domain, range, intercepts, end behavior, and intervals of increasing or decreasing. Finally, identify any symmetry present.

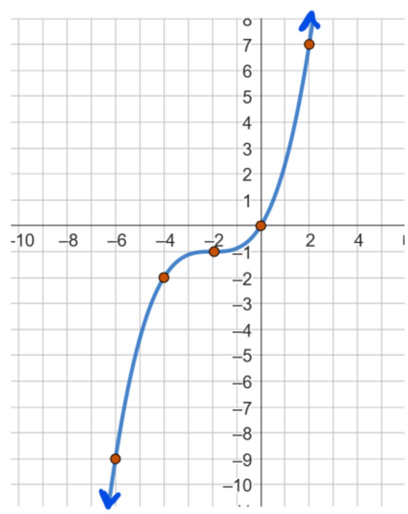
Solution:
This time we see that exponent of 3, which means we're dealing with the toolkit function $f(x) = x^3$. We want to find a, b, c, and d so that $g(x) = af(bx + c) + d = a(bx + c)^3 + d$.

We see $c = 1$, so we start with a shift left 1. Then $b = \frac{1}{2}$, so there is a horizontal stretch by a factor of 2. Finally, $d = -1$, so the graph shifts down 1. We show this process in the following table and graph:

$X \cdot \frac{2}{1}$	$X = x - 1$	x	$f(x)$	$y = f(x) - 1$
-6	-3	-2	-8	-9
-4	-2	-1	-1	-2
-2	-1	0	0	-1
0	0	1	1	0
2	1	2	8	7

The function has the following features:

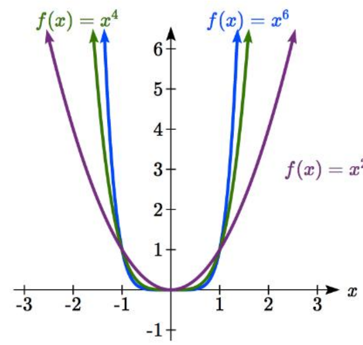
- Domain: $(-\infty, \infty)$
- Range: $(-\infty, \infty)$
- Intercepts: x and y-intercept at $(0,0)$.
- End behavior: As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to -\infty$.
- Increasing on $(-\infty, -2) \cup (-2, \infty)$
- Symmetry: Neither even nor odd.



Try it Now

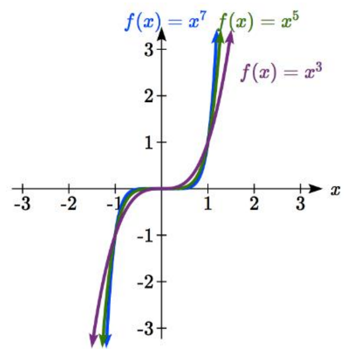
Graph $g(x) = -(x-2)^3 - 4$ using transformation techniques. Then identify the domain, range, intercepts, end behavior, and intervals of increasing or decreasing. Finally, identify any symmetry present.

Characteristics of Power Functions



Shown to the right are the graphs of $f(x) = x^2$, $f(x) = x^4$, and $f(x) = x^6$, all even whole number powers. Notice that all these graphs have a fairly similar shape, very similar to the quadratic toolkit, but as the power increases the graphs flatten somewhat near the origin, and become steeper away from the origin.

For all these even-powered power functions, we see the same end behavior: as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty, f(x) \to \infty$.



Shown here are the graphs of $f(x) = x^3$, $f(x) = x^5$, and $f(x) = x^7$, all odd whole number powers. Notice all these graphs look similar to the cubic toolkit, but again as the power increases the graphs flatten near the origin and become steeper away from the origin.

For these odd power functions, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

Example 3

Describe the long run behavior of the graph of $f(x) = x^8$.

Solution:

Since $f(x) = x^8$ has a whole, even power, we would expect this function to behave somewhat like the quadratic function. As the input gets large positive or negative, we would expect the output to grow without bound in the positive direction. In symbolic form, as $x \to \pm\infty$, $f(x) \to \infty$.

Example 4

Describe the long run behavior of the graph of $f(x) = -x^9$

Solution:

Since this function has a whole odd power, we would expect it to behave somewhat like the cubic function. The negative in front of the x^9 will cause a vertical reflection, so as the inputs grow large positive, the outputs will grow large in the negative direction, and as the inputs grow large negative, the outputs will grow large in the positive direction. In symbolic form, for the long run behavior we would write: as $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$.

Try it Now

Describe in words and symbols the long run behavior of $f(x) = -x^4$

Polynomials

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. If we wanted to write a formula for the area covered by the oil slick, we could do so by composing two functions together. The first is a formula for the radius, *r*, of the spill, which depends on the number of weeks, *w*, that have passed.

Hopefully you recognized that this relationship is linear:

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area, *A*, of a circle:

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks:

$$A(w) = A(r(w)) = A(24 + 8w) = \pi(24 + 8w)^2$$

Multiplying this out gives the formula

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a **polynomial**. A polynomial is simply the sum of terms each consisting of a vertically stretched or compressed power function with non-negative whole number power.

Terminology of Polynomial Functions

A **polynomial** is function that can be written as $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

Each of the a_i constants are called **coefficients** and can be positive, negative, or zero, and be whole numbers, decimals, or fractions.

A **term** of the polynomial is any one piece of the sum, that is any a_ix^i. Each individual term is a transformed power function.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

The **leading term** is the term containing the highest power of the variable: the term with the highest degree.

The **leading coefficient** is the coefficient of the leading term.

Because of the definition of the "leading" term we often rearrange polynomials so that the powers are descending.

$$f(x) = a_nx^n + \ldots.. + a_2x^2 + a_1x + a_0$$

End Behavior of Polynomials

For any polynomial, the **end behavior** of the polynomial will match the end behavior of the leading term.

Example 5

Identify the degree, leading term, and leading coefficient of these polynomials. Then describe the end behavior of the polynomial.

a) $f(x) = 3 + 2x^2 - 4x^3$
b) $g(t) = 5t^5 - 2t^3 + 7t$
c) $h(p) = 6p - p^4 - 2$

Solution:

a) For the function *f(x)*, the degree is 3, the highest power on *x*. The leading term is the term containing that power, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

Since we have an odd degree, we know that the two ends of the polynomial will point in opposite directions. Normally that's up on the right and down on the left, but here the leading coefficient is negative, so the graph has been flipped upside down. So, as $x \to \infty, f(x) \to -\infty$ and as $x \to -\infty, f(x) \to \infty$.

b) For *g(t)*, the degree is 5, the leading term is $5t^5$, and the leading coefficient is 5.

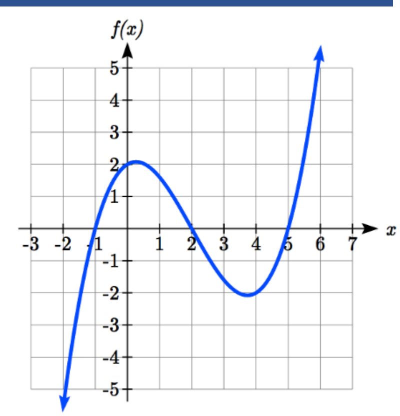
Since we have an odd degree, we know that the two ends of the polynomial will point in opposite directions. This time, the leading coefficient is positive, so as $t \to \infty, g(t) \to \infty$ and as $x \to -\infty, g(t) \to -\infty$.

c) For *h(p)*, the degree is 4, the leading term is $-p^4$, so the leading coefficient is -1.

With this even degree and negative leading coefficient, the end behavior is as $p \to \infty, h(p) \to -\infty$ and as $p \to -\infty, h(p) \to -\infty$.

Example 6

What can we determine about the long run behavior and degree of the equation for the polynomial graphed here?



Solution:
Since the output grows large and positive as the inputs grow large and positive, we describe the long run behavior symbolically by writing: as $x \to \infty, f(x) \to \infty$. Similarly, as $x \to -\infty, f(x) \to -\infty$.

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function which has not been reflected, so the degree of the polynomial creating this graph must be odd, and the leading coefficient would be positive.

Try it Now

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$ use your algebra skills to write the function in standard polynomial form (as a sum of terms) and determine the leading term, degree, and long run behavior of the function.

Short Run Behavior

Characteristics of the graph such as vertical and horizontal intercepts and the places the graph changes direction are part of the short run behavior of the polynomial.

Like with all functions, the vertical intercept is where the graph crosses the vertical axis, and occurs when the input value is zero. Since a polynomial is a function, there can only be one vertical intercept, which occurs at the point $(0, a_0)$. The horizontal intercepts occur at the input values that correspond with an output value of zero. It is possible to have more than one horizontal intercept.

Horizontal intercepts are also called **zeros**, or **roots** of the function.

Example 7

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the vertical and horizontal intercepts.

Solution:
The vertical intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= -2 \cdot 1 \cdot -4 \\ &= 8 \end{aligned}$$

The graph crosses the vertical axis at the point (0, 8).

The horizontal intercepts occur when the output is zero.
$0 = (x-2)(x+1)(x-4)$ when x = 2, -1, or 4, due to the Zero Product Property.
f(x) has zeros, or roots, at x = 2, -1, and 4.

The graph crosses the horizontal axis at the points (2, 0), (-1, 0), and (4, 0)

Notice that the polynomial in the previous example, which would be degree three if multiplied out, had three horizontal intercepts and two turning points – places where the graph changes direction. We will now make a general statement without justifying it – the reasons will become clear later in this chapter.

Intercepts and Turning Points of Polynomials

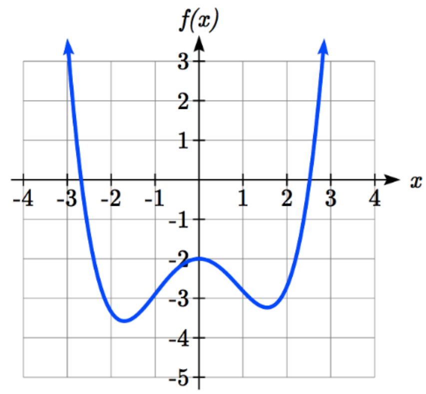
A polynomial of degree n will have:

At most n horizontal intercepts. An odd degree polynomial will always have at least one.

At most $n-1$ turning points

Example 8

What can we conclude about the graph of the polynomial shown here?



Solution:
Based on the long run behavior, with the graph becoming large positive on both ends of the graph, we can determine that this is the graph of an even degree polynomial. The graph has 2 horizontal intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4, so it is probably a fourth degree polynomial.

Try it Now

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the short run behavior.

Important Topics of this Section

Power Functions
Polynomials
Coefficients
Leading coefficient
Term
Leading Term
Degree of a polynomial
Long run behavior
Short run behavior

Section 2.6 Exercises

Conceptual Questions

1. How do you describe the end behavior for polynomials?
2. What are the main differences between $g(x) = -(x+1)^2 - 2$ and $h(x) = -(x+1)^3 - 2$? What are the similarities?
3. True or false: all polynomials must have at least one y-intercept.
4. True or false: all polynomials must have at least one x-intercept.

Practice Problems

Find the end behavior of each function as $\boldsymbol{x} \to \boldsymbol{\infty}$ and $\boldsymbol{x} \to -\boldsymbol{\infty}$

1. $f(x) = x^4$
2. $f(x) = x^6$
3. $f(x) = x^3$
4. $f(x) = x^5$
5. $f(x) = -x^2$
6. $f(x) = -x^4$
7. $f(x) = -x^7$
8. $f(x) = -x^9$

Find the degree and leading coefficient of each polynomial

9. $4x^7$
10. $5x^6$
11. $5 - x^2$
12. $6 + 3x - 4x^3$
13. $-2x^4 - 3x^2 + x - 1$
14. $6x^5 - 2x^4 + x^2 + 3$
15. $(2x+3)(x-4)(3x+1)$
16. $(3x+1)(x+1)(4x+3)$

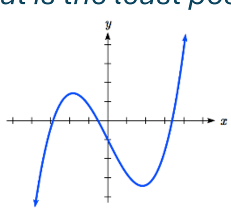
Find the end behavior of each function as $\boldsymbol{x} \to \boldsymbol{\infty}$ and $\boldsymbol{x} \to -\boldsymbol{\infty}$

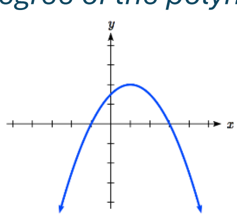
17. $-2x^4 - 3x^2 + x - 1$
18. $6x^5 - 2x^4 + x^2 + 3$
19. $3x^2 + x - 2$
20. $-2x^3 + x^2 - x + 3$

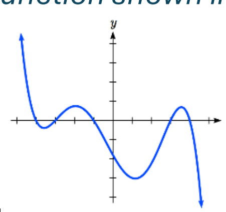
21. What is the maximum number of x-intercepts and turning points for a polynomial of degree 5?

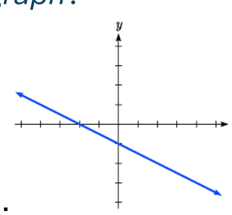
22. What is the maximum number of x-intercepts and turning points for a polynomial of degree 8?

What is the least possible degree of the polynomial function shown in each graph?

23. 

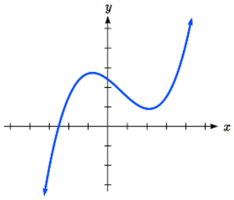
24. 

25. 

26. 

What is the least possible degree of the polynomial function shown in each graph?

27.



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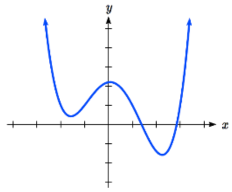

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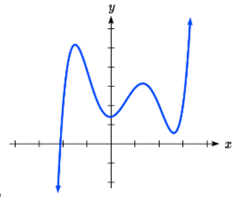

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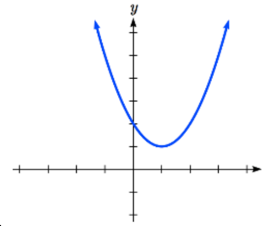


Find the vertical and horizontal intercepts of each function.

31. $f(t) = 2(t-1)(t+2)(t-3)$
32. $f(x) = 3(x+1)(x-4)(x+5)$
33. $g(n) = -2(3n-1)(2n+1)$
34. $k(u) = -3(4-n)(4n+3)$

Graph each function using transformation techniques. Then identify the domain, range, intercepts, end behavior, and intervals of increasing or decreasing. Finally, identify any symmetry present.

35. $g(x) = (x-4)^2$
36. $h(x) = 4 - x^2$
37. $j(x) = -2x^2$
38. $k(x) = (-2x)^2$
39. $g(x) = (x-1)^3$
40. $h(x) = x^3 - 1$
41. $j(x) = -\frac{1}{2}x^3$
42. $k(x) = (-\frac{1}{2}x^3)$
43. $g(x) = -(x+2)^2 - 1$
44. $h(x) = \left(\frac{1}{3}x - 1\right)^2$
45. $j(x) = -(x+2)^3 - 1$
46. $h(x) = \left(\frac{1}{3}x - 1\right)^3$

Section 2.7 Quadratic Functions

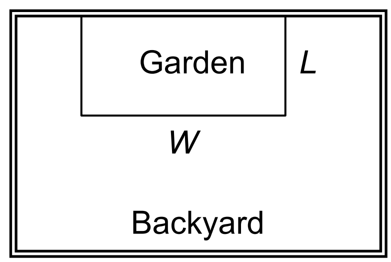
In this section, we will explore the family of 2nd degree polynomials, the quadratic functions. While they share many characteristics of polynomials in general, the calculations involved in working with quadratics is typically a little simpler, which makes them a good place to start our exploration of short run behavior. In addition, quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 1

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4th side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.

Solution:

In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, W, to represent the side of fencing parallel to the 4th side or backyard fence.



Since we know we only have 80 feet of fence available, we know that $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L:
$W = 80 - 2L$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L)$$
$$A(L) = 80L - 2L^2$$

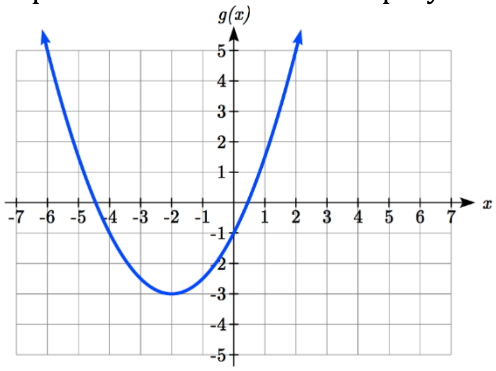
This formula represents the area of the fence in terms of the variable length L.

Describing Quadratics: Vertex

We now explore the interesting features of the graphs of quadratics. In addition to intercepts, quadratics have an interesting feature where they change direction, called the **vertex**. You probably noticed that all quadratics are related to transformations of the basic quadratic function $f(x) = x^2$.

Example 2

Write an equation for the quadratic graphed below as a transformation of $f(x) = x^2$, then expand the formula and simplify terms to write the equation in standard polynomial form.



Solution:

We can see the graph is the toolkit quadratic shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$. By plugging in a point that falls on the grid, such as (0,-1), we can solve for the stretch factor:

$$-1 = a(0+2)^2 - 3$$
$$2 = 4a$$
$$a = \frac{1}{2}$$

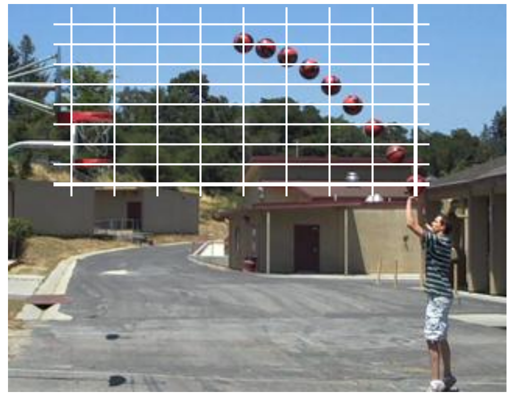
Written as a transformation, the equation for this formula is $g(x) = \frac{1}{2}(x+2)^2 - 3$. To write this in standard polynomial form, we can expand the formula and simplify terms:

$$g(x) = \frac{1}{2}(x+2)^2 - 3$$
$$g(x) = \frac{1}{2}(x+2)(x+2) - 3$$
$$g(x) = \frac{1}{2}(x^2 + 4x + 4) - 3$$
$$g(x) = \frac{1}{2}x^2 + 2x + 2 - 3$$
$$g(x) = \frac{1}{2}x^2 + 2x - 1$$

Notice that the horizontal and vertical shifts of the basic quadratic determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Try it Now

A coordinate grid has been superimposed over the quadratic path of a basketball[11]. Find an equation for the path of the ball. Does he make the basket?



Forms of Quadratic Functions

The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$

The **transformation form** of a quadratic function is $f(x) = a(x-h)^2 + k$

The **vertex** of the quadratic function is located at (h, k), where h and k are the numbers in the transformation form of the function. Because the vertex appears in the transformation form, it is often called the **vertex form.**

In the previous example, we saw that it is possible to rewrite a quadratic function given in transformation form and rewrite it in standard form by expanding the formula. It would be useful to reverse this process, since the transformation form reveals the vertex.

One way of approaching that challenge is by completing the square.

Example 3

Find the vertex of the quadratic $f(x) = 2x^2 - 6x + 7$ by rewriting the quadratic into transformation form (vertex form) using the strategy of completing the square. Determine whether the vertex gives a maximum or minimum value.

Solution:
When completing the square in this context, we'll focus only on the $2x^2 - 6x$ portion of the expression. The $+7$ will be there, but not involved in the completing the square process.

First, we'll factor out $a = 2$:

$$f(x) = 2(x^2 - 3x) + 7$$

Now, we determine n. Since $2n = -3$, we know $n = -\frac{3}{2}$.

[11] From http://blog.mrmeyer.com/?p=4778, © Dan Meyer, CC-BY

Therefore, $n^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$. We need to add a fancy form of 0 *inside* the parentheses to make our perfect square.

$$f(x) = 2\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) + 7$$

Now we factor the perfect square part:

$$f(x) = 2\left(\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right) + 7$$

And we'll finish by distributing the 2 and simplifying.

$$\begin{aligned} f(x) &= 2\left(x - \frac{3}{2}\right)^2 - 2 \cdot \frac{9}{4} + 7 \\ &= 2\left(x - \frac{3}{2}\right)^2 - \frac{9}{2} + \frac{14}{2} \\ &= 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} \end{aligned}$$

Now we see that the vertex of the parabola is at $\left(\frac{3}{2}, \frac{5}{2}\right)$.

We notice that because the a-value of 2 causes a vertical stretch, but no reflection. That tells us that the general shape of our graph will be an upward-opening parabola, so the vertex is a minimum.

There's another way of finding that vertex as well, if we compare the vertex and standard forms of the quadratic.

Expanding out the general transformation form of a quadratic gives:

$$f(x) = a(x - h)^2 + k = a(x - h)(x - h) + k$$
$$f(x) = a(x^2 - 2xh + h^2) + k = ax^2 - 2ahx + ah^2 + k$$

This should be equal to the standard form of the quadratic:

$$ax^2 - 2ahx + ah^2 + k = ax^2 + bx + c$$

The second degree terms are already equal. For the linear terms to be equal, the coefficients must be equal:

$$-2ah = b, \text{ so } h = -\frac{b}{2a}$$

This provides us a method to determine the horizontal shift of the quadratic from the standard form. We could likewise set the constant terms equal to find:

$$ah^2 + k = c, \text{ so } k = c - ah^2 = c - a\left(-\frac{b}{2a}\right)^2 = c - a\frac{b^2}{4a^2} = c - \frac{b^2}{4a}$$

In practice, though, it is usually easier to remember that *k* is the output value of the function when the input is *h*, so $k = f(h)$.

Finding the Vertex of a Quadratic

For a quadratic given in standard form, the vertex (h, k) is located at:

$$h = -\frac{b}{2a}, \quad k = f(h) = f\left(\frac{-b}{2a}\right)$$

Example 4

Find the vertex of the quadratic $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic into transformation form (vertex form).

Solution:

The horizontal coordinate of the vertex will be at $h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}$

The vertical coordinate of the vertex will be at $f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}$

Rewriting into transformation form, the stretch factor will be the same as the a in the original quadratic. Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

Try it Now

Given the equation $g(x) = 13 + x^2 - 6x$ write the equation in standard form and then in transformation/vertex form. Find the vertex, and determine whether it is a maximum or a minimum.

Features of a Quadratic

In addition to enabling us to more easily graph a quadratic written in standard form, finding the vertex serves another important purpose – it allows us to determine the maximum or minimum value of the function, depending on which way the graph opens.

Recall from transformations that the a-value will determine not only the vertical stretch, but also whether the parabola has been flipped vertically. Therefore, if we know the a-value of a quadratic, we can deduce a lot of additional information.

Features of a Quadratic

For a quadratic in either standard ($f(x) = ax^2 + bx + c$) or vertex ($f(x) = a(x - h)^2 + k$) form,

If $a > 0$, the parabola opens up, so

The vertex $(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ gives the location of a minimum.

- The range of the quadratic is $[k, \infty)$.
- The end behavior is as $x \to \pm\infty, f(x) \to \infty$.
- The function is decreasing on $(-\infty, h)$ and increasing on (h, ∞).
- The graph is concave up on $(-\infty, \infty)$.

If $a < 0$, the parabola opens down, so

- The vertex $(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ gives the location of a maximum.
- The range of the quadratic is $(-\infty, k]$
- The end behavior is as $x \to \pm\infty, f(x) \to -\infty$.
- The function is increasing on $(-\infty, h)$ and decreasing on (h, ∞).
- The graph is concave down on $(-\infty, \infty)$.

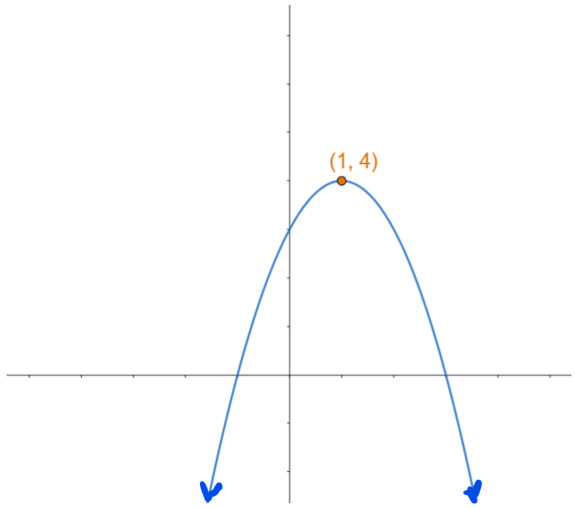
Example 5

Given the function $f(x) = -x^2 + 2x + 3$, describe its domain and range, end behavior, intervals of increasing and decreasing, concavity, maximum or minimum, and intercepts.

Solution:
Once we figure out the vertex, we'll know almost everything we need to know. This quadratic is in standard form, so $a = -1$, $b = 2$, and $c = 3$. That means the vertex is at

$$h = \frac{-2}{2(-1)} = \frac{-2}{-2} = 1$$
$$k = f(h) = -(1)^2 + 2(1) + 3 = 4$$
$$(1,4).$$

Since a is negative, the parabola opens down, so we create a rough picture in our minds that looks something like this:



That's plenty of information to describe the majority of the features of the function.
Domain: $(-\infty, \infty)$
Range: $(-\infty, 4]$
End behavior: as $x \to \pm\infty, f(x) \to -\infty$
Increasing on $(-\infty, 1)$, decreasing on $(1, \infty)$
Maximum value of 4 when $x = 1$
Concave down on $(-\infty, \infty)$

The only things that require slightly more work are the intercepts. To find the x-intercepts, we'll set the function equal to 0 and solve.

$$-x^2 + 2x + 3 = 0$$
$$-(x^2 - 2x - 3) = 0$$
$$-(x-3)(x+1) = 0$$
$$x = 3, -1$$

To find the y-intercept, we substitute in $x = 0$:

$$f(0) = -(0)^2 + 2(0) + 3$$
$$= 3$$

Try it Now
Given the function $f(x) = 2x^2 + 16x$, describe its domain and range, end behavior, intervals of increasing and decreasing, concavity, maximum or minimum, and intercepts.

Example 6

Returning to our backyard farmer from the beginning of the section, what dimensions should she make her garden to maximize the enclosed area?

Solution:

Earlier we determined the area she could enclose with 80 feet of fencing on three sides was given by the equation $A(L) = 80L - 2L^2$. Notice that quadratic has been vertically reflected, since the coefficient on the squared term is negative, so the graph will open downwards, and the vertex will be a maximum value for the area.

In finding the vertex, we take care since the equation is not written in standard polynomial form with decreasing powers. But we know that a is the coefficient on the squared term, so a = -2, b = 80, and c = 0.

Finding the vertex:

$h = -\frac{80}{2(-2)} = 20, \quad k = A(20) = 80(20) - 2(20)^2 = 800$

The maximum value of the function is an area of 800 square feet, which occurs when L = 20 feet. When the shorter sides are 20 feet, that leaves 40 feet of fencing for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet, and the longer side parallel to the existing fence has length 40 feet.

Example 7

A local newspaper currently has 84,000 subscribers, at a quarterly charge of $30. Market research has suggested that if they raised the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution:

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the charge per subscription times the number of subscribers. We can introduce variables, C for charge per subscription and S for the number subscribers, giving us the equation

Revenue = CS

Since the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently S = 84,000 and C = 30, and that if they raise the price to $32 they would lose 5,000 subscribers, giving a second pair of values, C = 32 and S = 79,000. From this we can find a linear equation relating the two quantities. Treating C as the input and S as the output, the equation will have form $S = mC + b$. The slope will be

$$m = \frac{79{,}000 - 84{,}000}{32 - 30} = \frac{-5{,}000}{2} = -2{,}500$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept

$S = -2500C + b$ Plug in the point $S = 84{,}000$ and $C = 30$
$84{,}000 = -2500(30) + b$ Solve for b
$b = 159{,}000$

This gives us the linear equation $S = -2{,}500C + 159{,}000$ relating cost and subscribers. We now return to our revenue equation.

Revenue $= CS$ Substituting the equation for S from above
Revenue $= C(-2{,}500C + 159{,}000)$ Expanding
Revenue $= -2{,}500C^2 + 159{,}000C$

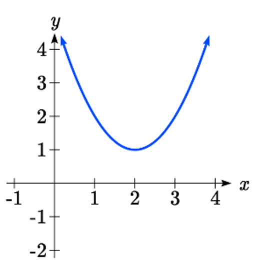
We now have a quadratic equation for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex:

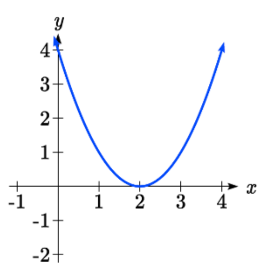
$$h = -\frac{159{,}000}{2(-2{,}500)} = 31.8$$

The model tells us that the maximum revenue will occur if the newspaper charges \$31.80 for a subscription. To find what the maximum revenue is, we can evaluate the revenue equation:
Maximum Revenue $= -2{,}500(31.8)^2 + 159{,}000(31.8) = \$2{,}528{,}100$

Describing Quadratics: Intercepts

As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.





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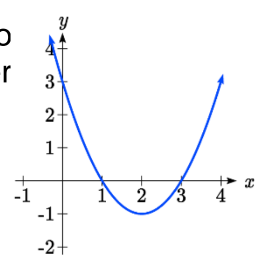


one horizontal
intercept

tw
int

Example 8

Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$

Solution:
We can find the vertical intercept by evaluating the function at an input of zero:
$f(0) = 3(0)^2 + 5(0) - 2 = -2$ Vertical intercept at (0,-2)

For the horizontal intercepts, we solve for when the output will be zero
$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution
$$0 = (3x - 1)(x + 2)$$
ZPP
$$\begin{array}{cc} 3x - 1 = 0 & x + 2 = 0 \\ 3x = 1 & x = -2 \\ x = \dfrac{1}{3} & \end{array}$$
Horizontal intercepts at $\left(\frac{1}{3}, 0\right)$ and (-2,0).

Notice that in the standard form of a quadratic, the constant term *c* reveals the vertical intercept of the graph.

Example 8

Find the horizontal intercepts of the quadratic $f(x) = 2x^2 + 4x - 4$

Solution:
Again we will solve for when the output will be zero
$$0 = 2x^2 + 4x - 4$$

Since the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic into transformation form.
$h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1 \qquad k = f(-1) = 2(-1)^2 + 4(-1) - 4 = -6$
$$f(x) = 2(x + 1)^2 - 6$$

Now we can solve for when the output will be zero
$$0 = 2(x + 1)^2 - 6$$
$$6 = 2(x + 1)^2$$
$$3 = (x + 1)^2$$
$$x + 1 = \pm\sqrt{3}$$
$$x = -1 \pm \sqrt{3}$$

The graph has horizontal intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$

Try it Now
In Try it Now problem 2 we found the standard & transformation form for the function $g(x) = 13 + x^2 - 6x$. Now find the Vertical & Horizontal intercepts (if any).

The process in the last example is done commonly enough that sometimes people find it easier to solve the problem once in general and remember the formula for the result, rather than repeating the process each time. Based on our previous work we showed that any quadratic in standard form can be written into transformation form as:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Solving for the horizontal intercepts using this general equation gives:

$0 = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$	start to solve for *x* by moving the constants to the other side
$\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$	divide both sides by *a*
$\frac{b^2}{4a^2} - \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2$	find a common denominator to combine fractions
$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$	combine the fractions on the left side of the equation
$\frac{b^2-4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$	take the square root of both sides
$\pm\sqrt{\frac{b^2-4ac}{4a^2}} = x + \frac{b}{2a}$	subtract *b*/2*a* from both sides
$-\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = x$	combining the fractions
$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Notice that this can yield two different answers for *x*

Quadratic Formula
For a quadratic function given in standard form $f(x) = ax^2 + bx + c$, the **quadratic formula** gives the horizontal intercepts of the graph of this function.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 9
A ball is thrown upwards from the top of a 40-foot-tall building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

What is the maximum height of the ball?

When does the ball hit the ground?

Solution:

To find the maximum height of the ball, we would need to know the vertex of the quadratic.
$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2}$, $k = H\left(\frac{5}{2}\right) = -16\left(\frac{5}{2}\right)^2 + 80\left(\frac{5}{2}\right) + 40 = 140$

The ball reaches a maximum height of 140 feet after 2.5 seconds.
To find when the ball hits the ground, we need to determine when the height is zero – when $H(t) = 0$. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32}$$

Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:
$t = \frac{-80-\sqrt{8960}}{-32} \approx 5.458$ or $t = \frac{-80+\sqrt{8960}}{-32} \approx -0.458$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

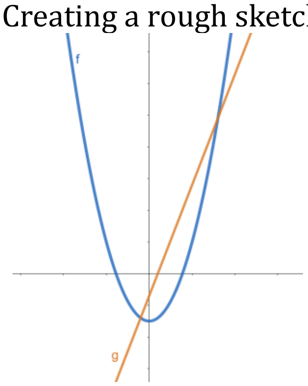
Solving Equations and Inequalities

We'll end with some examples of solving equations and inequalities involving quadratic functions. The graphs of the functions can help to ground these tasks and make sense of the answers.

Example 10

Find the values for which $f(x) = g(x)$ given $f(x) = x^2 - 15$ and $g(x) = 7x - 7$. Then, find the solution to $f(x) \leq g(x)$.

Solution:
We know quite a few things about the graphs of $f(x)$ and $g(x)$. For $f(x)$, we'll have a parabola which has been shifted down 15, and which opens upward. We know $g(x)$ is a line with y-intercept -7 and slope $7/1$. Creating a rough sketch to get our bearings, we see:



This sketch shows us we can expect 2 real solutions to $f(x) = g(x)$. Let's solve.

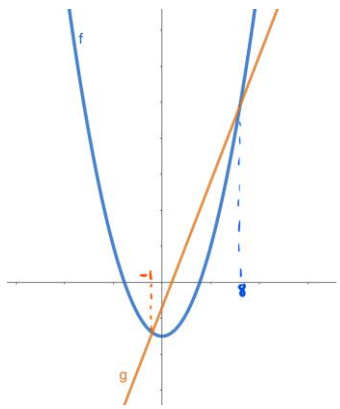
$$f(x) = g(x)$$
$$x^2 - 15 = 7x - 7$$

This is a quadratic equation. We'll move all the terms to the left side so the equation equals 0, then see if we can factor.

$$x^2 - 7x - 15 + 7 = 0$$
$$x^2 - 7x - 8 = 0$$
$$(x - 8)(x + 1) = 0$$

From the ZPP, we see that $x = 8, -1$, so the two functions intersect at these locations.

Now, if we want to solve $f(x) \leq g(x)$, our sketch will be very helpful. Let's add in the information that the functions intersect at $x = -1$ and 8, and investigate what we know from there:



We are looking for the places where the height of f is lower than the height of g. That's the middle region of the graph where the blue parabola is beneath the orange line. We know that region starts and ends where the two shapes intersect, so the solution to $f(x) \leq g(x)$ is $[-1, 8]$.

Try it Now

Given the functions $f(x) = x^2 - 9x + 5$ and $g(x) = -4x + 5$, find where $f(x) = g(x)$. Then, find where $f(x) > g(x)$.

Important Topics of this Section

Quadratic functions

- Standard form
- Transformation form/Vertex form
- Vertex as a maximum / Vertex as a minimum

Short run behavior

- Vertex / Horizontal & Vertical intercepts

Quadratic formula

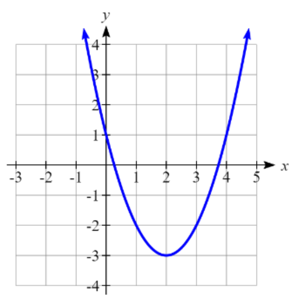
Section 2.7 Exercises

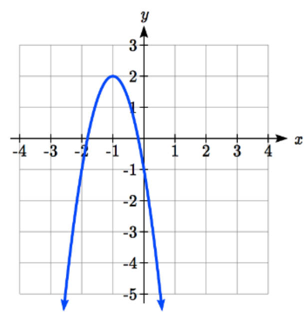
Conceptual Questions

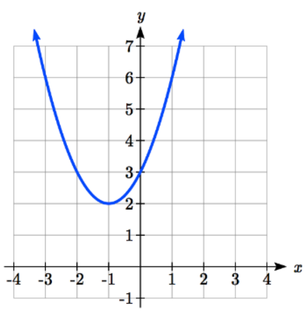
1. Describe how to find the maximum or minimum of a quadratic function. How do you know whether it's a max or min?
2. Describe how to find the end behavior of a quadratic function.
3. Describe how to find the intercepts of a quadratic function.
4. Describe the meaning of x-intercepts in a real world situation of your choosing.
5. Describe the meaning of a maximum in a real world situation of your choosing.
6. True or false: a quadratic function may have no x-intercepts.
7. True or false: a quadratic function may have no y-intercepts.
8. True or false: the domain and range of a quadratic function are both $(-\infty, \infty)$.

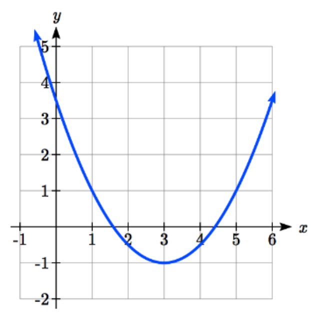
Practice Problems

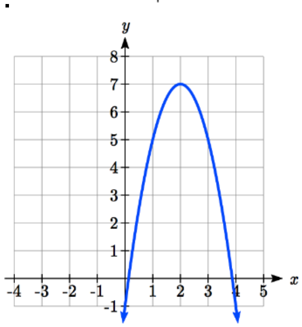
Write an equation for the quadratic function graphed.

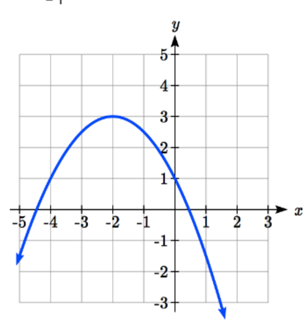
1. 

4. 

2. . 

5. 

3. 

6. . 

Graph each quadratic function. Then describe the intercepts, domain and range, intervals of increasing and decreasing, maximum or minimum, and end behavior.

7. $f(x) = -(x+4)^2 - 5$
8. $g(x) = 2x^2 + 2$
9. $h(x) = 3(x-2)^2 + 1$
10. $j(x) = -2(x+2)^2 + 2$

For each of the follow quadratic functions, find a) the vertex, b) the vertical intercept, and c) the horizontal intercepts.

11. $y(x) = 2x^2 + 10x + 12$
12. $z(p) = 3x^2 + 6x - 9$
13. $f(x) = 2x^2 - 10x + 4$
14. $g(x) = -2x^2 - 14x + 12$
15. $h(t) = -4t^2 + 6t - 1$
16. $k(t) = 2x^2 + 4x - 15$

Rewrite the quadratic function into vertex form. Then graph each quadratic function. Finally, describe the intercepts, domain and range, intervals of increasing and decreasing, maximum or minimum, and end behavior.

17. $f(x) = x^2 - 12x + 32$
18. $g(x) = x^2 + 2x - 3$
19. $h(x) = 2x^2 + 8x - 10$
20. $k(x) = 3x^2 - 6x - 9$

21. Find the values of *b* and *c* so $f(x) = -8x^2 + bx + c$ has vertex $(2, \ -7)$
22. Find the values of *b* and *c* so $f(x) = 6x^2 + bx + c$ has vertex $(7, \ -9)$

Write an equation for a quadratic with the given features

23. *x*-intercepts (-3, 0) and (1, 0), and *y* intercept (0, 2)
24. *x*-intercepts (2, 0) and (-5, 0), and *y* intercept (0, 3)
25. *x*-intercepts (2, 0) and (5, 0), and *y* intercept (0, 6)
26. *x*-intercepts (1, 0) and (3, 0), and *y* intercept (0, 4)
27. Vertex at (4, 0), and *y* intercept (0, -4)
28. Vertex at (5, 6), and *y* intercept (0, -1)
29. Vertex at (-3, 2), and passing through (3, -2)
30. Vertex at (1, -3), and passing through (-2, 3)

31. Given $f(x) = x^2 - 4x + 3$ and $g(x) = 8 - 8x$, find where the functions intersect. Then find where $f(x) \le g(x)$.

32. Given $f(x) = 5x$ and $g(x) = \ -7x^2$, find where the functions intersect. Then find where $f(x) > g(x)$.

33. Given $f(x) = 5x^2 + 7x$ and $g(x) = 8x + 4x^2$, find where the functions intersect. Then find where $f(x) < g(x)$.

34. Given $f(x) = 9x^2 + 16x - 11$ and $g(x) = 6x$, find where the functions intersect. Then find where $f(x) \geq g(x)$.

35. Given $f(x) = 11x^2 - 9x$ and $g(x) = -5 + 10x^2$, find where the functions intersect. Then find where $f(x) > g(x)$.

36. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$.
    a. From what height was the rocket launched?
    b. How high above sea level does the rocket reach its peak?
    c. Assuming the rocket will splash down in the ocean, at what time does splashdown occur?

37. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$.
    a. From what height was the ball thrown?
    b. How high above ground does the ball reach its peak?
    c. When does the ball hit the ground?

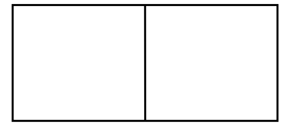
38. The height of a ball thrown in the air is given by $h(x) = -\frac{1}{12}x^2 + 6x + 3$, where *x* is the horizontal distance in feet from the point at which the ball is thrown.
    a. How high is the ball when it was thrown?
    b. What is the maximum height of the ball?
    c. How far from the thrower does the ball strike the ground?

39. A javelin is thrown in the air. Its height is given by $h(x) = -\frac{1}{20}x^2 + 8x + 6$, where *x* is the horizontal distance in feet from the point at which the javelin is thrown.
    a. How high is the javelin when it was thrown?
    b. What is the maximum height of the javelin?
    c. How far from the thrower does the javelin strike the ground?

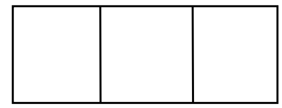
40. A box with a square base and no top is to be made from a square piece of cardboard by cutting 6 in. squares out of each corner and folding up the sides. The box needs to hold 1000 in^3. How big a piece of cardboard is needed?

41. A box with a square base and no top is to be made from a square piece of cardboard by cutting 4 in. squares out of each corner and folding up the sides. The box needs to hold 2700 in^3. How big a piece of cardboard is needed?

42. A farmer wishes to enclose two pens with fencing, as shown. If the farmer has 500 feet of fencing to work with, what dimensions will maximize the area enclosed?



43. A farmer wishes to enclose three pens with fencing, as shown. If the farmer has 700 feet of fencing to work with, what dimensions will maximize the area enclosed?



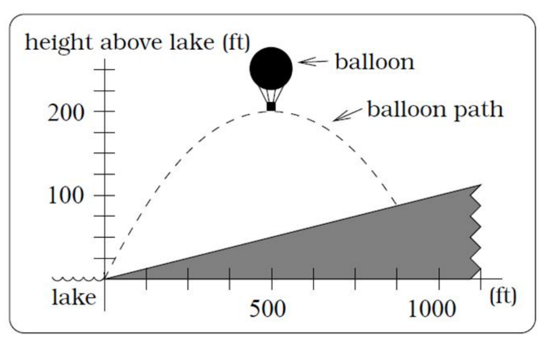
44. You have a wire that is 56 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a square. The other piece will be bent into the shape of a circle. Let *A* represent the total area enclosed by the square and the circle. What is the circumference of the circle when *A* is a minimum?

45. You have a wire that is 71 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a right triangle with legs of equal length. The other piece will be bent into the shape of a circle. Let *A* represent the total area enclosed by the triangle and the circle. What is the circumference of the circle when *A* is a minimum?

46. A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

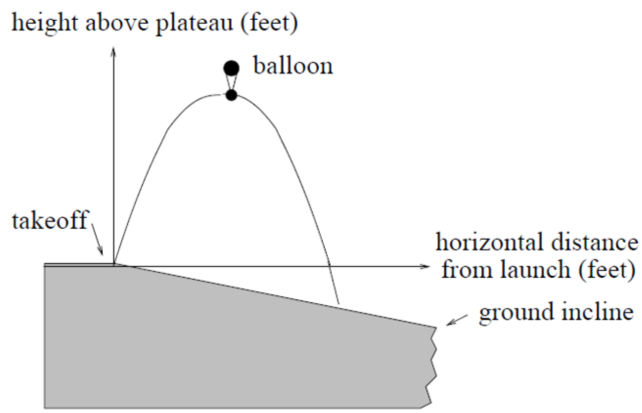
47. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

48. A hot air balloon takes off from the edge of a mountain lake. Impose a coordinate system as pictured and assume that the path of the balloon follows the graph of $f(x) = -\frac{2}{2500}x^2 + \frac{4}{5}x$. The land rises at a constant incline from the lake at the rate of 2 vertical feet for each 20 horizontal feet. [UW]
    a. What is the maximum height of the balloon above water level?



b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

49. A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows is the graph of the quadratic function $f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x$. The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. [UW]



a. What is the maximum height of the balloon above plateau level?
b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

Section 2.8 Graphs of Polynomial Functions

In the previous section, we explored the short run behavior of quadratics, a special case of polynomials. In this section, we will explore the short run behavior of polynomials in general.

Short run Behavior: Intercepts

As with any function, the vertical intercept can be found by evaluating the function at an input of zero. Since this is evaluation, it is relatively easy to do it for a polynomial of any degree.

To find horizontal intercepts, we need to solve for when the output will be zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and 4th degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

1) The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2) The polynomial is given in factored form.
3) Technology is used to determine the intercepts.

Other techniques for finding the intercepts of general polynomials will be explored later sections.

Example 1

Find the horizontal intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution:

We can attempt to factor this polynomial to find solutions for *f(x)* = 0.

$x^6 - 3x^4 + 2x^2 = 0$ Factoring out the greatest common factor

$x^2(x^4 - 3x^2 + 2) = 0$ Factoring the inside as a quadratic in x^2

$x^2(x^2 - 1)(x^2 - 2) = 0$ Then break apart to find solutions

$$\begin{array}{c} x^2 = 0 \\ x = 0 \end{array} \quad or \quad \begin{array}{c} x^2 - 1 = 0 \\ x^2 = 1 \\ x = \pm 1 \end{array} \quad or \quad \begin{array}{c} x^2 - 2 = 0 \\ x^2 = 2 \\ x = \pm\sqrt{2} \end{array}$$

This gives us 5 horizontal intercepts.

Example 2

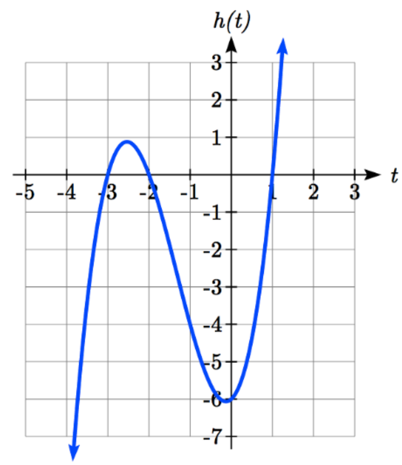
Find the vertical and horizontal intercepts of $g(t) = (t - 2)^2(2t + 3)$

Solution:
The vertical intercept can be found by evaluating $g(0)$.
$$g(0) = (0-2)^2(2(0)+3) = 12$$

The horizontal intercepts can be found by solving $g(t) = 0$.
$$(t-2)^2(2t+3) = 0$$
Since this is already factored, we can break it apart using the ZPP:
$$(t-2)^2 = 0 \qquad 2t+3 = 0$$
$$t-2 = 0 \quad or \quad t = -\frac{3}{2}$$
$$t = 2$$
We can always check our answers are reasonable by graphing the polynomial.

Example 3

Find the horizontal intercepts of $h(t) = t^3 + 4t^2 + t - 6$



Solution:
Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.

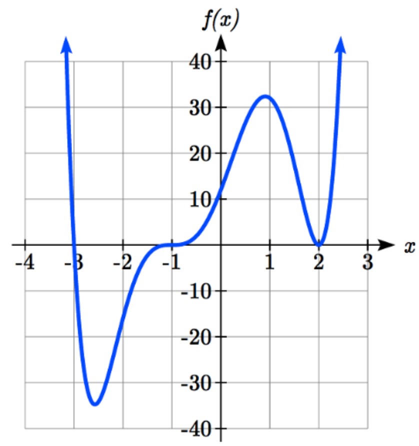
Graphing this function, it appears there are horizontal intercepts at $t = -3, -2,$ and 1.

We could check these are correct by plugging in these values for t and verifying that $h(-3) = h(-2) = h(1) = 0$.

Try it Now

Find the vertical and horizontal intercepts of the function $f(t) = t^4 - 4t^2$.

Graphical Behavior at Intercepts



If we graph the function $f(x) = (x+3)(x-2)^2(x+1)^3$, notice that the behavior at each of the horizontal intercepts is different.

At the horizontal intercept $x = -3$, coming from the $(x+3)$ factor of the polynomial, the graph passes directly through the horizontal intercept.
The factor $(x+3)$ is linear (has a power of 1), so the behavior near the intercept is like that of a line - it passes directly through the intercept. We call this a single zero, since the zero corresponds to a single factor of the function.

At the horizontal intercept *x* = 2, coming from the $(x-2)^2$ factor of the polynomial, the graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic – it bounces off the horizontal axis at the intercept. Since $(x-2)^2 = (x-2)(x-2)$, the factor is repeated twice, so we call this a double zero. We could also say the zero has **multiplicity** 2.

At the horizontal intercept *x* = -1, coming from the $(x+1)^3$ factor of the polynomial, the graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic, with the same "S" type shape near the intercept that the toolkit x^3 has. We call this a triple zero. We could also say the zero has multiplicity 3.

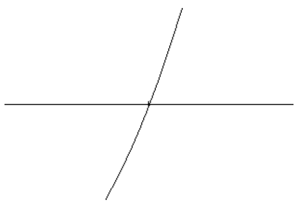
By utilizing these behaviors, we can sketch a reasonable graph of a factored polynomial function without needing technology.

Multiplicity
If a function f is fully factored, and $(x-r)^m$ is one of those factors, then $x = r$ is a horizontal asymptote with multiplicity m.

Graphical Behavior of Polynomials at Horizontal Intercepts

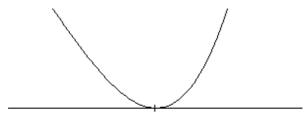
If a polynomial contains a factor of the form $(x - r)^p$, the behavior near the horizontal intercept r is determined by the power on the factor, that is, the multiplicity of the intercept.

$p = 1$



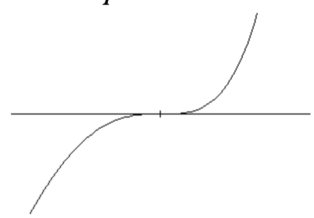
Single zero
Multiplicity 1

$p = 2$



Double zero
Multiplicity 2

$p = 3$



Triple zero
Multiplicity 3

For higher even powers 4,6,8 etc.... the graph will still bounce off the horizontal axis but the graph will appear flatter with each increasing even power as it approaches and leaves the axis.

For higher odd powers, 5,7,9 etc... the graph will still pass through the horizontal axis but the graph will appear flatter with each increasing odd power as it approaches and leaves the axis.

Example 4

Sketch a graph of $f(x) = -2(x + 3)^2(x - 5)$.

Solution:
This graph has two horizontal intercepts. At $x = -3$, the factor is squared, indicating the graph will bounce at this horizontal intercept. At $x = 5$, the factor is not squared, indicating the graph will pass through the axis at this intercept.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the long-run behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs get large positive, and the inputs increasing as the inputs get large negative.

To sketch this we consider the following:
As $x \to -\infty$ the function $f(x) \to \infty$ so we know the graph starts in the 2nd quadrant and is decreasing toward the horizontal axis.

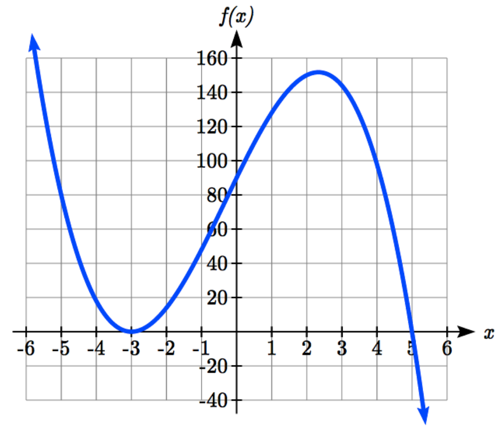
At (-3, 0) the graph bounces off the horizontal axis and so the function must start increasing.

At $(0, 90)$ the graph crosses the vertical axis at the vertical intercept.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis since the graph passes through the next intercept at $(5,0)$.

As $x \to \infty$ the function $f(x) \to -\infty$ so we know the graph continues to decrease and we can stop drawing the graph in the 4th quadrant.

Using technology we can verify the shape of the graph.



Try it Now

Given the function $g(x) = x^3 - x^2 - 6x$ use the methods that we have learned so far to find the vertical & horizontal intercepts, determine where the function is negative and positive, describe the long run behavior and sketch the graph without technology.

Solving Polynomial Inequalities

One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative. We can solve polynomial inequalities by either utilizing the graph, or by using test values.

Example 5

Solve $(x+3)(x+1)^2(x-4) > 0$

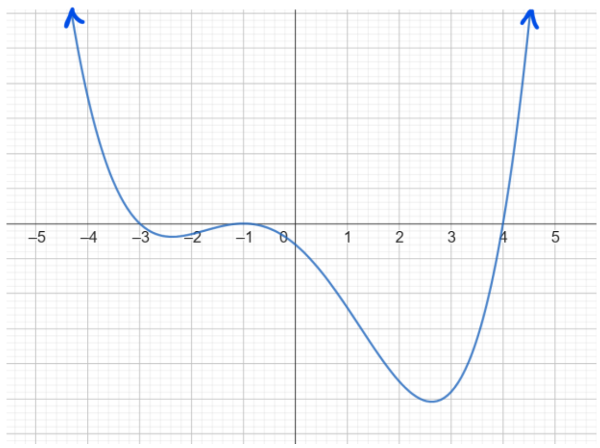
Solution:

As with all inequalities, we start by solving the equality $(x+3)(x+1)^2(x-4) = 0$, which has solutions at $x = -3, -1$, and 4.

We can sketch the function to help us determine where its height is greater than 0. We know that the function has the following intercepts and corresponding multiplicities:

$x = -3$	multiplicity 1	pass through
$x = -1$	multiplicity 2	bounce
$x = 4$	multiplicity 1	pass through

This is a degree 4 polynomial with a positive leading coefficient, so as $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to \infty$. Therefore, the graph looks something like this:



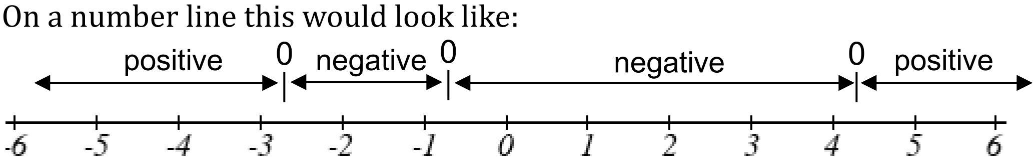
Therefore, the function is greater than 0 (has heights above the x-axis) when $x < -3$ or when $x > 4$. In interval notation, that's $(-\infty, -3) \cup (4, \infty)$.

Alternatively, we know the function can only change from positive to negative at $x = -3, -1$, and 4, so these divide the inputs into 4 intervals.

We could choose a test value in each interval and evaluate the function $f(x) = (x+3)(x+1)^2(x-4)$ at each test value to determine if the function is positive or negative in that interval

Interval	Test *x* in interval	*f*(test value)	>0 or <0?
x < -3	-4	72	> 0
-3 < *x* < -1	-2	-6	< 0
-1 < *x* < 4	0	-12	< 0
x > 4	5	288	> 0

On a number line this would look like:



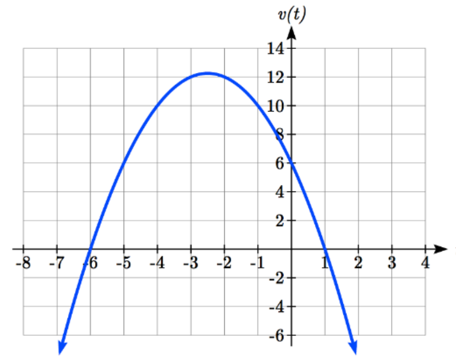
From our test values, we can determine this function is positive when $x <$ -3 or $x >$ 4, or in interval notation, $(-\infty, -3) \cup (4, \infty)$.

Example 6

Solve $6 - 5t - t^2 \geq 0$.

Solution:

We start by solving the equality $6 - 5t - t^2 = 0$. While we could use the quadratic formula, this equation factors nicely to $(6 + t)(1 - t) = 0$, giving horizontal intercepts $t = 1$ and $t = -6$.



Sketching a graph of this quadratic will allow us to determine when it is positive.

From the graph we can see this function is positive for inputs between the intercepts. So $6 - 5t - t^2 \geq 0$ for $-6 \leq t \leq 1$.

Writing Equations using Intercepts

Since a polynomial function written in factored form will have a horizontal intercept where each factor is equal to zero, we can form a function that will pass through a set of horizontal intercepts by introducing a corresponding set of factors.

Writing an Equation for a Polynomial from Intercepts

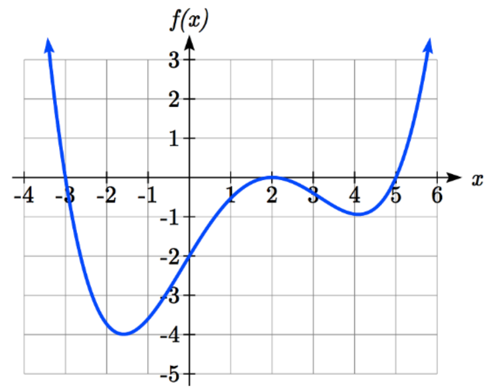
If we know the horizontal intercepts and the behavior or multiplicity at those intercepts, we can write a polynomial of minimal degree with those intercepts.

1. Determine the horizontal intercepts $x = x_1, x_2, \ldots, x_k$
2. Examine the behavior at each intercept to determine the corresponding multiplicity of each intercept, $p_1, p_2, \ldots, p_k$
3. Write the polynomial in factored form $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_k)^{p_k}$
4. Use another point on the graph to solve for the stretch factor a

Notice the degree of the polynomial will be the sum of the multiplicities p_i.

Example 7

Write a formula for the polynomial function graphed here.



Solution:
This graph has three horizontal intercepts: x = -3, 2, and 5. At x = -3 and 5 the graph passes through the axis, suggesting the corresponding factors of the polynomial will be linear. At x = 2 the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be 2nd degree (quadratic).

Together, this gives us:

$$f(x) = a(x+3)(x-2)^2(x-5)$$

To determine the stretch factor, we can utilize another point on the graph. Here, the vertical intercept appears to be (0,-2), so we can plug in those values to solve for a:

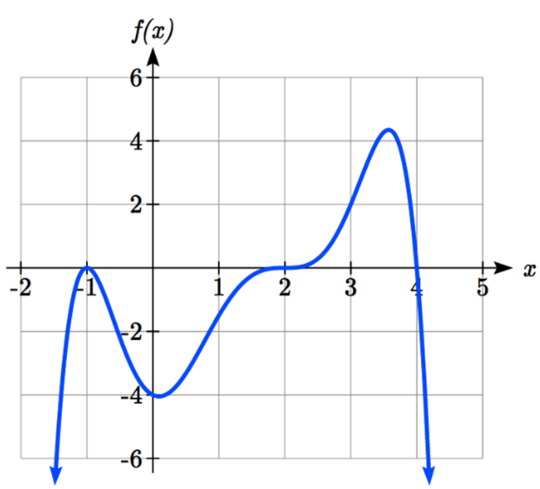
$$-2 = a(0+3)(0-2)^2(0-5)$$
$$-2 = -60a$$
$$a = \frac{1}{30}$$

The graphed polynomial appears to represent the function

$$f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5).$$

Try it Now

Given the graph, write a formula for the function shown.

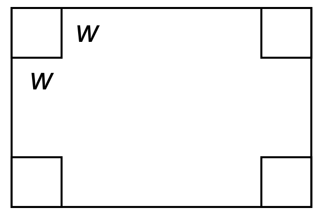


Estimating Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Example 8

An open-top box is to be constructed by cutting out squares from each corner of a 14cm by 20cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

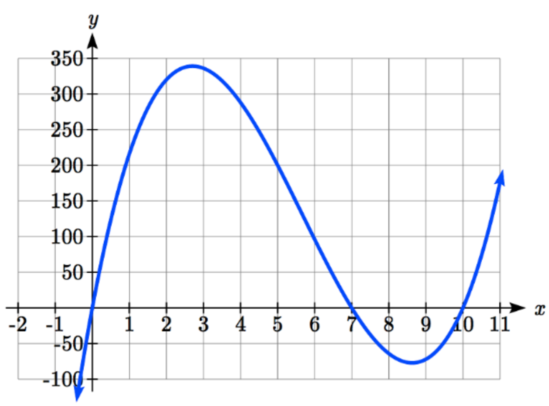


Solution:
We will start this problem by drawing a picture, labeling the width of the cut-out squares with a variable, w.

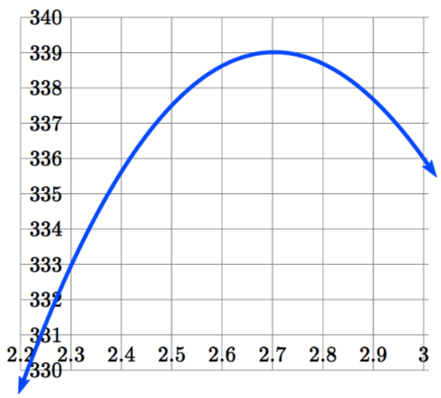
Notice that after a square is cut out from each end, it leaves a $(14 - 2w)$ cm by $(120 - 2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume:

$$V(w) = (14 - 2w)(20 - 2w)w = 280w - 68w^2 + 4w^3$$

Using technology to sketch a graph allows us to estimate the maximum value for the volume, restricted to reasonable values for w: values from 0 to 7.



From this graph, we can estimate the maximum value is around 340, and occurs when the squares are about 2.75cm square. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph.



From this zoomed-in view, we can refine our estimate for the max volume to about 339, when the squares are 2.7cm square.

Try it Now

Use technology to find the maximum and minimum values on the interval [-1, 4] of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Important Topics of this Section
Short Run Behavior
Intercepts (Horizontal & Vertical)
Methods to find Horizontal intercepts
Factoring Methods
Factored Forms
Technology
Graphical Behavior at intercepts
Single, Double and Triple zeros (or multiplicity 1, 2, and 3 behaviors)
Solving polynomial inequalities using test values & graphing techniques
Writing equations using intercepts
Estimating extrema

Section 2.8 Exercises

Conceptual Questions

1. A fully-factored polynomial has factors including $(x-r)^2$ and $(x-s)^3$. Describe how the graph will differ at the roots $x=r$ and $x=s$.
2. Describe how to find the solution to $f(x) \geq 0$ using the graph of f, a polynomial.
3. True or false: A polynomial of degree 3 must have at least one root.
4. True or false: A polynomial of degree 4 must have four roots.

Practice Problems

Find the C and t intercepts of each function.

1. $C(t) = 2(t-4)(t+1)(t-6)$
2. $C(t) = 3(t+2)(t-3)(t+5)$
3. $C(t) = 4t(t-2)^2(t+1)$
4. $C(t) = 2t(t-3)(t+1)^2$
5. $C(t) = 2t^4 - 8t^3 + 6t^2$
6. $C(t) = 4t^4 + 12t^3 - 40t^2$

Use your calculator or other graphing technology to solve graphically for the zeros of the function.

7. $f(x) = x^3 - 7x^2 + 4x + 30$
8. $g(x) = x^3 - 6x^2 + x + 28$

Find the end behavior of each function as $\boldsymbol{t} \to \boldsymbol{\infty}$ and $\boldsymbol{t} \to \boldsymbol{-\infty}$

9. $h(t) = 3(t-5)^3(t-3)^3(t-2)$
10. $k(t) = 2(t-3)^2(t+1)^3(t+2)$
11. $p(t) = -2t(t-1)(3-t)^2$
12. $q(t) = -4t(2-t)(t+1)^3$

Sketch a graph of each equation.

13. $f(x) = (x+3)^2(x-2)$
14. $g(x) = (x+4)(x-1)^2$
15. $h(x) = (x-1)^3(x+3)^2$
16. $k(x) = (x-3)^3(x-2)^2$
17. $m(x) = -2x(x-1)(x+3)$
18. $n(x) = -3x(x+2)(x-4)$

Solve each inequality.

19. $(x-3)(x-2)^2 > 0$
20. $(x-5)(x+1)^2 > 0$
21. $(x-1)(x+2)(x-3) < 0$
22. $(x-4)(x+3)(x+6) < 0$

Find the domain of each function.

23. $f(x) = \sqrt{-42 + 19x - 2x^2}$
24. $g(x) = \sqrt{28 - 17x - 3x^2}$
25. $h(x) = \sqrt{4 - 5x + x^2}$
26. $k(x) = \sqrt{2 + 7x + 3x^2}$

27. $n(x) = \sqrt{(x-3)(x+2)^2}$

28. $m(x) = \sqrt{(x-1)^2(x+3)}$

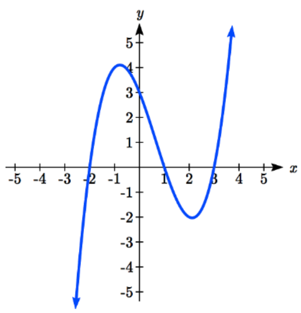
29. $p(t) = \frac{1}{t^2+2t-8}$

30. $q(t) = \frac{4}{x^2-4x-5}$

Write an equation for a polynomial the given features.

31. Degree 3. Zeros at *x* = -2, *x* = 1, and *x* = 3. Vertical intercept at (0, -4)
32. Degree 3. Zeros at *x* = -5, *x* = -2, and *x* = 1. Vertical intercept at (0, 6)
33. Degree 5. Roots of multiplicity 2 at *x* = 3 and *x* = 1, and a root of multiplicity 1 at *x* = -3. Vertical intercept at (0, 9)
34. Degree 4. Root of multiplicity 2 at *x* = 4, and a roots of multiplicity 1 at *x* = 1 and *x* = -2. Vertical intercept at (0, -3)
35. Degree 5. Double zero at *x* = 1, and triple zero at *x* = 3. Passes through the point (2, 15)
36. Degree 5. Single zero at *x* = -2 and *x* = 3, and triple zero at *x* = 1. Passes through the point (2, 4)

Write a formula for each polynomial function graphed.

37.


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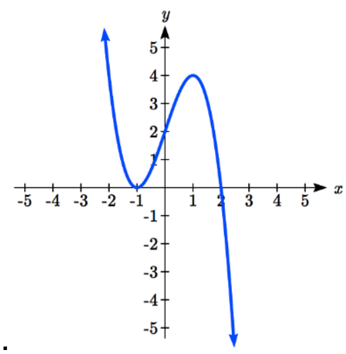

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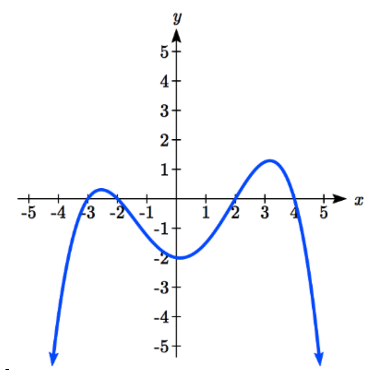

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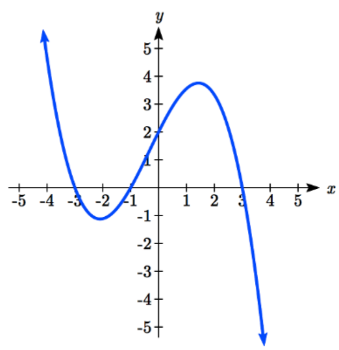

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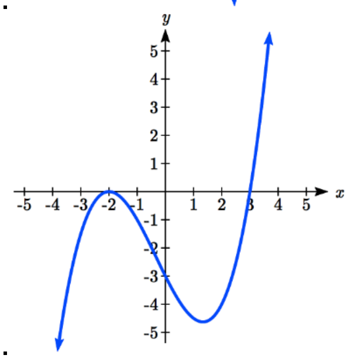

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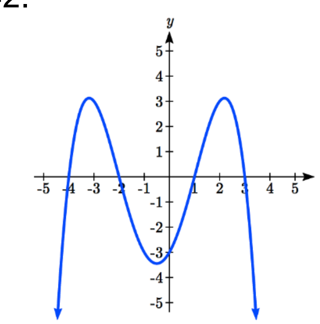
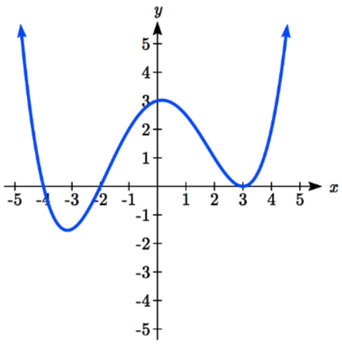


Write a formula for each polynomial function graphed.



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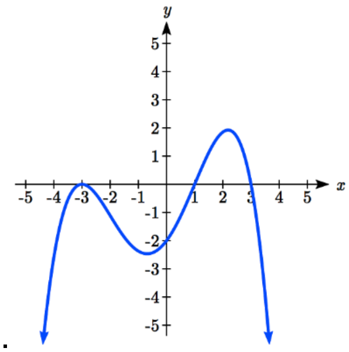

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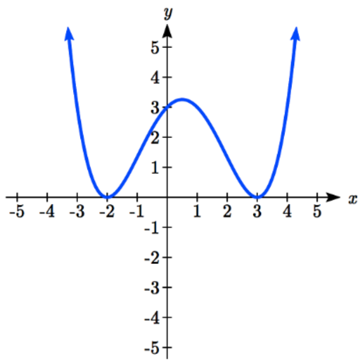

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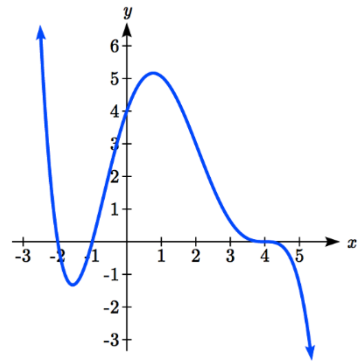

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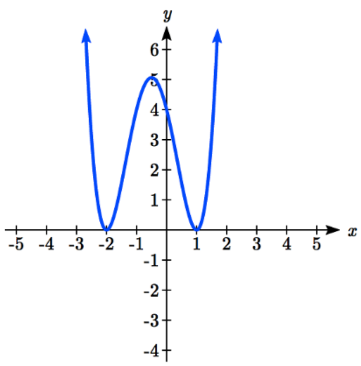

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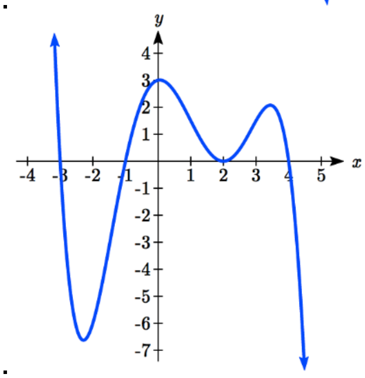

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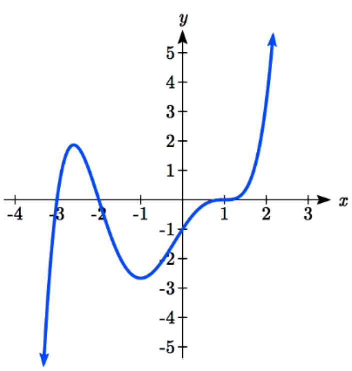

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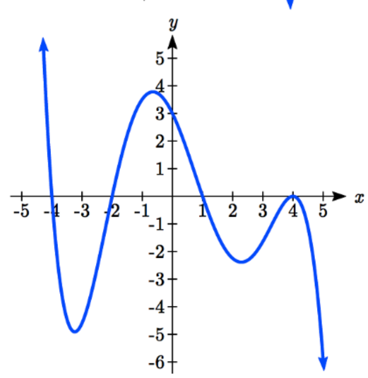


50.

51. A rectangle is inscribed with its base on the *x* axis and its upper corners on the parabola $y = 5 - x^2$. What are the dimensions of such a rectangle that has the greatest possible area?

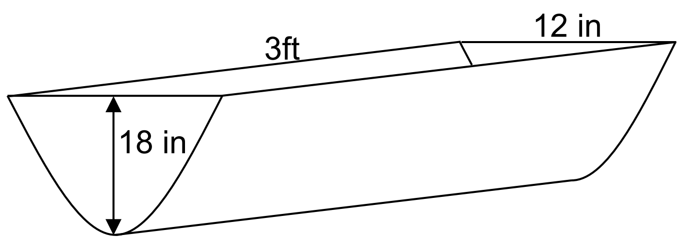
52. A rectangle is inscribed with its base on the *x* axis and its upper corners on the curve $y = 16 - x^4$. What are the dimensions of such a rectangle that has the greatest possible area?

Section 2.9 Inverses and Radical Functions

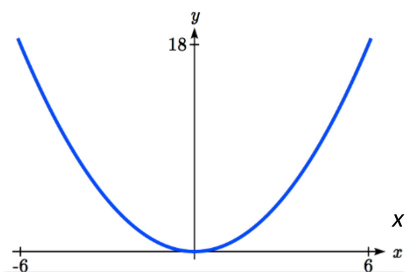
In this section, we will explore the inverses of polynomial and rational functions, and in particular the radical functions that arise in the process.

Example 1

A water runoff collector is built in the shape of a parabolic trough as shown below. Find the surface area of the water in the trough as a function of the depth of the water.



Since it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with x measured horizontally and y measured vertically, with the origin at the vertex of the parabola.



From this we find an equation for the parabolic shape. Since we placed the origin at the vertex of the parabola, we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point (6,18), from which we can solve for the stretch factor a:

$$18 = a6^2$$
$$a = \frac{18}{36} = \frac{1}{2}$$

Our parabolic cross section has equation $y(x) = \frac{1}{2}x^2$

Since we are interested in the surface area of the water, we are interested in determining the width at the top of the water as a function of the water depth. For any depth y the width will be given by $2x$, so we need to solve the equation above for x. However notice that the original function is not one-to-one, and indeed given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive x values. On this domain, we can find an inverse by solving for the input variable:

$$y = \frac{1}{2}x^2$$
$$2y = x^2$$
$$x = \pm\sqrt{2y}$$

This is not a function as written. Since we are limiting ourselves to positive x values, we eliminate the negative solution, giving us the inverse function we're looking for

$$x(y) = \sqrt{2y}$$

Since x measures from the center out, the entire width of the water at the top will be $2x$. Since the trough is 3 feet (36 inches) long, the surface area will then be $36(2x)$, or in terms of y:

$$Area = 72x = 72\sqrt{2y}$$

The previous example illustrated two important things:

1) When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2) The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called **radical functions**.

Tookit Function

Square Root Toolkit Function

The toolkit power function with an exponent of $\frac{1}{2}$ is

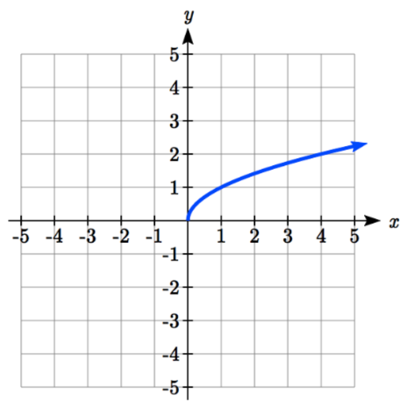
$$f(x) = x^{\frac{1}{2}} = \sqrt{x}$$

It is called the square root function.

This function passes through the points

x	$f(x)$
0	0
1	1
4	2
9	3

and its graph looks as follows:



The square root function has the following features:

Domain: $[0, \infty)$

Range: $[0, \infty)$

Intercepts: x and y intercepts at $(0,0)$.

End behavior: As $x \to \infty, f(x) \to \infty$

Increasing on $(0, \infty)$

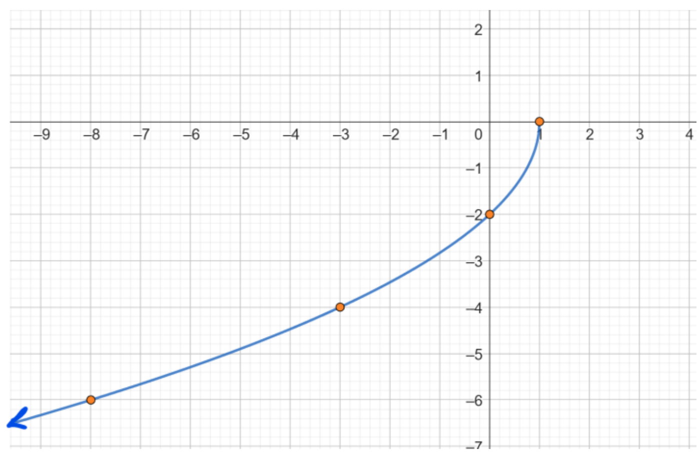
Symmetry: no symmetry

Example 2

Graph the function $g(x) = -2\sqrt{-x+1}$ using transformations. Then describe the function's domain, range, intercepts, end behavior, and intervals of increasing/decreasing.

Solution:
We see the square root in our function, telling us we should use the parent function $f(x) = \sqrt{x}$. We need to identify our $a, b, c,$ and d values for $g(x) = af(bx + c) + d = a\sqrt{bx + c} + d$. Starting with $c = 1$, we have a shift left 1. Then since $b = -1$, there is a horizontal reflection. Finally, with $a = -2$, there is a vertical reflection and a vertical stretch by a factor of 2. Applying these transformations, we produce the following table and graph:

$-1 \cdot X$	$X = x - 1$	$\leftarrow x$	$f(x) \rightarrow$	$-2 \cdot f(x)$
1	-1	0	0	0
0	0	1	1	-1
-3	3	4	2	-4
-8	8	9	3	-6



This function has the following features:
Domain: $(-\infty, 1]$
Range: $(-\infty, 0]$
Intercepts: x-intercept $(1, 0)$, y-intercept $(0, -2)$
End behavior: As $x \to -\infty, f(x) \to -\infty$
Increasing on $(-\infty, 1]$

Notice that square root functions are the first functions we're seeing where the domain is *not* $(-\infty, \infty)$. If we sketch the graph, we can identify the domain, but that's not always the easiest way to proceed if all we need is the domain.

If we want to algebraically consider the domain, we must remember that the square root of a negative number is not a real number. That means, to find the domain of a square root function, we need to only include input values which make the inside of the square root 0 or positive.

Domain of Square Root functions

To find the domain of a square root function in the form

$$\sqrt{\text{stuff}}$$

Solve the inequality

$$\text{stuff} \geq 0$$

Example 3

Find the domain of $f(x) = \sqrt{3 - 2x}$.

Solution:

We want to ensure that we're only trying to square root 0 or a positive number, so we want to find the solution to

$$3 - 2x \geq 0$$

This is a linear inequality, so we'll isolate x. BUT we'll reverse the inequality if we multiply or divide by a negative!

$$3 - 2x \geq 0$$
$$-2x \geq -3$$
$$x \leq \frac{3}{2}$$

Therefore, the domain of $f(x)$ is $\left(-\infty, \frac{3}{2}\right]$.

Example 4

Find the domain of $f(x) = \sqrt{x^2 - 4}$.

Solution:

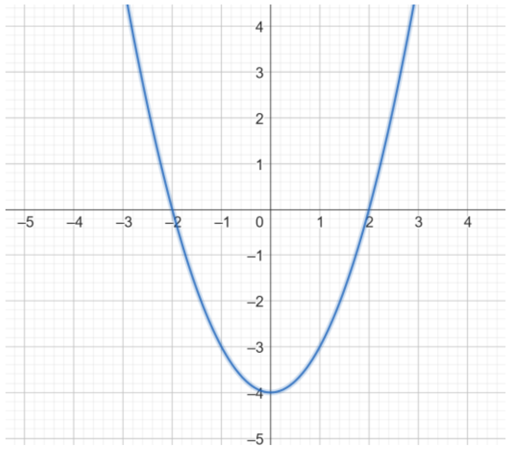
Again, we want to ensure that the inputs to the square root are 0 or positive, so

$$x^2 - 4 \geq 0$$

This time, we have a polynomial (quadratic) inequality. Let's find where $x^2 - 4 = 0$ and strategize from there.

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

So, we know we have a parabola with x-intercepts at 2 and -2. We also know the parabola opens up, since the a-value is positive 1.



We want to know when this function is greater than or equal to 0; that happens when the parabola is above the x-axis, so the solution to $x^2 - 4 \geq 0$ is $(-\infty, -2] \cup [2, \infty)$.

Therefore, the domain of $f(x) = \sqrt{x^2 - 4}$ is $(-\infty, -2] \cup [2, \infty)$.

Try it Now

Find the domain of each function.

a. $f(x) = \sqrt{5x - 7}$

b. $g(x) = \sqrt{-x^2 - 5x - 4}$

Square Roots as Inverses of Quadratics

Example 5

Find the inverse of $f(x) = (x-2)^2 - 3 = x^2 - 4x + 1$

Solution

From the transformation form of the function, we can see this is a transformed quadratic with vertex at (2,-3) that opens upwards. Since the graph will be decreasing on one side of the vertex, and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we will use the vertex form of the quadratic. We start by replacing the *f(x)* with a simple variable *y*, swap x and y, and solve for y.

y	$= (x-2)^2 - 3$	Replace $f(x)$with y
x	$= (y-2)^2 - 3$	Swap $x's$ and $y's$
$x + 3$	$= (y-2)^2$	Add 3 to both sides
$\pm\sqrt{x+3}$	$= y - 2$	square root both sides
$2 \pm \sqrt{x+3}$	$= y$	Add 2 to both sides

As currently written, this is not a function. Since we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the positive case:

$$y = 2 + \sqrt{x+3}$$

Now, we finish by writing this as an inverse function, $f^{-1}(x) = 2 + \sqrt{x+3}$

Example 6

Compare the features of the functions $f(x) = (x-2)^2 - 3$ and its inverse $f^{(-1)}(x) = \sqrt{x+3} - 2$.

Solution:
Let's start with the graphical transformations. The function $f(x)$ has a shift right 2 and down 3. The inverse $f^{-1}(x)$ has a shift *down* 2 and *left* 3. This makes some intuitive sense if we think about the fact that the function-inverse relationship means we have swapped all inputs and outputs; horizontal moves become vertical and vice versa.

Let's look at domains and ranges next. The domain of $f(x)$ is $(-\infty, \infty)$, but we limited that to $[2, \infty)$ to make it a one-to-one function. The range is $[-3, \infty)$. The domain of $f^{(-1)}(x)$ is $[-3, \infty)$ and its range is $[2, \infty)$. Again, we see the swapping of the input/output relationship, since the domains and ranges have swapped!

Try it Now

Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

Finding an inverse of a quadratic function

To find the inverse of a quadratic, $f(x)$:

1. Limit the domain to a one-to-one portion, which still contains the full range.
2. Replace $f(x)$ with a y.
3. Swap all x's for $y's$ and all $y's$ for x's.
4. Isolate y
5. Rename the final function as $f^{-1}(x)$

The inverse of a quadratic will be a square root function.

While it is not possible to find an inverse of most polynomial functions, some other basic polynomials are invertible.

Example 7

Find the inverse of the function $f(x) = 5x^3 + 1$.

Solution:
This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by swapping the x's and y's, then solving for y.

$$y = 5x^3 + 1$$
$$x = 5y^3 + 1$$
$$x - 1 = 5y^3$$
$$\frac{x-1}{5} = y^3$$
$$y = f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$$

Notice that this inverse is also a transformation of a power function with a fractional power, $x^{1/3}$.

Try it Now

Find the inverse of the function $f(x) = 2(x-1)^3$.

Besides being important as an inverse function, radical functions are common in important physical models.

Example 8

The velocity, v in feet per second, of a car that slammed on its brakes can be determined based on the length of skid marks that the tires left on the ground. This relationship is given by

$$v(d) = \sqrt{2gfd}$$

In this formula, g represents acceleration due to gravity (32 ft/sec^2), d is the length of the skid marks in feet, and f is a constant representing the friction of the surface. A car lost control on wet asphalt, with a friction coefficient of 0.5, leaving 200 foot skid marks. How fast was the car travelling when it lost control?

Solution:
Using the given values of $f = 0.5$ and $d = 200$, we can evaluate the given formula:
$v(200) = \sqrt{2(32)(0.5)(200)} = 80$ ft/sec, which is about 54.5 miles per hour.

Solving Radical Equations

To solve radical equations, we'll follow an approach similar to absolute value equations. That is, we'll isolate the square root first, then undo it, then proceed with solving.

Example 9

Find the x-intercepts of the function $f(x) = \sqrt{-2x-14} - 6$.

Solution:
To answer this question, we *could* graph the function, or we could solve $\sqrt{-2x-15} - 6 = 0$. This is a radical equation, so we'll isolate the radical, then square both sides of the equation to undo the root.

$$\begin{aligned} \sqrt{-2x-14} - 6 &= 0 & & \\ \sqrt{-2x-14} &= 6 & & \text{Add 6 to both sides} \\ \left(\sqrt{-2x-14}\right)^2 &= 6^2 & & \text{Square both sides} \\ -2x-14 &= 36 & & \end{aligned}$$

Now that the root is gone, we reassess. This is now a linear equation, so our only remaining job is to isolate x.

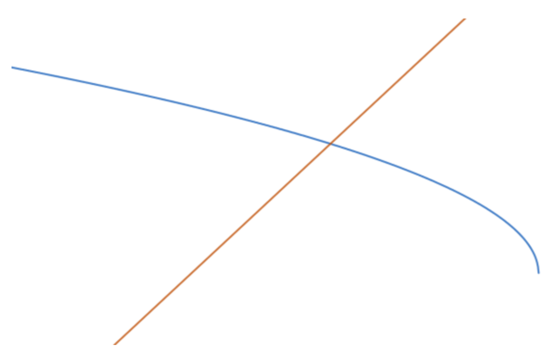
$$-2x - 14 = 36$$
$$-2x = 50$$
$$x = -25$$

So, the function $f(x) = \sqrt{-2x-14} - 6$ has an x-intercept at $(-25,0)$.

Example 10

Find the intersection of f(x)=$\sqrt{15-2x}$ and $g(x) = x$.

Solution:
Here we have a root function that's been shifted left 15, horizontally shrunk by a factor of 2, and horizontally reflected. We want to see where that function will intersect with a straight line, x. The situation looks something like this:



To find where these functions intersect, we must solve $\sqrt{15-2x} = x$. Let's ask our solving questions.

What type of equation is this? A radical (root) equation.
What are our solving strategies? We must isolate the root, square both sides to undo the square root, then reassess the situation.

The root is already alone on the left hand side, so let's proceed with squaring.

$$\left(\sqrt{15-2x}\right)^2 = x^2$$
$$15 - 2x = x^2$$

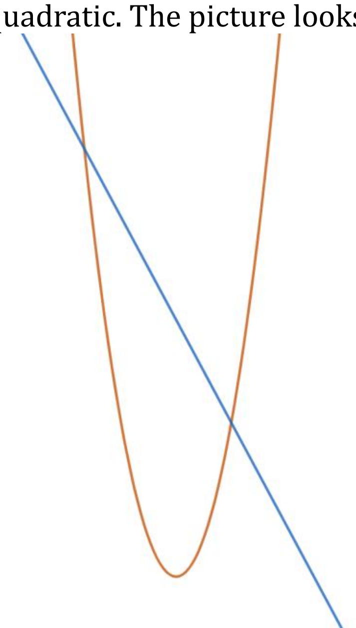
Now, we ask our questions again!

What type of equation is this? Quadratic
How can we solve? This equation is a good candidate for either factoring or the quadratic formula. Either way, we should move all the terms to one side so we have an =0.

$$0 = x^2 + 2x - 15$$
$$0 = (x-3)(x+5)$$
$$\text{ZPP}$$
$$\begin{array}{ccc} x - 3 = 0 & & x + 5 = 0 \\ & or & \\ x = 3 & & x = -5 \end{array}$$

Now, wait a second. Our original sketch only had one intersection, but we're finding two answers. What happened?

Well, when we squared both sides of the equation, we changed the picture. Now we have $15 - 2x$, a straight line, and x^2, a quadratic. The picture looks like this:



So, in our solving process, we accidentally invented a whole second solution that we shouldn't have! To figure out which of our solutions is the extra one, we'll check our answers in the original equation.

Check $x = 3$ as a solution for $\sqrt{15 - 2x} = x$:

$$\begin{aligned} \sqrt{15 - 2 \cdot 3} &\;?= 3 \\ \sqrt{15 - 6} &\;?= 3 \\ \sqrt{9} &= 3 \end{aligned}$$

This solution checks out!

Now, check $x = -5$:

$$\begin{aligned} \sqrt{15 - 2 \cdot -5} &\;?= -5 \\ \sqrt{15 + 10} &\;?= -5 \\ \sqrt{25} &\;?= -5 \\ 5 &\neq -5 \end{aligned}$$

We ended with a false statement, so $x = -5$ is not a solution to the original equation! We call this an **extraneous solution.**

The final solution, and x-value for the intersection of our functions, is $x = 3$.
The full point of intersection is $(3,3)$.

We learned a couple of important things in those last two examples. Let's summarize before we solve more.

Solving Square Root Equations

To solve an equation where the variable is under the square root:

1. Isolate the root if possible
2. Square both sides to undo the root
3. Solve the resulting equation.

When solving a square root equation, we must always check for **extraneous** solutions!

Example 11

Solve $\sqrt{x + 3} = 3x - 1$

Solution:
What type of equation is it? Radical (square root)
How can we solve? Isolate the root, square both sides to undo the root

$$\begin{aligned} \sqrt{x + 3} &= 3x - 1 \\ \left(\sqrt{x + 3}\right)^2 &= (3x - 1)^2 \\ x + 3 &= (3x - 1)^2 \end{aligned}$$

Before we proceed, it's a good idea to simplify the right hand side. Keep in mind, that square tells us to multiply the full $3x-1$ by itself!

$$\begin{aligned} x+3 &= (3x-1)(3x-1) \\ x+3 &= 9x^2-6x+1 \end{aligned}$$

Now, we reassess the situation!
What type of equation is this now? Quadratic.
How can we solve? Get $=0$, then use factoring or the quadratic formula

$$0 = 9x^2-7x-2$$
$$0 = 9x^2-9x+2x-2$$
$$0 = 9x(x-1)+2(x-1)$$
$$0 = (x-1)(9x+2)$$
$$\text{ZPP}$$
$$\begin{array}{ccc} x-1=0 & & 9x+2=0 \\ x=1 & or & 9x=-2 \\ & & x=-\frac{2}{9} \end{array}$$

Don't forget to check for extraneous solutions!

Check $x=1$ in $\sqrt{x+3}=3x-1$

$$\begin{array}{cc} \sqrt{1+3} & ?= 3\cdot 1-1 \\ \sqrt{4} & ?= 3-1 \\ 2 & = 2 \end{array}$$

Check $x=-\frac{2}{9}$

$$\begin{array}{cc} \sqrt{-\frac{2}{9}+3} & ?= 3\cdot -\frac{2}{9}-1 \\ \sqrt{-\frac{2}{9}+\frac{27}{9}} & ?= -\frac{2}{3}-\frac{3}{3} \\ \sqrt{\frac{25}{9}} & ?= -\frac{5}{3} \\ \frac{5}{3} & \neq -\frac{5}{3} \end{array}$$

The only solution to the equation is $x=1$.

Try it Now

Find where $f(x) = \sqrt{x-1}$ and $g(x) = x - 7$ intersect.

Try it Now

Find the horizontal intercepts of $f(x) = \sqrt{3x-1} - 2$

Important Topics of this Section
Finding an inverse function
Restricting the domain
Invertible toolkit functions
Radical Functions
Finding Intercepts of Radical Functions
Solving Radical Equations

Section 2.9 Exercises

Conceptual Questions

1. Why must we limit the domain of a quadratic function in order to find its inverse?
2. How does our toolkit function for the square root differ from our other toolkit functions so far?
3. Why do we sometimes find extraneous solutions when solving square root equations?
4. What is the general strategy for solving radical equations?
5. True or false: a square root function can never have a maximum.
6. True or false: a square root function must always have exactly one x-intercept
7. True or false: a square root function must always have exactly one y-intercept.

Practice Problems

Graph each function using transformation techniques. Then describe the domain, range, intercepts, end behavior, and intervals of increasing/decreasing.

1. $g(x) = 4\sqrt{x}$
2. $h(x) = \sqrt{4x}$
3. $j(x) = \sqrt{x+4}$
4. $k(x) = \sqrt{x} - 4$
5. $f(x) = -\sqrt{x-2}$
6. $g(x) = 2\sqrt{-x}$
7. $h(x) = -\sqrt{2x+1} - 3$
8. $j(x) = 2\sqrt{-x+1} + 3$

For each function, find a domain on which the function is one-to-one and non-decreasing, then find an inverse of the function on this domain.

9. $f(x) = (x-4)^2$
10. $f(x) = (x+2)^2$
11. $f(x) = 12 - x^2$
12. $f(x) = 9 - x^2$
13. $f(x) = 3x^3 + 1$
14. $f(x) = 4 - 2x^3$

Find the inverse of each function.

15. $f(x) = 9 + \sqrt{4x-4}$
16. $f(x) = \sqrt{6x-8} + 5$
17. $f(x) = 9 + 2\sqrt[3]{x}$
18. $f(x) = 3 - \sqrt[3]{x}$

Find the x-intercept(s) of each function.

19. $h(x) = \sqrt{\frac{x}{3}} - 3$
20. $g(x) = -3 + \sqrt{x+1}$
21. $k(x) = \frac{1}{2}\sqrt{\frac{x}{3}}$
22. $f(x) = \sqrt{-5x-9} - 6$
23. $g(x) = \sqrt{9x+1} - 8$
24. $k(x) = \sqrt{9-x} - 3$

Find the intersection of each pair of functions.

25. $f(x) = \sqrt{30-x}$ and $g(x) = x$
26. $f(x) = \sqrt{-1+2x}$ and $g(x) = x$
27. $f(x) = \sqrt{x}$ and $g(x) = x$
28. $f(x) = \sqrt{2x-2}$ and $g(x) = x-1$
29. $f(x) = \sqrt{2x+1}$ and $g(x) = x+1$
30. $f(x) = \sqrt{11-2x}$ and $g(x) = x-4$

31. Police use the formula $v = \sqrt{20L}$ to estimate the speed of a car, v, in miles per hour, based on the length, L, in feet, of its skid marks when suddenly braking on a dry, asphalt road.

32. At the scene of an accident, a police officer measures a car's skid marks to be 215 feet long. Approximately how fast was the car traveling?

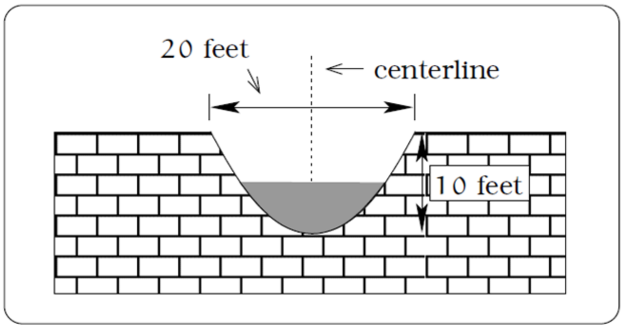
33. At the scene of an accident, a police officer measures a car's skid marks to be 135 feet long. Approximately how fast was the car traveling?

34. The formula $v = \sqrt{2.7r}$ models the maximum safe speed, v, in miles per hour, at which a car can travel on a curved road with radius of curvature r, in feet.

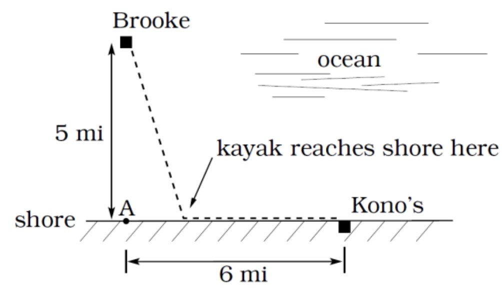
35. A highway crew measures the radius of curvature at an exit ramp on a highway as 430 feet. What is the maximum safe speed?

36. A highway crew measures the radius of curvature at a tight corner on a highway as 900 feet. What is the maximum safe speed?

37. A drainage canal has a cross-section in the shape of a parabola. Suppose that the canal is 10 feet deep and 20 feet wide at the top. If the water depth in the ditch is 5 feet, how wide is the surface of the water in the ditch? [UW]



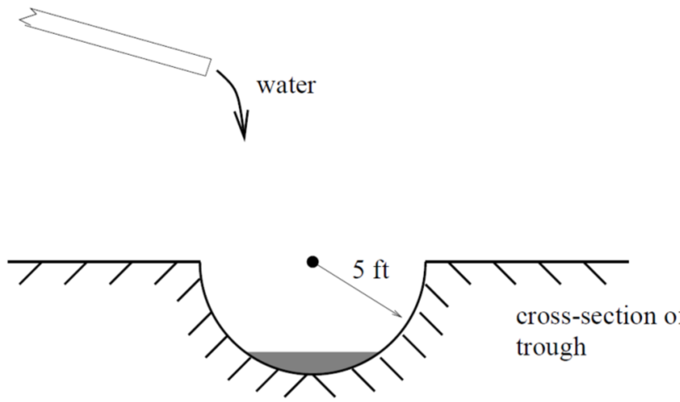
38. Brooke is located 5 miles out from the nearest point *A* along a straight shoreline in her sea kayak. Hunger strikes and she wants to make it to Kono's for lunch; see picture. Brooke can paddle 2 mph and walk 4 mph. [UW]



a. If she paddles along a straight line course to the shore, find an expression that computes the total time to reach lunch in terms of the location where Brooke beaches her kayak.
b. Determine the total time to reach Kono's if she paddles directly to the point *A*.
c. Determine the total time to reach Kono's if she paddles directly to Kono's.
d. Do you think your answer to b or c is the minimum time required for Brooke to reach lunch?
e. Determine the total time to reach Kono's if she paddles directly to a point on the shore half way between point A and Kono's. How does this time compare to the times in parts b or c? Do you need to modify your answer to part d?

39. Clovis is standing at the edge of a dropoff, which slopes 4 feet downward from him for every 1 horizontal foot. He launches a small model rocket from where he is standing. With the origin of the coordinate system located where he is standing, and the *x*-axis extending horizontally, the path of the rocket is described by the formula $y = -2x^2 + 120x$. [UW]

f. Give a function $h = f(x)$ relating the height *h* of the rocket above the sloping ground to its *x*-coordinate.
g. Find the maximum height of the rocket above the sloping ground. What is its *x*-coordinate when it is at its maximum height?
h. Clovis measures the height *h* of the rocket above the sloping ground while it is going up. Give a function $x = g(h)$ relating the *x*-coordinate of the rocket to *h*.
i. Does the function from (c) still work when the rocket is going down? Explain.



40. A trough has a semicircular cross section with a radius of 5 feet. Water starts flowing into the trough in such a way that the depth of the water is increasing at a rate of 2 inches per hour. [UW]
    j. Give a function $w = f(t)$ relating the width *w* of the surface of the water to the time *t*, in hours. Make sure to specify the domain and compute the range too.
    k. After how many hours will the surface of the water have width of 6 feet?
    l. Give a function $t = f^{-1}(w)$ relating the time to the width of the surface of the water. Make sure to specify the domain and compute the range too.

Section 2.10 Factor Theorem and Remainder Theorem

In the last section, we limited ourselves to finding the intercepts, or zeros, of polynomials that factored simply, or we turned to technology. In this section, we will look at algebraic techniques for finding the zeros of polynomials like $h(t) = t^3 + 4t^2 + t - 6$.

Long Division

In the last section we saw that we could write a polynomial as a product of factors, each corresponding to a horizontal intercept. If we knew that *x* = 2 was an intercept of the polynomial $x^3 + 4x^2 - 5x - 14$, we might guess that the polynomial could be factored as $x^3 + 4x^2 - 5x - 14 = (x - 2)(\text{something})$. To find that "something," we can use polynomial division.

Example 1

Divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$

Solution:
Start by writing the problem out in long division form

$$x - 2\ \overline{)x^3 + 4x^2 - 5x - 14}$$

Now we divide the leading terms: $x^3 \div x = x^2$. It is best to align it above the same-powered term in the dividend. Now, multiply that x^2 by $x - 2$ and write the result below the dividend.

$$\begin{array}{rrrrr} & & x^2 & & \\ \cline{2-5} x - 2 &)x^3 & +4x^2 & -5x & -14 \\ & x^3 & -2x^2 & & \end{array}$$

Now subtract that expression from the dividend.

$$\begin{array}{rrrrr} & & x^2 & & \\ \cline{2-5} x - 2 &)x^3 & +4x^2 & -5x & -14 \\ & -(x^3 & -2x^2) & & \\ \cline{2-3} & 0 & 6x^2 & -5x & -14 \end{array}$$

Again, divide the leading term of the remainder by the leading term of the divisor. $6x^2 \div x = 6x$. We add this to the result, multiply $6x$ by $x-2$, and subtract.

$$\begin{array}{rrrrr}
 & & x^2 & +6x & \\ \cline{2-5}
x-2 &)x^3 & +4x^2 & -5x & -14 \\
 & -(x^3 & -2x^2) & & \\ \cline{2-3}
 & 0 & 6x^2 & -5x & -14 \\
 & & -(6x^2 & -12x) & \\ \cline{3-4}
 & & & 7x & -14
\end{array}$$

Repeat the process one last time.

$$\begin{array}{rrrrr}
 & & x^2 & +6x & +7 \\ \cline{2-5}
x-2 &)x^3 & +4x^2 & -5x & -14 \\
 & -(x^3 & -2x^2) & & \\ \cline{2-3}
 & 0 & 6x^2 & -5x & -14 \\
 & & -(6x^2 & -12x) & \\ \cline{3-4}
 & & & 7x & -14 \\
 & & & -(7x & -14) \\ \cline{4-5}
 & & & & 0
\end{array}$$

This tells us $x^3 + 4x^2 - 5x - 14$ divided by $x - 2$ is $x^2 + 6x + 7$, with a remainder of zero. This also means that we can factor $x^3 + 4x^2 - 5x - 14$ as $(x-2)(x^2 + 6x + 7)$.

This factored form gives us a way to find the horizontal intercepts of this polynomial.

Example 2

Find the horizontal intercepts of $h(x) = x^3 + 4x^2 - 5x - 14$.

Solution:
To find the horizontal intercepts, we need to solve $h(x) = 0$. From the previous example, we know the function can be factored as $h(x) = (x-2)(x^2 + 6x + 7)$.

$h(x) = (x-2)(x^2 + 6x + 7) = 0$ when $x = 2$ or when $x^2 + 6x + 7 = 0$. This second equation doesn't factor nicely, but we could use the quadratic formula to find the remaining two zeros.

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)} = -3 \pm \sqrt{2}.$$

The horizontal intercepts will be at (2,0), $\left(-3 - \sqrt{2}, 0\right)$, and $\left(-3 + \sqrt{2}, 0\right)$.

Try it Now

Divide $2x^3 - 7x + 3$ by $x + 3$ using long division.

The Factor and Remainder Theorems

When we divide a polynomial, $p(x)$ by some divisor polynomial $d(x)$, we will get a quotient polynomial $q(x)$ and possibly a remainder $r(x)$. In other words,
$p(x) = d(x)q(x) + r(x)$.

Because of the division, the remainder will either be zero, or a polynomial of lower degree than *d(x)*. Because of this, if we divide a polynomial by a term of the form $x - c$, then the remainder will be zero or a constant.

If $p(x) = (x - c)q(x) + r$, then $p(c) = (c - c)q(c) + r = 0 + r = r$, which establishes the Remainder Theorem.

The Remainder Theorem
If $p(x)$ is a polynomial of degree 1 or greater and c is a real number, then when $p(x)$ is divided by $x - c$, the remainder is $p(c)$.

If $x - c$ is a factor of the polynomial *p*, then $p(x) = (x - c)q(x)$ for some polynomial *q*. Then $p(c) = (c - c)q(c) = 0$, showing *c* is a zero of the polynomial. This shouldn't surprise us - we already knew that if the polynomial factors it reveals the roots.

If $p(c) = 0$, then the remainder theorem tells us that if p is divided by $x - c$, then the remainder will be zero, which means $x - c$ is a factor of p.

The Factor Theorem
If $p(x)$ is a nonzero polynomial, then the real number c is a zero of $p(x)$ if and only if $x - c$ is a factor of $p(x)$.

Synthetic Division

Since dividing by $x - c$ is a way to check if a number is a zero of the polynomial, it would be nice to have a faster way to divide by $x - c$ than having to use long division every time. Happily, quicker ways have been discovered.

Let's look back at the long division we did in Example 1 and try to streamline it. First, let's change all the subtractions into additions by distributing through the negatives.

$$
\begin{array}{rrrrr}
 & & x^2 & +6x & +7 \\ \cline{2-5}
x-2 & \big)\,x^3 & +4x^2 & -5x & -14 \\
 & -(x^3 & -2x^2) & & \\ \cline{2-3}
 & 0 & 6x^2 & -5x & -14 \\
 & & -(6x^2 & -12x) & \\ \cline{3-4}
 & & & 7x & -14 \\
 & & & -(7x & -14) \\ \cline{4-5}
 & & & & 0
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$, and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and −14) aren't really necessary to recopy, so we omit them, too.

$$
\begin{array}{rrrrr}
 & & x^2 & +6x & +7 \\ \cline{2-5}
x-2 & \big)\,x^3 & +4x^2 & -5x & -14 \\
 & & 2x^2 & & \\ \cline{2-3}
 & & 6x^2 & & \\
 & & & 12x & \\ \cline{3-4}
 & & & 7x & \\
 & & & & 14 \\ \cline{5-5}
 & & & & 0
\end{array}
$$

Now, let's move things up a bit and, for reasons which will become clear in a moment, copy the x^3 into the last row.

$$
\begin{array}{rrrrr}
 & & x^2 & +6x & +7 \\ \cline{2-5}
x-2 & \big)\,x^3 & +4x^2 & -5x & -14 \\
 & & 2x^2 & 12x & 14 \\ \cline{2-5}
 & x^3 & 6x^2 & 7x & 0
\end{array}
$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial.
This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$
\begin{array}{rrrrr}
 & & & & \\ \cline{2-5}
x-2 & \big)\,x^3 & +4x^2 & -5x & -14 \\
 & & 2x^2 & 12x & 14 \\ \cline{2-5}
 & x^3 & 6x^2 & 7x & 0
\end{array}
$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the −2 in x − 2, then by −1 when we changed the subtraction to addition. Multiplying by −2 then by −1 is the same as multiplying by 2, so we replace the −2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

We have constructed a **synthetic division** tableau for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we write 2 in the place of the divisor and the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend. Then "bring down" the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was "brought down" to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the −5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the −14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x - c$. It is important to note that it works only for these kinds of divisors. Also take note that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division.

Example 3

Use synthetic division to divide $5x^3 - 2x^2 + 1$ by $x - 3$.

Solution:
When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend. Doing so gives

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \boxed{118} \end{array}$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$. This means
$5x^3 - 2x^2 + 1 = (x - 3)(5x^2 + 13x + 39) + 118.$

It also means that $x - 3$ is *not* a factor of $5x^3 - 2x^2 + 1$.

Example 4

Divide $x^3 + 8$ by $x + 2$

Solution:
For this division, we rewrite $x + 2$ as $x - (-2)$ and proceed as before.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

The quotient is $x^2 - 2x + 4$ and the remainder is zero. Since the remainder is zero, $x + 2$ is a factor of $x^3 + 8$.

$$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$$

Try it Now

Divide $4x^4 - 8x^2 - 5x$ by $x - 3$ using synthetic division.

Using this process allows us to find the real zeros of polynomials, presuming we can figure out at least one root. We'll explore how to do that in the next section.

Example 5

The polynomial $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$ has a horizontal intercept at $x = \frac{1}{2}$ with multiplicity 2. Find the other intercepts of $p(x)$.

Solution:

Since $x = \frac{1}{2}$ is an intercept with multiplicity 2, then $x - \frac{1}{2}$ is a factor twice. Use synthetic division to divide by $x - \frac{1}{2}$ twice.

$$\begin{array}{c|ccccc} \frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\ & \downarrow & 2 & -1 & -6 & 3 \\ \hline & 4 & -2 & -12 & 6 & \boxed{0} \end{array}$$

$$\begin{array}{c|cccc} \frac{1}{2} & 4 & -2 & -12 & 6 \\ & \downarrow & 2 & 0 & -6 \\ \hline & 4 & 0 & -12 & \boxed{0} \end{array}$$

From the first division, we get $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)(4x^3 - 2x^2 - 12x - 6)$

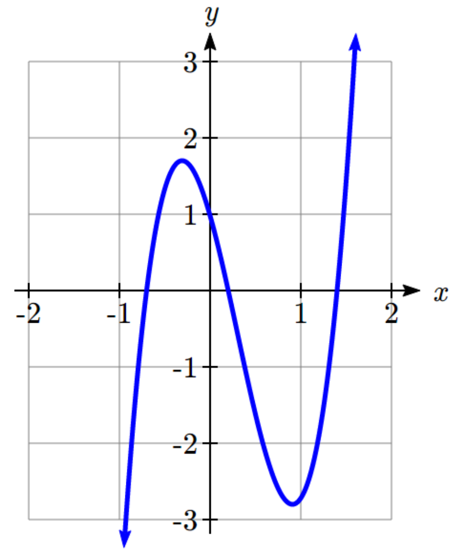
The second division tells us
$4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)(4x^2 - 12)$.

To find the remaining intercepts, we set $4x^2 - 12 = 0$ and get $x = \pm\sqrt{3}$.

Note this also means $4x^4 - 4x^3 - 11x^2 + 12x - 3 = 4\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)(x - \sqrt{3})(x + \sqrt{3})$.

Finding Possible Real Zeros

So far in this section, we have seen how to determine if a real number was a zero of a polynomial. Now we will learn how to find good candidates to test using synthetic division. In the days before graphing technology was commonplace, mathematicians discovered a lot of clever tricks for determining the likely locations of zeros. Technology has provided a much simpler approach to narrow down potential candidates, but it is not always sufficient by itself. For example, the function shown to the right does not have any clear intercepts.



There are two results that can help us identify where the zeros of a polynomial are. The first gives us an interval on which all the real zeros of a polynomial can be found.

Cauchy's Bound

Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, let M be the largest of the coefficients in absolute value. Then all the real zeros of $f(x)$ lie in the interval

$$\left[-\frac{M}{|a_n|} - 1, \quad \frac{M}{|a_n|} + 1\right]$$

Example 6

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Determine an interval which contains all the real zeros of f.

Solution:

To find the M from Cauchy's Bound, we take the absolute value of the coefficients and pick the largest, in this case $|-6| = 6$. Divide this by the absolute value of the leading coefficient, 2, to get 3. All the real zeros of f lie in the interval $\left[-\frac{6}{|2|} - 1, \quad \frac{6}{|2|} + 1\right] = [-3 - 1, \quad 3 + 1] = [-4, \ 4]$.

Knowing this bound can be very helpful when using a graphing calculator, since we can use it to set the display bounds. This helps avoid missing a zero because it is graphed outside of the viewing window.

Try it Now

Determine an interval which contains all the real zeros of $f(x) = 3x^3 - 12x^2 + 6x - 8$

Now that we know *where* we can find the real zeros, we still need a list of *possible* real zeros. The Rational Roots Theorem provides us a list of potential integer and rational zeros.

Rational Roots Theorem

Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients, if r is a rational zero of f, then r is of the form $r = \pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient, a_n.

This gives us a list of numbers to try in our synthetic division, which is a nicer place to start than simply guessing. If none of the numbers in the list are zeros, then either the polynomial has no real zeros at all, or all the real zeros are irrational numbers.

Example 7

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Roots Theorem to list all the possible rational zeros of $f(x)$.

Solution:

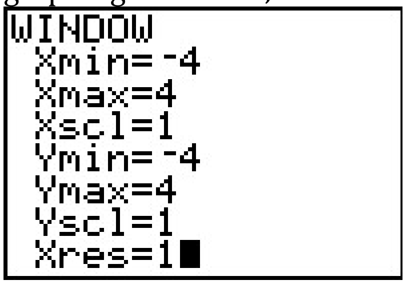
To generate a complete list of rational zeros, we need to take each of the factors of the
constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$. The factors of -3 are ± 1 and ± 3. Since the Rational Roots Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Roots Theorem gives the list $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\}$, or $\left\{\pm 1, \pm\frac{1}{2}, \pm 3, \pm\frac{3}{2}\right\}$

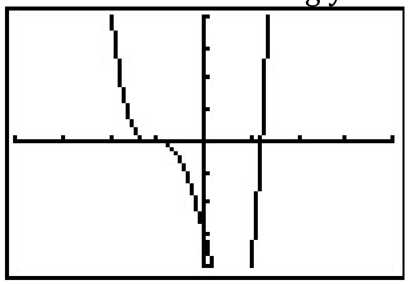
Now we can use synthetic division to test these possible zeros. To narrow the list first, we could use graphing technology to help us identify some good possibilities.

Example 8

Find the horizontal intercepts of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

Solution:
From Example 1, we know that the real zeros lie in the interval [-4, 4]. Using a graphing calculator, we could set the window accordingly and get the graph below.





In the previous example, we learned that any rational zero must be on the list $\left\{\pm 1, \pm\frac{1}{2}, \pm 3, \pm\frac{3}{2}\right\}$. From the graph, it looks like -1 is a good possibility, so we try that using synthetic division.

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \boxed{0} \end{array}$$

Success! Remembering that f was a fourth degree polynomial, we know that our quotient is a third degree polynomial. If we can do one more successful division, we will have knocked the quotient down to a quadratic, and, if all else fails, we can use the quadratic formula to find the last two zeros. Since there seems to be no other rational zeros to try, we continue with -1. Also, the shape of the crossing at $x = -1$ leads us to wonder if the zero $x = -1$ has multiplicity 3.

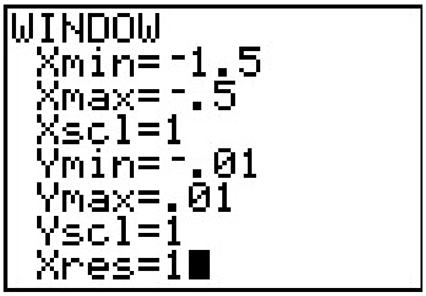
$$\begin{array}{r|rrrr} -1 & 2 & 2 & -3 & -3 \\ & \downarrow & -2 & 0 & 3 \\ \hline & 2 & 0 & -3 & \boxed{0} \end{array}$$

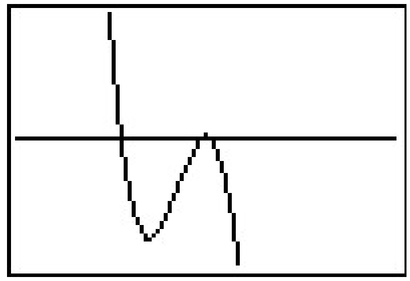
Success again! Our quotient polynomial is now $2x^2 - 3$. Setting this to zero gives $2x^2 - 3 = 0$, giving $x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$. Since a fourth degree polynomial can have at most four zeros, including multiplicities, then the intercept $x = -1$ must only have multiplicity 2, which we had found through division, and not 3 as we had guessed.

It is interesting to note that we could greatly improve on the graph of $y = f(x)$ in the previous example given to us by the calculator. For instance, from our determination of the zeros of *f* and their multiplicities, we know the graph crosses at $x = -\frac{\sqrt{6}}{2} \approx -1.22$ then turns back upwards to touch the x-axis at $x = -1$.

This tells us that, despite what the calculator showed us the first time, there is a relative maximum occurring at $x = -1$ and not a "flattened crossing" as we originally believed.

After resizing the window, we see not only the relative maximum but also a relative minimum just to the left of $x = -1$.





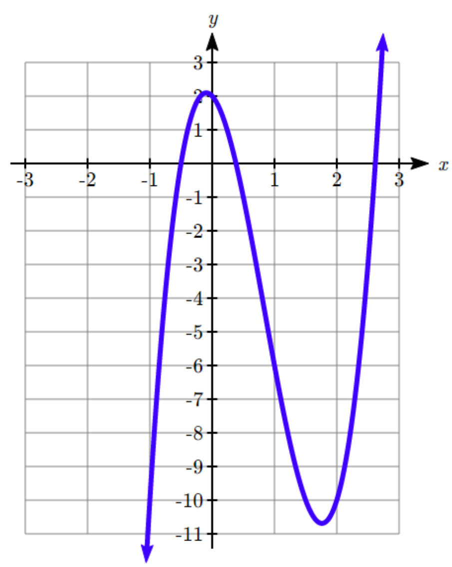
In this case, mathematics helped reveal something that was hidden in the initial graph.

Example 9

Find the real zeros of $f(x) = 4x^3 - 10x^2 - 2x + 2$.

Solution:

Cauchy's Bound tells us that the real zeros lie in the interval $\left[-\frac{10}{|4|} - 1,\ \ \frac{10}{|4|} + 1\right] = [-3.5,\ 3.5]$.



Graphing on this interval reveals no clear integer zeros. Turning to the rational roots theorem, we need to take each of the factors of the constant term, $a_0 = 2$, and divide them by each of the factors of the leading coefficient $a_3 = 4$. The factors of 2 are 1 and 2. The factors of 4 are 1, 2, and 4, so the Rational Roots Theorem gives the list $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{2}{4}\right\}$, or $\left\{\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2\right\}$

The two most likely candidates are $\pm\frac{1}{2}$.

Trying $\frac{1}{2}$,

$$\begin{array}{c|cccc} \frac{1}{2} & 4 & -10 & -2 & 2 \\ & \downarrow & 2 & -4 & -3 \\ \hline & 4 & -8 & -6 & -1 \end{array}$$

The remainder is not zero, so this is not a zero. Trying $-\frac{1}{2}$,

$$\begin{array}{r|rrrr} -\frac{1}{2} & 4 & -10 & -2 & 2 \\ & \downarrow & -2 & 6 & -2 \\ \hline & 4 & -12 & 4 & 0 \end{array}$$

Success! This tells us $4x^3 - 10x^2 - 2x + 2 = \left(x + \frac{1}{2}\right)(4x^2 - 12x + 4)$, and that the graph has a horizontal intercept at $x = -\frac{1}{2}$.

To find the remaining two intercepts, we can use the quadratic equation, setting $4x^2 - 12x + 4 = 0$. First, we might pull out the common factor, $4(x^2 - 3x + 1) = 0$.

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2} \approx 2.618, \quad 0.382$$

Try it Now

Find the real zeros of $f(x) = 3x^3 - x^2 - 6x + 2$

Important Topics of this Section
Long division of polynomials
Remainder Theorem
Factor Theorem
Synthetic division of polynomials
Cauchy's Bound for all real zeros of a polynomial
Rational Roots Theorem
Finding real zeros of a polynomial

Section 2.10 Exercises

Conceptual Questions

1. Describe the process to divide polynomials using long division.
2. If we divide two polynomials and get a remainder of zero, what does that tell us?
3. Explain Cauchy's Bound.
4. Describe how to use Cauchy's Bound and the Rational Roots Theorem to find roots of a polynomial.

Practice Problems

Use polynomial long division to perform the indicated division.

1. $(4x^2 + 3x - 1) \div (x - 3)$
2. $(2x^3 - x + 1) \div (x^2 + x + 1)$
3. $(5x^4 - 3x^3 + 2x^2 - 1) \div (x^2 + 4)$
4. $(-x^5 + 7x^3 - x) \div (x^3 - x^2 + 1)$
5. $(9x^3 + 5) \div (2x - 3)$
6. $(4x^2 - x - 23) \div (x^2 - 1)$

Use synthetic division to perform the indicated division.

7. $(3x^2 - 2x + 1) \div (x - 1)$
8. $(x^2 - 5) \div (x - 5)$
9. $(3 - 4x - 2x^2) \div (x + 1)$
10. $(4x^2 - 5x + 3) \div (x + 3)$
11. $(x^3 + 8) \div (x + 2)$
12. $(4x^3 + 2x - 3) \div (x - 3)$
13. $(18x^2 - 15x - 25) \div \left(x - \frac{5}{3}\right)$
14. $(4x^2 - 1) \div \left(x - \frac{1}{2}\right)$
15. $(2x^3 + x^2 + 2x + 1) \div \left(x + \frac{1}{2}\right)$
16. $(3x^3 - x + 4) \div \left(x - \frac{2}{3}\right)$
17. $(2x^3 - 3x + 1) \div \left(x - \frac{1}{2}\right)$
18. $(4x^4 - 12x^3 + 13x^2 - 12x + 9) \div \left(x - \frac{3}{2}\right)$
19. $(x^4 - 6x^2 + 9) \div (x - \sqrt{3})$
20. $(x^6 - 6x^4 + 12x^2 - 8) \div (x + \sqrt{2})$

Below you are given a polynomial and one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

21. $x^3 - 6x^2 + 11x - 6, c = 1$
22. $x^3 - 24x^2 + 192x - 512, c = 8$
23. $3x^3 + 4x^2 - x - 2, c = \frac{2}{3}$
24. $2x^3 - 3x^2 - 11x + 6, c = \frac{1}{2}$
25. $x^3 + 2x^2 - 3x - 6, c = -2$
26. $2x^3 - x^2 - 10x + 5, c = \frac{1}{2}$
27. $4x^4 - 28x^3 + 61x^2 - 42x + 9, c = \frac{1}{2}$ is a zero of multiplicity 2
28. $x^5 + 2x^4 - 12x^3 - 38x^2 - 37x - 12, c = -1$ is a zero of multiplicity 3

For each of the following polynomials, use Cauchy's Bound to find an interval containing all the real zeros, then use Rational Roots Theorem to make a list of possible rational zeros.

29. $f(x) = x^3 - 2x^2 - 5x + 6$
30. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
31. $f(x) = x^4 - 9x^2 - 4x + 12$
32. $f(x) = x^3 + 4x^2 - 11x + 6$
33. $f(x) = x^3 - 7x^2 + x - 7$
34. $f(x) = -2x^3 + 19x^2 - 49x + 20$
35. $f(x) = -17x^3 + 5x^2 + 34x - 10$
36. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
37. $f(x) = 3x^3 + 3x^2 - 11x - 10$
38. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

Find the real zeros of each polynomial.

39. $f(x) = x^3 - 2x^2 - 5x + 6$
40. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
41. $f(x) = x^4 - 9x^2 - 4x + 12$
42. $f(x) = x^3 + 4x^2 - 11x + 6$
43. $f(x) = x^3 - 7x^2 + x - 7$
44. $f(x) = -2x^3 + 19x^2 - 49x + 20$
45. $f(x) = -17x^3 + 5x^2 + 34x - 10$
46. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
47. $f(x) = 3x^3 + 3x^2 - 11x - 10$
48. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
49. $f(x) = 9x^3 - 5x^2 - x$
50. $f(x) = 6x^4 - 5x^3 - 9x^2$
51. $f(x) = x^4 + 2x^2 - 15$
52. $f(x) = x^4 - 9x^2 + 14$
53. $f(x) = 3x^4 - 14x^2 - 5$
54. $f(x) = 2x^4 - 7x^2 + 6$
55. $f(x) = x^6 - 3x^3 - 10$
56. $f(x) = 2x^6 - 9x^3 + 10$
57. $f(x) = x^5 - 2x^4 - 4x + 8$
58. $f(x) = 2x^5 + 3x^4 - 18x - 27$
59. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$
60. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$

Section 2.11 Rational Functions

In the previous sections, we have built polynomials based on the positive whole number power functions. In this section, we explore functions based on power functions with negative integer powers, called rational functions.

Example 1

You plan to drive 100 miles. Find a formula for the time the trip will take as a function of the speed you drive.

Solution:

You may recall that multiplying speed by time will give you distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which we drive, then $vt =$ distance. Since our distance is fixed at 100 miles, $vt = 100$. Solving this relationship for the time gives us the function we desired:

$$t(v) = \frac{100}{v} = 100v^{-1}$$

While this type of relationship can be written using the negative exponent, it is more common to see it written as a fraction.

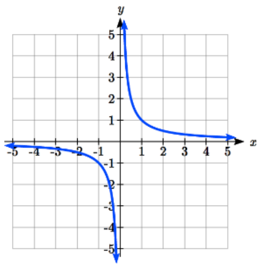
This particular example is one of an **inversely proportional** relationship – where one quantity is a constant divided by the other quantity, like $y = \frac{5}{x}$.

Notice that this is a transformation of the reciprocal toolkit function, $f(x) = \frac{1}{x}$.

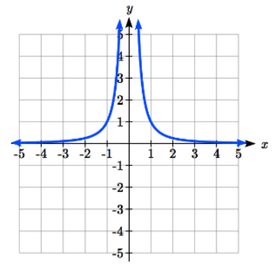
Several natural phenomena, such as gravitational force and volume of sound, behave in a manner **inversely proportional to the square** of another quantity. For example, the volume, *V*, of a sound heard at a distance *d* from the source would be related by $V = \frac{k}{d^2}$ for some constant value *k*.

These functions are transformations of the reciprocal squared toolkit function $f(x) = \frac{1}{x^2}$.

The graphs of these toolkit functions have several important features.



$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. As you well know, dividing by zero is not allowed and therefore zero is not in the domain, and so the function is undefined at an input of zero.

Short run behavior:

As the input values approach zero from the left side (taking on very small, negative values), the function values become very large in the negative direction (in other words, they approach negative infinity).
We write: as $x \to 0^-$, $f(x) \to -\infty$.

As we approach zero from the right side (small, positive input values), the function values become very large in the positive direction (approaching infinity).
We write: as $x \to 0^+$, $f(x) \to \infty$.

This behavior creates a **vertical asymptote**. An asymptote is a line that the graph approaches. In this case the graph is approaching the vertical line *x* = 0 as the input becomes close to zero.

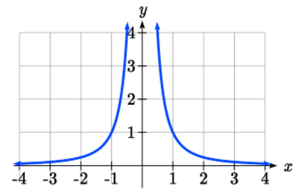
Long run behavior:

As the values of *x* approach infinity, the function values approach 0.
As the values of *x* approach negative infinity, the function values approach 0.
Symbolically: as $x \to \pm\infty$, $f(x) \to 0$.

Based on this long run behavior and the graph we can see that the function approaches 0 but never actually reaches 0, it just "levels off" as the inputs become large. This behavior creates a **horizontal asymptote**. In this case the graph is approaching the horizontal line $f(x) = 0$ as the input becomes very large in the negative and positive directions.

Vertical and Horizontal Asymptotes
A **vertical asymptote** of a graph is a vertical line $x = c$ where the graph tends towards positive or negative infinity as the inputs approach c. As $x \to c$, $f(x) \to \pm\infty$.   A **horizontal asymptote** of a graph is a horizontal line $y = d$ where the graph approaches the line as the inputs get large. As $x \to \pm\infty$, $f(x) \to d$.

Try it Now:
Use symbolic notation to describe the long run behavior and short run behavior for the reciprocal squared function.

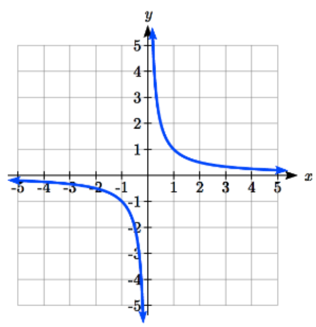


Now we summarize the toolkit information for the reciprocal and reciprocal squared functions.

Toolkit (Parent) Reciprocal Function

The parent function $f(x) = \frac{1}{x}$ can be described with the following table and graph.

x	$f(x)$
Horizontal Asymptote (HA)	0
-1	-1
0	Vertical Asymptote (VA)
1	1



The graph of the reciprocal function has the following features:

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

Intercepts: none

End Behavior: As $x \to \pm\infty$, $f(x) \to 0$ (Horizontal Asymptote at $y = 0$)

Vertical asymptote at $x = 0$ (as $x \to 0^+, f(x) \to \infty$ and as $x \to 0^-, f(x) \to -\infty$)

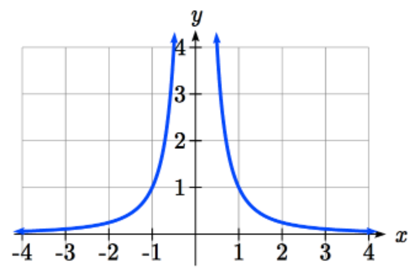
Decreasing on $(-\infty, 0) \cup (0, \infty)$

Concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$

Toolkit (Parent) Reciprocal Squared Function

The parent function $f(x) = \frac{1}{x^2}$ can be described with the following table and graph.

x	$f(x)$
Horizontal Asymptote (HA)	0
-1	1
0	Vertical Asymptote (VA)
1	1



The graph has the following features:

Domain: $(-\infty, 0 \cup (0, \infty)$

Range: $(0, \infty)$

Intercepts: none

End Behavior: As $x \to \pm\infty,\ f(x) \to 0$ (Horizontal Asymptote at $y = 0$)

Vertical asymptote at $x = 0$ (as $x \to 0^{\pm}, f(x) \to \infty$)

Increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$

Concave up on $(-\infty, 0) \cup (0, \infty)$

Example 2

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any. Then, describe all the features of the graph.

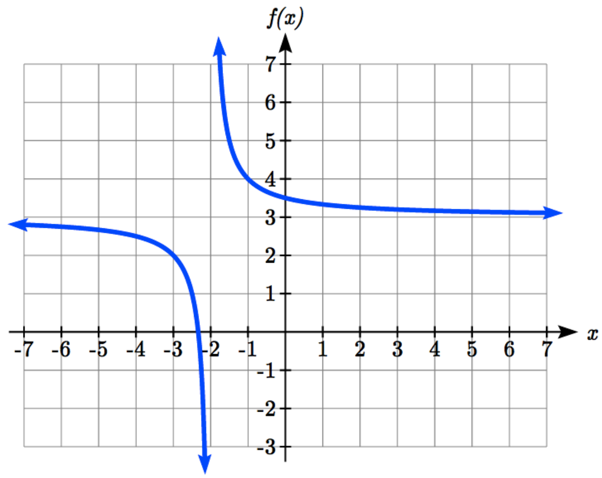
Solution:

Transforming the graph left 2 and up 3 would result in the function $g(x) = \frac{1}{x+2} + 3$, or equivalently, by giving the terms a common denominator, $g(x) = \frac{(3x+7)}{x+2}$.

Working with the first form of the function, though, we can use our C-BAD process to graph, where $c = 2$ and $d = 3$

$x-2$	x	$f(x)$	$f(x)+3$
HA	HA	0	3
-3	-1	-1	2
-2	0	VA	VA
-1	1	1	4

The key here is that when we try to apply transformations to asymptotes, nothing happens to them. They're infinite beings; subtracting two or adding three won't change them! We can now use our table and our knowledge of the shape of the reciprocal function to sketch the graph.



Notice that this equation is undefined at x = -2, and the graph also is showing a vertical asymptote at x= -2.
As $x \to -2^-, g(x) \to -\infty$, and
as $x \to -2^+, g(x) \to \infty$

As the inputs grow large, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.
As $x \to \pm\infty, g(x) \to\ 3$

Notice that horizontal and vertical asymptotes get shifted left 2 and up 3 along with the function.

This graph has the following features:
Intercepts: x-intercept of about -2.1 (more on that soon), y-intercept of about 3.5
Domain: $(-\infty, -2) \cup (-2, \infty)$
Range: $(-\infty, 3) \cup (3, \infty)$
Decreasing on $(-\infty, -2) \cup (-2, \infty)$
Concave down on $(-\infty, -2)$, concave up on $(-2, \infty)$

Try it Now

Sketch the graph and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

In a previous example, we shifted a toolkit function in a way that resulted in a function of the form $f(x) = \frac{3x+7}{x+2}$. This is an example of a more general rational function.

Rational Function

A **rational function** is a function that can be written as the ratio of two polynomials, $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p}{b_0 + b_1 x + b_2 x^2 + \cdots + b_q x^q}$$

Example 3

A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Solution:

Notice that the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\text{water} = 100 + 10t$$
$$\text{sugar} = 5 + 1t$$

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5 + t}{100 + 10t}$$

Finding Asymptotes and Intercepts

Given a rational function, as part of investigating the short run behavior we are interested in finding any vertical and horizontal asymptotes, as well as finding any vertical or horizontal intercepts, as we have done in the past.

To find vertical asymptotes, we notice that the vertical asymptotes in our examples occur when the denominator is zero, so the function is undefined. With one exception, a vertical asymptote will occur whenever the denominator is zero.

Example 4

Find the vertical asymptotes of the function $k(x) = \frac{5+2x^2}{2-x-x^2}$.

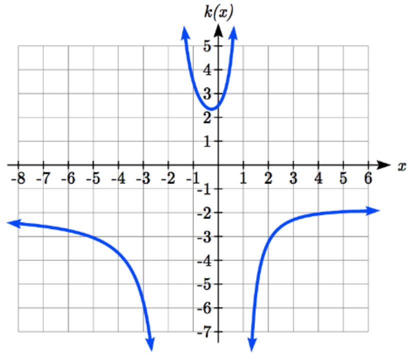
Solution:

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$2 - x - x^2 = 0$$
$$(2 + x)(1 - x) = 0$$

$$x = -2, 1$$

This indicates two vertical asymptotes, which a look at a graph confirms.



The exception to this rule can occur when both the numerator and denominator of a rational function are zero at the same input.

Example 5

Find the vertical asymptotes of the function $k(x) = \frac{x-2}{x^2-4}$.

Solution:

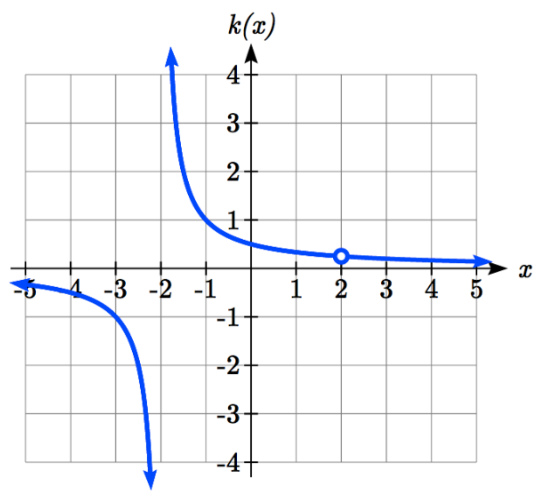
To find the vertical asymptotes, we determine where this function will be undefined by finding where denominator will be zero. We can do this by factoring:

$$k(x) = \frac{x-2}{x^2-4}$$
$$= \frac{x-2}{(x+2)(x-2)}$$

So the denominator is zero at $x = 2, -2$.

However, the numerator of this function is also equal to zero when $x = 2$. The function will still be undefined at 2, since $\frac{0}{0}$ is undefined, but if $x \neq 2$ then $\frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2}$ which does not have a vertical asymptote at $x = 2$.

The graph of $k(x)$ will have a vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole: a single point where the graph is not defined, indicated by an open circle.



Vertical Asymptotes and Holes of Rational Functions

At points where the denominator of a rational function equals zero and the numerator is not zero, the rational function has a **vertical asymptote.**

At points where both the numerator and denominator of a rational function equal zero, factor the numerator and denominator and simplify: either the simplified function will have a vertical asymptote (and the original function will as well) or the graph of the original function will have a **hole.**

To find horizontal asymptotes, we are interested in the behavior of the function as the input grows large, so we consider long run behavior of the numerator and denominator separately. Recall that a polynomial's long run behavior will mirror that of the leading term. Likewise, a rational function's long run behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

There are three distinct outcomes when this analysis is done:

Case 1: The degree of the denominator > degree of the numerator

Example: $f(x) = \frac{3x+2}{x^2+4x-5}$

In this case, the long run behavior is $f(x) \approx \frac{3x}{x^2} = \frac{3}{x}$. This tells us that as the inputs grow large, this function will behave similarly to the function $g(x) = \frac{3}{x}$. As the inputs grow large, the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$.

As $x \to \pm\infty, f(x) \to 0$

Case 2: The degree of the denominator < degree of the numerator

Example: $f(x) = \frac{3x^2+2}{x-5}$

In this case, the long run behavior is $f(x) \approx \frac{3x^2}{x} = 3x$. This tells us that as the inputs grow large, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote.

As $x \to \pm\infty, f(x) \to \pm\infty$, respectively.

Case 3: The degree of the denominator = degree of the numerator

Example: $f(x) = \frac{3x^2+2}{x^2+4x-5}$

In this case, the long run behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line.

As $x \to \pm\infty, f(x) \to 3$ resulting in a horizontal asymptote at $y = 3$.

Horizontal Asymptote of Rational Functions

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

Degree of denominator $>$ degree of numerator: Horizontal asymptote at $y = 0$

Degree of denominator $<$ degree of numerator: No horizontal asymptote

Degree of denominator $=$ degree of numerator: Horizontal asymptote at ratio of leading coefficients.

Example 6

In the sugar concentration problem from earlier, we created the equation $C(t) = \frac{5+t}{100+10t}$.

Find the horizontal asymptote and interpret it in context of the scenario.

Solution:

Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values: As $t \to \infty, C(t) \to \frac{1}{10}$. This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the input gets large, the output values will approach 1/10. In context, this means that as more time goes by, the concentration of sugar in the tank will approach one tenth of a pound of sugar per gallon of water or 1/10 pounds per gallon.

Example 7

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

First, note this function has no inputs that make both the numerator and denominator zero, so there are no potential holes. The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at x= 1, -2, and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than that of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty, f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$.

Try it Now

Find the vertical and horizontal asymptotes of the function $f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$

Intercepts

As with all functions, a rational function will have a vertical intercept when the input is zero, if the function is defined at zero. It is possible for a rational function to not have a vertical intercept if the function is undefined at zero.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be zero (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. Since a fraction is only equal to zero when the numerator is zero, horizontal intercepts will occur when the numerator of the rational function is equal to zero.

Example 8

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Solution:

We can find the vertical intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)}$$
$$= \frac{-6}{10}$$
$$= -\frac{3}{5}$$

The horizontal intercepts will occur when the function is equal to zero:

$$\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} = 0$$

This is zero when the numerator is zero.

$$(x-2)(x+3) = 0$$
$$x = 2, -3$$

Sometimes we're going to need to do a bit of work to find intercepts of our rational functions. We'll outline a solving strategy and do a bit of practice just in finding intercepts.

Solving Rational Equations

To solve a rational equation (where the variable is in the denominator):

1. Create a common denominator for every term
2. Simplify each side so that we have a fraction equal to a fraction
3. Set the numerators equal to each other to solve
4. Set the denominator equal to 0 to find excluded values

Example 9

Find the x- and y-intercepts of the function $f(x) = \frac{1}{x+2} + 3$

Solution:
To find the x-intercepts, we must solve $f(x) = 0$.

$$\frac{1}{x+2} + 3 \quad = 0$$

This is a rational equation, since the variable is in the denominator. So, we'll create a common denominator, add, then finish solving.

$$\begin{aligned} \frac{1}{x+2} + 3 &= 0 \\ \frac{1}{x+2} + 3 \cdot \frac{x+2}{x+2} &= 0 \cdot \frac{x+2}{x+2} && \text{Multiply by a fancy form of 1} \\ \frac{1}{x+2} + \frac{3(x+2)}{x+2} &= \frac{0(x+2)}{x+2} \\ \frac{1+3(x+2)}{x+2} &= \frac{0(x+2)}{x+2} && \text{Add} \end{aligned}$$

Now, here's the idea: the denominators are already the same on both sides of the equation. That means, if we want to solve the equation, we don't have to worry about the denominator at all! We'll just set the numerators equal to each other and finish solving.

$$1 + 3(x+2) = 0(x+2)$$

This is a linear function, so we'll simplify, move all the variables to the same side, and then isolate the variable.

$$\begin{aligned} 1 + 3x + 6 &= 0 \\ 3x + 7 &= 0 \\ 3x &= -7 \\ x &= -\frac{7}{3} \end{aligned}$$

Now we know our x-intercept was exactly $-7/3$.

For the y-intercept, we'll still evaluate $f(0)$, so

$$\begin{aligned} f(0) &= \frac{1}{0+2} + 3 \\ &= \frac{1}{2} + \frac{6}{2} \\ &= \frac{7}{2} \end{aligned}$$

The y-intercept is exactly $7/2$.

Example 10

Find the intercepts of $f(x) = \frac{-2}{x^3 - 3x^2} - \frac{1}{x}$

Solution:

To find the x intercepts, we want to solve $f(x) = 0$:

$$\frac{-2}{x^3 - 3x^2} - \frac{1}{x} = 0$$

This is a rational equation, since the variable is in the denominators. We'll need to choose a common denominator. That will be easier if we first factor our denominators.

$$\frac{-2}{x^2(x-3)} - \frac{1}{x} = 0$$

The least common denominator is $x^2(x-3)$, so we'll multiply each term by a fancy form of 1 to create that denominator.

$$\begin{gathered} \frac{-2}{x^2(x-3)} - \frac{1}{x} \cdot \frac{x(x-3)}{x(x-3)} = 0 \cdot \frac{x^2(x-3)}{x^2(x-3)} \\ \frac{-2}{x^2(x-3)} - \frac{1x(x-3)}{x^2(x-3)} = \frac{0}{x^2(x-3)} \\ \frac{-2 - x(x-3)}{x^2(x-3)} = \frac{0}{x^2(x-3)} \end{gathered}$$

Now the denominators are equal, so we only have to worry about solving for $-2 - x(x-3) = 0$. If we expand a little, we have $-2 - x^2 + 3x = 0$, a quadratic. Let's rearrange so we can factor and use the Zero Product Property!

$$\begin{gathered} -x^2 + 3x - 2 = 0 \\ -1(x^2 - 3x + 2) = 0 \end{gathered}$$

We're looking for two numbers which multiply to make 2 and add to make -3. That's -2 and -1.

$$\begin{gathered} -1(x-2)(x-1) = 0 \\ \text{ZPP} \\ \begin{matrix} x - 2 = 0 \\ x = 2 \end{matrix} \quad or \quad \begin{matrix} x - 1 = 0 \\ x = 1 \end{matrix} \end{gathered}$$

Now we check for extraneous solutions. We would have a division by zero error if the denominator was 0, so we need to exclude x-values which make $x^2(x-3)=0$. This is already factored, so a quick application of the ZPP tells us $x \neq 0, 3$. Our intercepts were 2 and 1, so we don't have to exclude either of these.

For the y-intercept, we evaluate $f(0)$:

$$f(0) = -\frac{2}{0^3 - 3 \cdot 0^2} - \frac{1}{0}$$

Wait a second! Division by zero is undefined. There is no y-intercept, since there's a vertical asymptote at $x = 0$.

Try it Now

Find the intercepts of each function.

a. $f(x) = \frac{2}{x+6} - 6$

b. $g(x) = \frac{x-1}{x} - \frac{1}{x^2-x}$

Our solving strategies can also help us to find where rational functions intersect with other functions.

Example 11

Find where the functions $f(x) = \frac{2}{x-2}$ and $g(x) = x - 3$ intersect.

Solution:

To answer this, we'll need to solve $f(x) = g(x)$:

$$\frac{2}{x-2} = x - 3$$

This qualifies as a rational equation, since we have a fraction with a variable in the denominator.

$$\begin{aligned} \frac{2}{x-2} &= x - 3 \\ \frac{2}{x-2} &= (x-3) \cdot \frac{x-2}{x-2} && \text{Make a common denominator} \\ \frac{2}{x-2} &= \frac{(x-3)(x-2)}{(x-2)} \end{aligned}$$

Now, the denominators are equal, so we only need to solve $2 = (x-3)(x-2)$.

$$\begin{aligned}
2 &= (x-3)(x-2) && \\
2 &= x^2 - 5x + 6 && \text{Expand} \\
0 &= x^2 - 5x + 4 && \text{Quadratic:get equal to 0} \\
0 &= (x-4)(x-1) && \text{Factor} \\
x &= 4,1 && \text{ZPP}
\end{aligned}$$

We should also solve for $x - 2 = 0$ to look out for excluded values. The only thing we have to be concerned about is $x = 2$, so our solutions of $x = 4,1$ are both valid.

The functions intersect when x is 4 and 1.

Try it Now

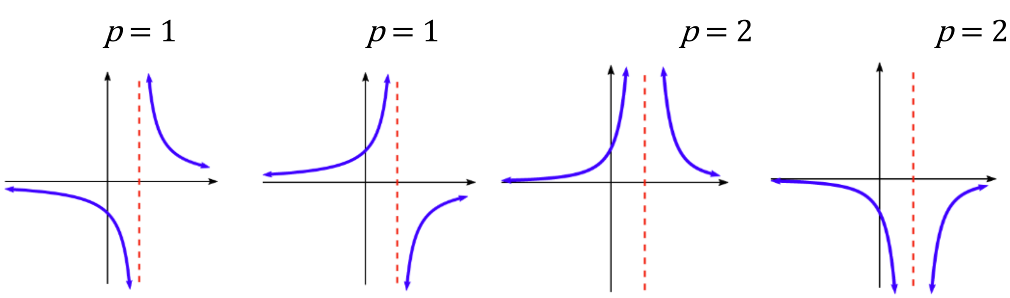
Find where the functions $f(x) = \frac{x-3}{2x-4} - \frac{x+1}{x-3}$ and $g(x) = \frac{1}{2x^2-10x+12}$ intersect.

Graphical Behavior at Intercepts and Vertical Asymptotes

As with polynomials, factors of the numerator may have integer powers greater than one. Happily, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials: if the factor giving the intercept is not squared, the graph passes through the axis; if the factor is squared, the graph will bounce off the axis at that intercept. The behavior at vertical asymptotes also depends on the power on the factor.

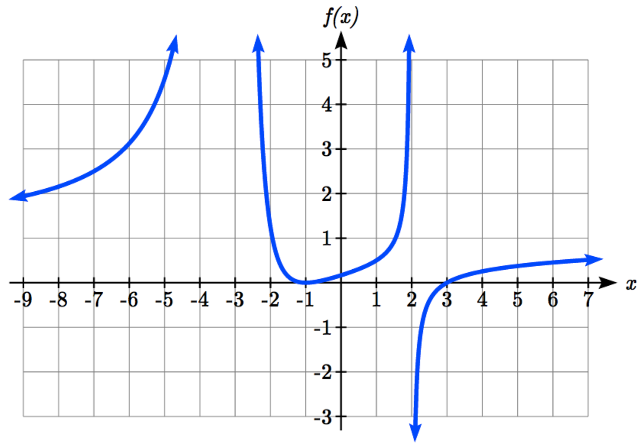
Graphical Behavior of Rational Functions at Vertical Asymptotes

If a rational function contains a factor of the form $(x-h)^p$ in the denominator, the behavior near the asymptote h is determined by the power on the factor.



When the factor is not squared, on one side of the asymptote the graph heads towards positive infinity and on the other side the graph heads towards negative infinity.

When the factor is squared, the graph either heads toward positive infinity on both sides of the vertical asymptote, or heads toward negative infinity on both sides.



For example, the graph of $f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown here.

At the horizontal intercept *x* = -1 corresponding to the $(x+1)^2$ factor of the numerator, the graph bounces at the intercept, consistent with the quadratic nature of the factor.

At the horizontal intercept *x* = 3 corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we'd expect from a linear factor.

At the vertical asymptote *x* = -3 corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the $\frac{1}{x^2}$ toolkit.
At the vertical asymptote *x* = 2 corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the $\frac{1}{x}$ toolkit.

Example 12

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution:
We can start our sketch by finding intercepts and asymptotes. Evaluating the function at zero gives the vertical intercept:

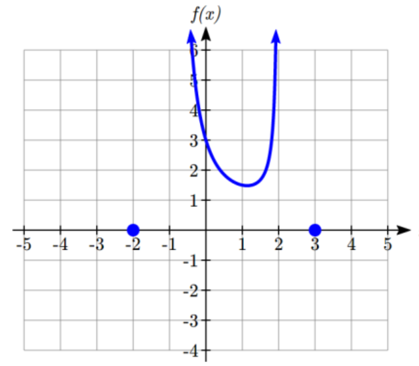
$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)}$$

$$= 3$$

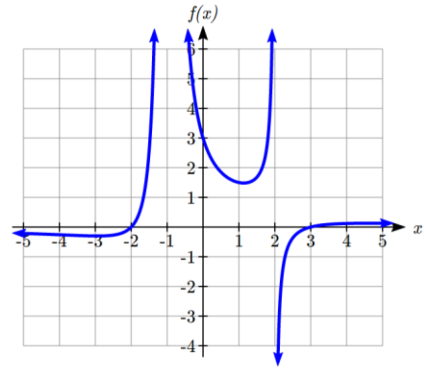
Looking at when the numerator of the function is zero, we can determine the graph will have horizontal intercepts at $x =$ -2 and $x =$ 3. At each, the behavior will be linear, with the graph passing through the intercept.

Looking at when the denominator of the function is zero, we can determine the graph will have vertical asymptotes at $x =$ -1 and $x =$ 2.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.
To sketch the graph, we might start by plotting the three intercepts. Since the graph has no horizontal intercepts between the vertical asymptotes, and the vertical intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph.



Since the factor associated with the vertical asymptote at $x = -1$ was squared, we know the graph will have the same behavior on both sides of the asymptote. Since the graph heads towards positive infinity as the inputs approach the asymptote on the right, the graph will head towards positive infinity on the left as well. For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote.



After passing through the horizontal intercepts, the graph will then level off towards an output of zero, as indicated by the horizontal asymptote.

Try it Now

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Since a rational function written in factored form will have a horizontal intercept where each factor of the numerator is equal to zero, we can form a numerator that will pass through a set of horizontal intercepts by introducing a corresponding set of factors. Likewise, since the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

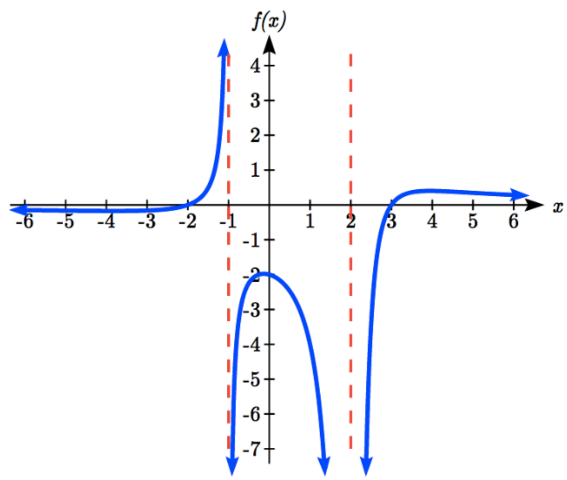
If a rational function has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, and vertical asymptotes at $x = v_1, v_2, \ldots, v_m$ then the function can be written in the form

$$f(x) = a\frac{(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}}{(x - v_1)^{q_1}(x - v_2)^{q_2} \cdots (x - v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the horizontal intercept, or by the horizontal asymptote if it is nonzero.

Example 13

Write an equation for the rational function graphed here.



Solution:
The graph appears to have horizontal intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors.

The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other.
The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote.

Utilizing this information indicates an function of the form

$$f(x) = a\frac{(x + 2)(x - 3)}{(x + 1)(x - 2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the vertical intercept (0,-2):

$$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$
$$-2 = a \cdot -\frac{6}{4}$$
$$a = -2 \div -\frac{6}{4}$$
$$a = -2 \cdot -\frac{4}{6} = \frac{4}{3}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

Oblique Asymptotes

Earlier we saw graphs of rational functions that had no horizontal asymptote, which occurs when the degree of the numerator is larger than the degree of the denominator. We can, however, describe in more detail the long-run behavior of a rational function.

Example 14

Describe the long-run behavior of $f(x) = \frac{3x^2+2}{x-5}$.

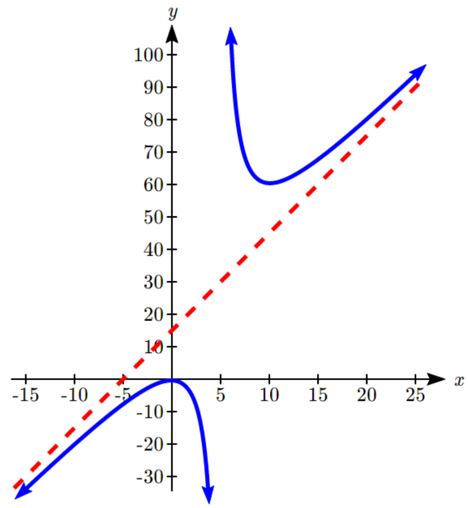
Solution:
Earlier we explored this function when discussing horizontal asymptotes. We found the long-run behavior is $f(x) \approx \frac{3x^2}{x} = 3x$, meaning that $x \to \pm\infty$, $f(x) \to \pm\infty$, respectively, and there is no horizontal asymptote.

If we were to do polynomial long division, we could get a better understanding of the behavior as $x \to \pm\infty$.

$$\begin{array}{r}
3x + 15 \\
x-5\overline{)3x^2 + 0x + 2} \\
-(3x^2 - 15x) \\
\hline
15x + 2 \\
-(15x - 75) \\
\hline
77
\end{array}$$

This means $f(x) = \frac{3x^2+2}{x-5}$ can be rewritten as $f(x) = 3x + 15 + \frac{77}{x-5}$.

As $x \to \pm\infty$, the term $\frac{77}{x-5}$ will become very small and approach zero, becoming insignificant. The remaining $3x + 15$ then describes the long-run behavior of the function: as $x \to \pm\infty$, $f(x) \to 3x + 15$.



We call this equation $y = 3x + 15$ the **oblique asymptote** of the function.

In the graph, you can see how the function is approaching the line on the far left and far right.

Oblique Asymptotes

To explore the long-run behavior of a rational function,

1) Perform polynomial long division (or synthetic division)

2) The quotient will describe the asymptotic behavior of the function

When this result is a line, we call it an **oblique asymptote**, or **slant asymptote**.

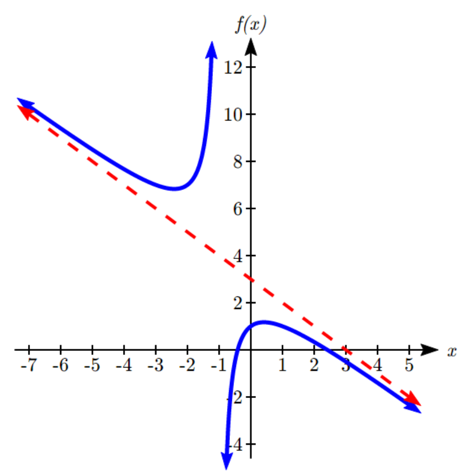
Example 15

Find the oblique asymptote of $f(x) = \frac{-x^2+2x+1}{x+1}$

Solution:

Performing polynomial long division:

$$\begin{array}{r} -x + 3 \\ x+1 \overline{) -x^2 + 2x + 1} \\ \underline{-(-x^2 - x)} \\ 3x \;\; + 1 \\ \underline{-(3x + 3)} \\ -2 \end{array}$$



This allows us to rewrite the function as $f(x) = -x + 3 - \frac{2}{x+1}$.

The quotient, $y = -x + 3$, is the oblique asymptote of $f(x)$. Just like functions we saw earlier approached their horizontal asymptote in the long run, this function will approach this oblique asymptote in the long run.

Try it Now

Find the oblique asymptote of $f(x) = \frac{1+7x-2x^2}{x-2}$

While we primarily concern ourselves with oblique asymptotes, this same approach can describe other asymptotic behavior.

Example 16

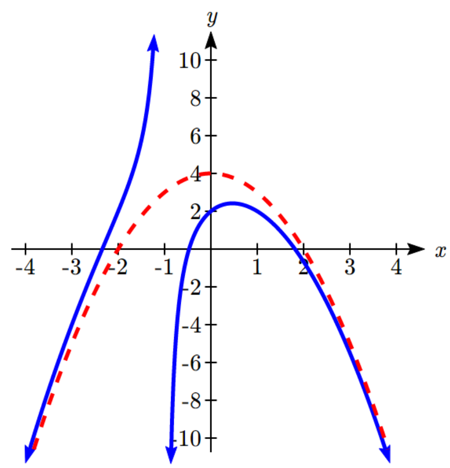
Describe the long-run shape of $f(x) = \frac{-x^3-x^2+4x+2}{x+1}$

Solution:

We could rewrite this using long division as $f(x) = -x^2 + 4 + \frac{2}{x+1}$.

Just looking at the quotient gives us the asymptote, $y = -x^2 + 4$.

This suggests that in the long run, the function will behave like a downwards opening parabola. The function will also have a vertical asymptote at $x = -1$.



Important Topics of this Section
Inversely proportional; Reciprocal toolkit function
Inversely proportional to the square; Reciprocal squared toolkit function
Horizontal Asymptotes
Vertical Asymptotes
Rational Functions
Finding intercepts, asymptotes, and holes.
Given equation sketch the graph
Identifying a function from its graph
Oblique Asymptotes

Section 2.11 Exercises

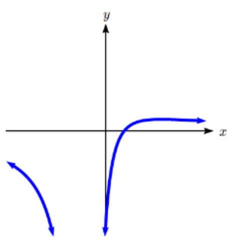
Conceptual Questions

1. Describe how to find vertical asymptotes of a rational function using its equation.
2. Describe how to find horizontal asymptotes of a rational function using its equation.
3. Describe how to find intercepts of a rational function using its equation.
4. Describe how to sketch a graph of a rational function.

Practice Problems

Match each equation form with one of the graphs.

1. $f(x) = \frac{x-A}{x-B}$
2. $g(x) = \frac{(x-A)^2}{x-B}$
3. $h(x) = \frac{x-A}{(x-B)^2}$
4. $k(x) = \frac{(x-A)^2}{(x-B)^2}$



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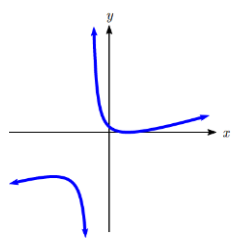

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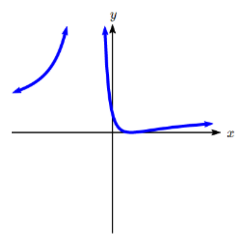

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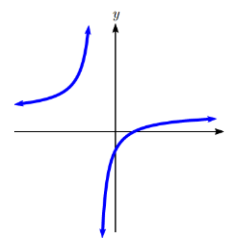


D

For each function, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal asymptote. Use that information to sketch a graph.

5. $p(x) = \frac{2x-3}{x+4}$
6. $q(x) = \frac{x-5}{3x-1}$
7. $s(x) = \frac{4}{(x-2)^2}$
8. $r(x) = \frac{5}{(x+1)^2}$
9. $f(x) = \frac{3x^2-14x-5}{3x^2+8x-16}$
10. $g(x) = \frac{2x^2+7x-15}{3x^2-14x+15}$
11. $a(x) = \frac{x^2+2x-3}{x^2-1}$
12. $b(x) = \frac{x^2-x-6}{x^2-4}$
13. $h(x) = \frac{2x^2+x-1}{x-4}$
14. $k(x) = \frac{2x^2-3x-20}{x-5}$
15. $n(x) = \frac{3x^2+4x-4}{x^3-4x^2}$
16. $m(x) = \frac{5-x}{2x^2+7x+3}$
17. $w(x) = \frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$
18. $z(x) = \frac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

Find where each pair of functions intersect.

19. $f(x) = \frac{4}{x}$ and $g(x) = 1 - \frac{3x-9}{2}$
20. $f(x) = \frac{x+2}{x}$ and $g(x) = \frac{3x+3}{x} - \frac{1}{3}$
21. $f(x) = \frac{1}{4x} - 1$ and $g(x) = \frac{x+3}{4x}$
22. $f(x) = \frac{-2}{5x^2+x}$ and $g(x) = \frac{1}{x}$
23. $f(x) = \frac{1}{3x^2} + \frac{1}{3}$ and $g(x) = \frac{x+3}{3x}$
24. $f(x) = \frac{4}{x^2} - \frac{1}{4}$ and $g(x) = \frac{x^2-8x+16}{x^2}$
25. $f(x) = \frac{1}{x+2} - 1$ and $g(x) = \frac{5}{x^2-3x-10}$
26. $f(x) = \frac{3}{x^2-x}$ and $g(x) = \frac{1}{x^2-x} + 1$

Write an equation for a rational function with the given characteristics.

27. Vertical asymptotes at $x = 5$ and $x = -5$
    x intercepts at $(2,\ 0)$ and $(-1,\ 0)$ *y* intercept at $(0,\ 4)$

28. Vertical asymptotes at $x = -4$ and $x = -1$
    x intercepts at $(1,\ 0)$ and $(5,\ 0)$ *y* intercept at $(0,\ 7)$

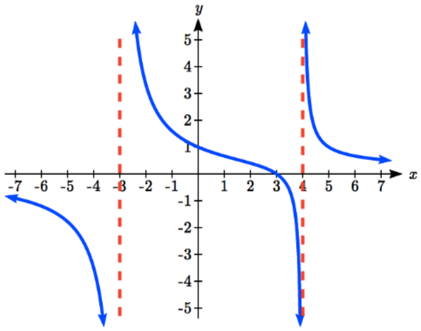
29. Vertical asymptotes at $x = -4$ and $x = -5$
    x intercepts at $(4,\ 0)$ and $(-6,\ 0)$ Horizontal asymptote at $y = 7$

30. Vertical asymptotes at $x = -3$ and $x = 6$
    x intercepts at $(-2,\ 0)$ and $(1,\ 0)$ Horizontal asymptote at $y = -2$

31. Vertical asymptote at $x = -1$
    Double zero at $x = 2$ *y* intercept at $(0,\ 2)$

32. Vertical asymptote at $x = 3$
    Double zero at $x = 1$ *y* intercept at $(0,\ 4)$

Write an equation for the function graphed.



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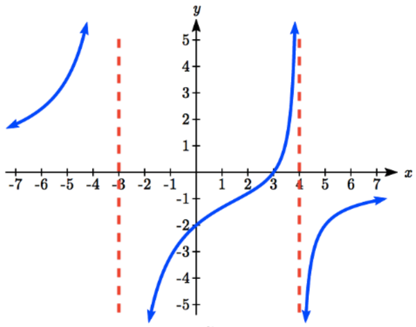

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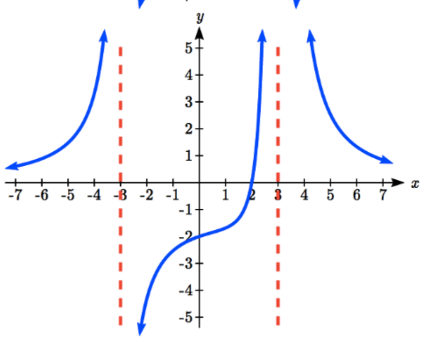

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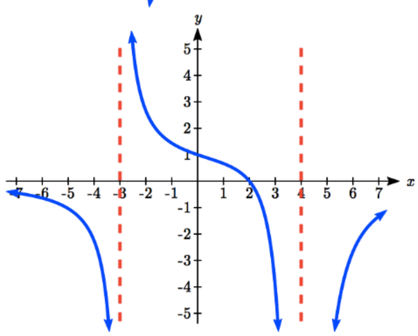

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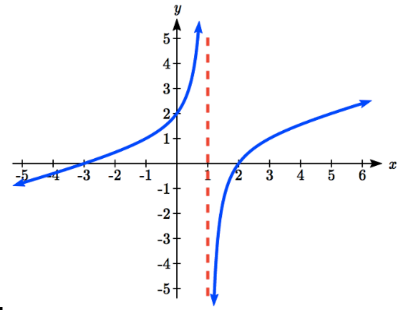

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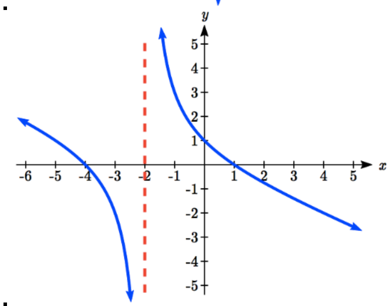

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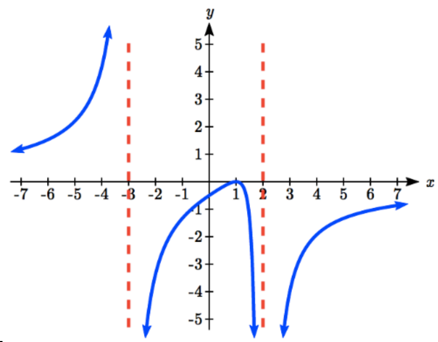

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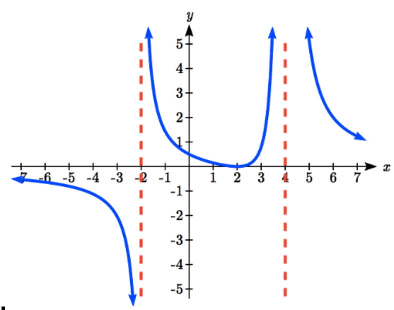
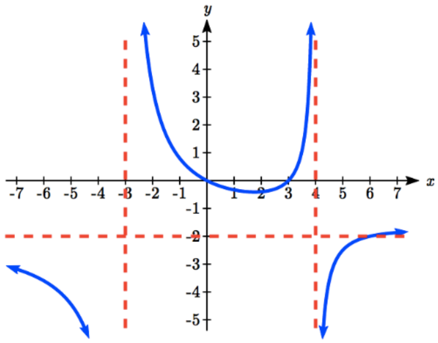
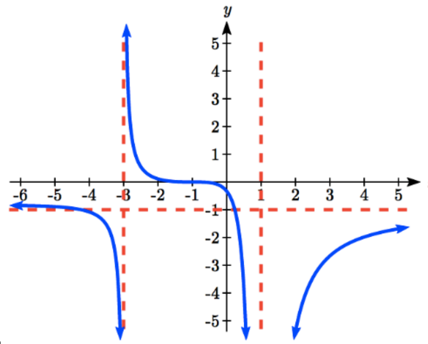
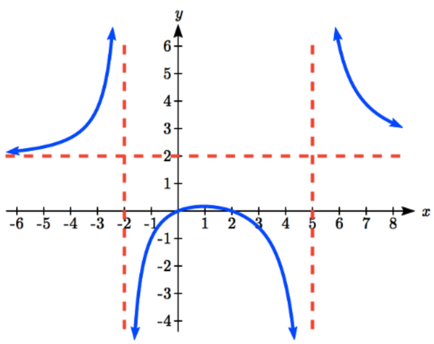


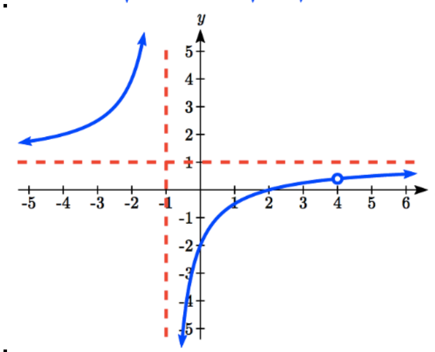
40..

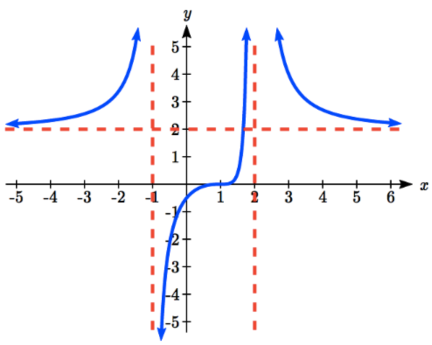
Write an equation for the function graphed.

41. 

44. 

42. 

45. 

43. 

Find the oblique asymptote of each function.

46. $g(x) = \frac{2x^2+3x-8}{x-1}$

47. $f(x) = \frac{3x^2+4x}{x+2}$

48. $k(x) = \frac{5+x-2x^2}{2x+1}$

49. $h(x) = \frac{x^2-x-3}{2x-6}$

50. $n(x) = \frac{2x^3+x^2+x}{x^2+x+1}$

51. $m(x) = \frac{-2x^3+x^2-6x+7}{x^2+3}$

52. A scientist has a beaker containing 20 mL of a solution containing 20% acid. To dilute this, she adds pure water.
    e. Write an equation for the concentration in the beaker after adding *n* mL of water.
    f. Find the concentration if 10 mL of water has been added.
    g. How many mL of water must be added to obtain a 4% solution?
    h. What is the behavior as $n \to \infty$, and what is the physical significance of this?

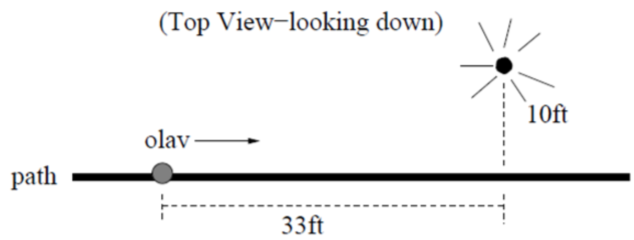
53. A scientist has a beaker containing 30 mL of a solution containing 3 grams of potassium hydroxide. To this, she mixes a solution containing 8 milligrams per mL of potassium hydroxide.
    a. Write an equation for the concentration in the tank after adding *n* mL of the second solution.
    b. Find the concentration if 10 mL of the second solution has been added.
    c. How many mL of water must be added to obtain a 50 mg/mL solution?
    d. What is the behavior as $n \to \infty$, and what is the physical significance of this?

54. Oscar is hunting magnetic fields with his gauss meter, a device for measuring the strength and polarity of magnetic fields. The reading on the meter will increase as Oscar gets closer to a magnet. Oscar is in a long hallway at the end of which is a room containing an extremely strong magnet. When he is far down the hallway from the room, the meter reads a level of 0.2. He then walks down the hallway and enters the room. When he has gone 6 feet into the room, the meter reads 2.3. Eight feet into the room, the meter reads 4.4. [UW]
    e. Give a rational model of form $m(x) = \frac{ax+b}{cx+d}$ relating the meter reading $m(x)$ to how many feet *x* Oscar has gone into the room.
    f. How far must he go for the meter to reach 10? 100?
    g. Considering your function from part (a) and the results of part (b), how far into the room do you think the magnet is?

55. The more you study for a certain exam, the better your performance on it. If you study for 10 hours, your score will be 65%. If you study for 20 hours, your score will be 95%. You can get as close as you want to a perfect score just by studying long enough. Assume your percentage score, $p(n)$, is a function of

the number of hours, n, that you study in the form $p(n) = \frac{an+b}{cn+d}$. If you want a score of 80%, how long do you need to study? [UW]

56. A street light is 10 feet north of a straight bike path that runs east-west. Olav is bicycling down the path at a rate of 15 miles per hour. At noon, Olav is 33 feet west of the point on the bike path closest to the street light. (See the picture). The relationship between the intensity C of light (in candlepower) and the distance d (in feet) from the light source is given by $C = \frac{k}{d^2}$, where k is a constant depending on the light source. [UW]



h. From 20 feet away, the street light has an intensity of 1 candle. What is k?
i. Find a function which gives the intensity of the light shining on Olav as a function of time, in seconds.
j. When will the light on Olav have maximum intensity?
k. When will the intensity of the light be 2 candles?

Chapter 3: Exponential and Logarithmic Functions

Photo by Mihály Köles on Unsplash

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Section 3.1: Exponential Functions and Their Graphs

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year[12]. We might ask if we can find a formula to model the population, *P*, as a function of time, *t*, in years after 2008, if the population continues to grow at this rate.

In linear growth, we had a constant rate of change – a constant *number* that the output increased for each increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by three each time the input increases by one. This population scenario is different – we have a *percent* rate of change rather than a constant number of people as our rate of change.

To see the significance of this difference consider these two companies:

Company *A* has 100 stores, and expands by opening 50 new stores a year

Company *B* has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

Looking at a few years of growth for these companies:

Year	Stores, company *A*		Stores, company *B*
0	100	Starting with 100 each	100
1	100 + 50 = 150	They both grow by 50 stores in the first year.	100 + 50% of 100 100 + 0.50(100) = 150
2	150 + 50 = 200	Store A grows by 50, Store B grows by 75	150 + 50% of 150 150 + 0.50(150) = 225
3	200 + 50 = 250	Store A grows by 50, Store B grows by 112.5	225 + 50% of 225 225 + 0.50(225) = 337.5

[12] World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 20, 2010

Notice that with the percent growth, each year the company grows by 50% of the current year's total, so as the company grows larger, the number of stores added in a year grows as well.

To try to simplify the calculations, notice that after 1 year the number of stores for company *B* was:
$100 + 0.50(100)$ or equivalently by factoring
$100(1 + 0.50) = 150$

We can think of this as "the new number of stores is the original 100% plus another 50%".

After 2 years, the number of stores was:
$150 + 0.50(150)$ or equivalently by factoring
$150(1 + 0.50)$

Now recall the 150 came from 100(1+0.50). Substituting that,
$100(1 + 0.50)(1 + 0.50) = 100(1 + 0.50)^2 = 225$

After 3 years, the number of stores was:
$225 + 0.50(225)$ or equivalently by factoring
$225(1 + 0.50)$

Now recall the 225 came from $100(1 + 0.50)^2$. Substituting that,
$100(1 + 0.50)^2(1 + 0.50) = 100(1 + 0.50)^3 = 337.5$

From this, we can generalize, noticing that to show a 50% increase, each year we multiply by a factor of (1+0.50), so after *n* years, our equation would be

$$B(n) = 100(1 + 0.50)^n$$

In this equation, the 100 represented the initial quantity, and the 0.50 was the percent growth rate. Generalizing further, we arrive at the general form of exponential functions.

Exponential Function

An **exponential growth or decay function** is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form
$f(x) = a(1 + r)^x$ or $f(x) = an^x$ where n = 1+r

Where
a is the initial or starting value of the function
r is the percent growth or decay rate, written as a decimal
n is the growth factor or growth multiplier. Since powers of negative numbers behave strangely, we limit n to positive values.

To see more clearly the difference between exponential and linear growth, compare the two tables and graphs below, which illustrate the growth of company *A* and *B* described above over a longer time frame if the growth patterns were to continue.

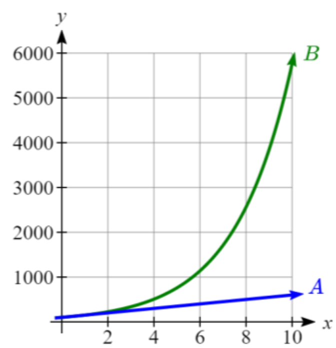
years	Company *A*	Company *B*
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767

Example 1

Write an exponential function for India's population, and use it to predict the population in 2020.

Solution:

At the beginning of the chapter we were given India's population of 1.14 billion in the year 2008 and a percent growth rate of 1.34%. Using 2008 as our starting time ($t = 0$), our initial population will be 1.14 billion. Since the percent growth rate was 1.34%, our value for r is 0.0134.



Using the basic formula for exponential growth $f(x) = a(1 + r)^x$ we can write the formula, $f(t) = 1.14(1 + 0.0134)^t$

To estimate the population in 2020, we evaluate the function at $t = 12$, since 2020 is 12 years after 2008.
$f(12) = 1.14(1 + 0.0134)^{12} \approx 1.337$billion people in 2020

For comparison, the actual population of India in 2020 was about 1.380 billion.

Try it Now

Given the three statements below, identify which represent exponential functions.

A. The cost of living allowance for state employees increases salaries by 3.1% each year.
B. State employees can expect a \$300 raise each year they work for the state.
C. Tuition costs have increased by 2.8% each year for the last 3 years.

Graphs

Like with linear functions, the graph of an exponential function is determined by the values for the parameters in the function's formula.

To get a sense for the behavior of exponentials, let us begin by looking more closely at the function $f(x) = 2^x$. Listing a table of values for this function:

x	-3	-2	-1	0	1	2	3
f(x)	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notice that:

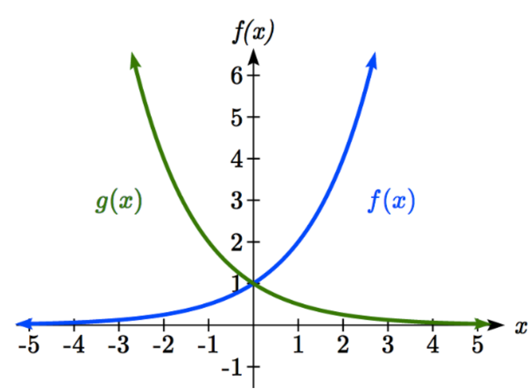
1) This function is positive for all values of *x*.
2) As *x* increases, the function grows faster and faster (the rate of change increases).
3) As *x* decreases, the function values grow smaller, approaching zero.
4) This is an example of exponential growth.

Looking at the function $g(x) = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3
g(x)	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Note this function is also positive for all values of *x*, but in this case grows as *x* decreases, and decreases towards zero as *x* increases. This is an example of exponential decay. You may notice from the table that this function appears to be the horizontal reflection of the $f(x) = 2^x$ table. This is in fact the case:

$$f(-x) = 2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x = g(x)$$



Looking at the graphs also confirms this relationship.

Consider a function of the form $f(x) = an^x$. Since *a* is the function value at an input of zero, *a* will give us the vertical intercept of the graph, or the initial value.

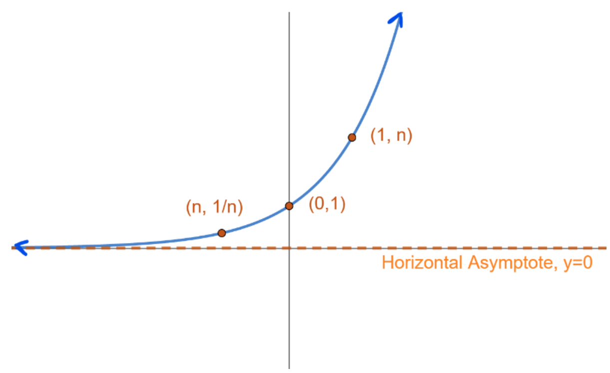
From the graphs above, we can see that an exponential graph will have a horizontal asymptote on one side of the graph, and can either increase or decrease, depending upon the growth factor. This horizontal asymptote will also help us determine the long run behavior and is easy to determine from the graph.

The graph will grow when the growth rate is positive, which will make the growth factor n larger than one. When it's negative, the growth factor will be less than one.

The Toolkit Exponential Function

Given a function $f(x) = n^x$, when $n > 1$, we can create the following table and graph:

x	$f(x)$
-1	$\frac{1}{n}$
0	1
1	n
Horizontal Asymptote	0



Features of the Toolkit Exponential Function

For $f(x) = n^x$ when $n > 1$

Intercepts: y-intercept of $(0,1)$; no x-intercept

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Increasing on $(-\infty, \infty)$

Concave up on $(-\infty, \infty)$

End Behavior: As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to 0$ (there is a horizontal asymptote at $y = 0$).

No symmetry

No maximum or minimum value

Notice that this toolkit function will change slightly as the base of the function, n, changes. As long as $n > 1$, though, most of the behavior of the function remains the same.

Example 2

Graph $g(x) = 4(3)^x + 1$. Then describe the features of the function.

Solution:

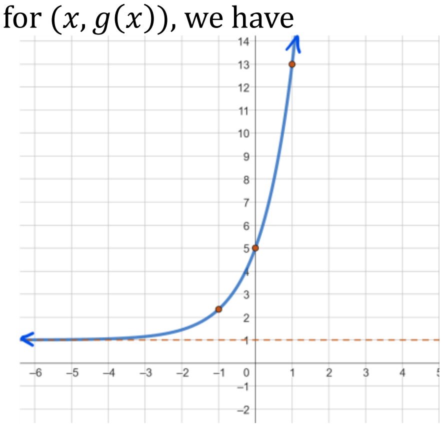
We can still use our transformation approach, we just need to be careful not to get distracted by the 3! That 3 is the base of our exponential toolkit function, so our initial table will be for $f(x) = 3^x$:

x	$f(x) = 3^x$
-1	$1/3$
0	1
1	3
HA	0

The next thing we need to be aware of is that the "inside" of an exponential function is the exponent! That's where we input each of our x values. That means, from a transformational point of view, we are looking for $g(x) = af(bx + c) + d = a3^{bx+c} + d$. In this case, then $a = 4$ and $d = 1$, so we'll see a vertical stretch by a factor of 4 and a vertical shift up 1.

x	$f(x) \to$	$y = 4 \cdot f(x) \to$	$g(x) = y + 1$
-1	$1/3$	$4/3$	$7/3$
0	1	4	5
1	3	12	13
HA	0	0	1

Graphing our final values for $(x, g(x))$, we have



The last row of the table seems a little funny at first. If we follow our transformations procedure, that last row will tell us the height for our horizontal asymptote.

Now, for the features. We see the following:

- Intercept: y-intercept at $(0,5)$
- Domain: $(-\infty, \infty)$
- Range: $(1, \infty)$
- End behavior: As $x \to \infty, g(x) \to \infty$ and as $x \to -\infty, g(x) \to 1$
- The graph is always concave up, and always increasing

Example 3

Graph $g(x) = 4\left(\frac{1}{3}\right)^x + 1$. Then describe the features of the function.

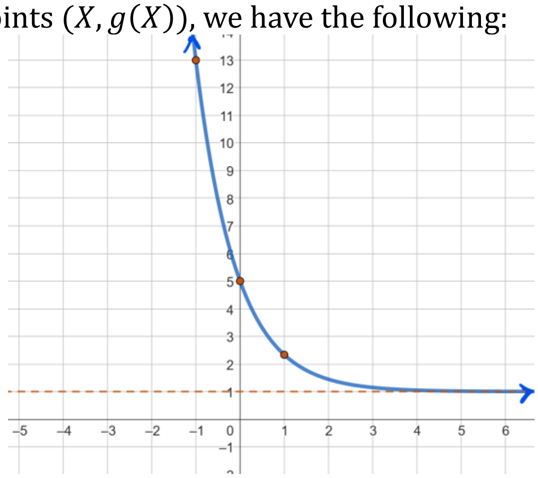
Solution:

There are a couple of ways we can think this one through. We could build a toolkit table based off of $f(x) = \left(\frac{1}{3}\right)^x$. We could also think of the function as $f(x) = 4(3^{-1})^x + 1 = 4(3)^{-x} + 1$. We'll try each approach here.

If we want to graph $g(x) = 4\left(\frac{1}{3}\right)^x + 1 = 4(3)^{-x} + 1$, we'll use the same parent table as the previous example. Now, for $g(x) = a3^{bx+c} + d$, we have $a = 4$, $b = -1$, and $d = 1$. So in addition to our vertical transformations, we'll have a reflection about the y-axis. Here's what that looks like in a table.

$X = -1 \cdot x$	x	$f(x) = 3^x$	$y = a \cdot f(x)$	$g(X) = y + 1$
1	-1	1/3	4/3	7/3
0	0	1	4	5
-1	1	3	12	13
HA	HA	0	0	1

When we graph the points $(X, g(X))$, we have the following:



Alternatively, we could have chosen to start with parent function $f(x) = \left(\frac{1}{3}\right)^x$. That has a table as follows:

x	$f(x)$
-1	$\left(\frac{1}{3}\right)^{-1} = 3$
0	$\left(\frac{1}{3}\right)^{0} = 1$
1	$\left(\frac{1}{3}\right)^{1} = \frac{1}{3}$
HA	0

Now, when we consider our transformations, we have $g(x) = af(bx + c) + d = a\left(\frac{1}{3}\right)^{bx+c} + d$, so we only need to consider the a and d values. When we apply the vertical stretch of 4 and the vertical shift of 1, we should end up with the same transformed points and graph as before.

Finally, we describe the features:

- Intercept: y-intercept at $(0,5)$
- Domain: $(-\infty, \infty)$
- Range: $(1, \infty)$
- End behavior: As $x \to \infty, g(x) \to 1$ and as $x \to -\infty, g(x) \to \infty$
- The graph is always concave up, and always decreasing

Try it Now:

Graph each function, using transformations. Then describe the features.

a. $g(x) = -4^{x+1}$

b. $h(x) = -\left(\frac{1}{4}\right)^{x+1}$

The last series of examples shows that the behavior of an exponential function will be very different when the base is greater than 1, versus when the base is a number between 0 and 1 (a small fraction). We'll summarize what we saw here.

Graphical Features of Exponential Functions

Graphically, in the function $f(x) = a\,n^x$

a is the vertical intercept of the graph

n determines the rate at which the graph grows. When a is positive,

the function will increase if $n > 1$

the function will decrease if $0 < n < 1$

The graph will have a horizontal asymptote at $y = 0$

The graph will be concave up if $a > 0$; concave down if $a < 0$.

The domain of the function is all real numbers

The range of the function is $(0, \infty)$

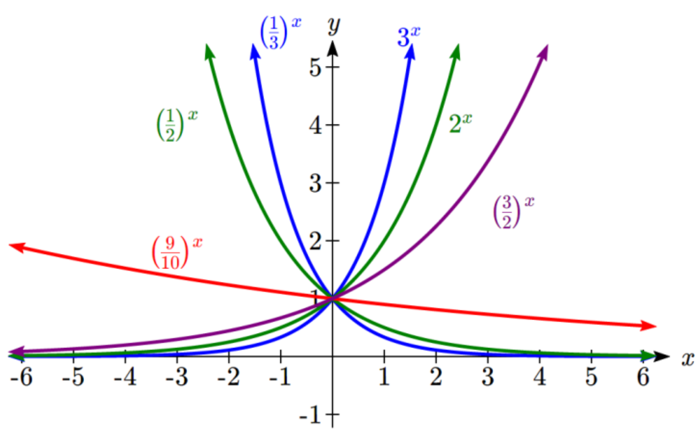
The value n will determine the function's long run behavior:

If $\mathrm{n} > 1$, as $x \to \infty, f(x) \to \infty$, and as $x \to -\infty, f(x) \to 0$.

If $0 < \mathrm{n} < 1$, as $x \to \infty, f(x) \to 0$, and as $x \to -\infty, f(x) \to \infty$.

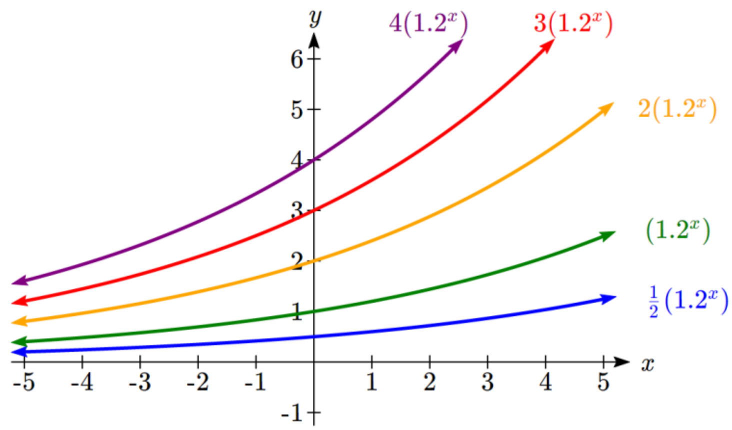
When sketching the graph of an exponential function, it can be helpful to remember that the graph will pass through the points (0, *a*) and (1, *an*).

To get a better feeling for the effect of *a* and *b* on the graph, examine the sets of graphs below. The first set shows various graphs, where *a* remains the same and we only change the value for *n*.



Notice that the closer the value of *n* is to 1, the less steep the graph will be.

In the next set of graphs, *a* is altered and our value for *n* remains the same.

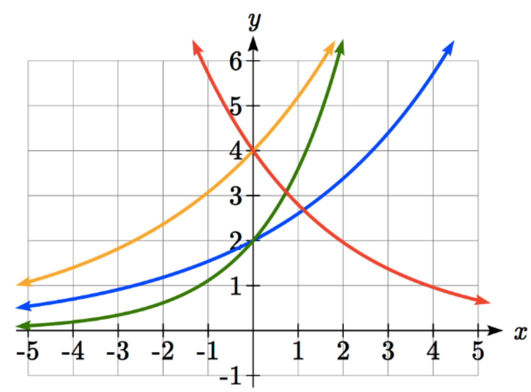


Notice that changing the value for *a* changes the vertical intercept. Since *a* is multiplying the n^x term, *a* acts as a vertical stretch factor, not as a shift. Notice also that the long run behavior for all of these functions is the same because the growth factor did not change and none of these *a* values introduced a vertical flip.

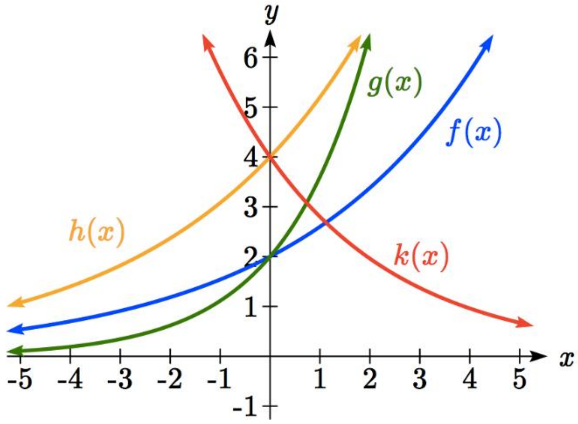
Example 2

Match each equation with its graph.

$$f(x) = 2(1.3)^x$$
$$g(x) = 2(1.8)^x$$
$$h(x) = 4(1.3)^x$$
$$k(x) = 4(0.7)^x$$



The graph of *k(x)* is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph. The graph of *h(x)* can be identified as the only growing exponential function with a vertical intercept at (0,4). The graphs of *f(x)* and *g(x)* both have a vertical intercept at (0,2), but since *g(x)* has a larger growth factor, we can identify it as the graph increasing faster.



Try it Now

Graph the following functions on the same axis:

$$f(x) = 2^x$$
$$g(x) = 2(2)^x$$
$$h(x) = 2\left(\frac{1}{2}\right)^x$$

Rewriting Transformations of Exponential Graphs

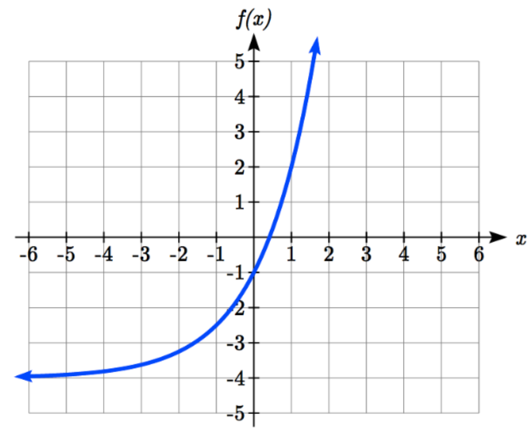
While exponential functions can be transformed following the same rules as any function, there are a few interesting features of transformations that can be identified. The first was seen at the beginning of the section – that a horizontal reflection is equivalent to a change in the growth factor. Likewise, since *a* is itself a stretch factor, a vertical stretch of an exponential corresponds with a change in the initial value of the function.

Next consider the effect of a horizontal shift on an exponential function. Shifting the function $f(x) = 3(2)^x$ four units to the left would give $f(x+4) = 3(2)^{x+4}$. Employing exponent rules, we could rewrite this:

$$f(x+4) = 3(2)^{x+4} = 3(2)^x(2^4) = 48(2)^x$$

Interestingly, it turns out that a horizontal shift of an exponential function corresponds with a change in initial value of the function.

Lastly, consider the effect of a vertical shift on an exponential function. Shifting $f(x) = 3(2)^x$ down 4 units would give the equation $f(x) = 3(2)^x - 4$.



Graphing that, notice it is substantially different than the basic exponential graph. Unlike a basic exponential, this graph does not have a horizontal asymptote at *y* = 0; due to the vertical shift, the horizontal asymptote has also shifted to *y* = -4. We can see that as $x \to \infty, f(x) \to \infty$, and as $x \to -\infty, f(x) \to -4$.

We have determined that a vertical shift is the only transformation of an exponential function that changes the graph in a way that cannot be achieved by altering the parameters *a* and *n* in the basic exponential function $f(x) = an^x$.

Transformations of Exponentials

Any transformed exponential can be written in the form

$$f(x) = an^x + c$$

where $y = c$ is the horizontal asymptote.

Finding Equations of Exponential Functions

In the previous examples, we were able to write equations for exponential functions since we knew the initial quantity and the growth rate. If we do not know the growth rate, but instead know only some input and output pairs of values, we can still construct an exponential function.

Example 4

In 2009, 80 deer were reintroduced into a wildlife refuge area from which the population had previously been hunted to elimination. By 2015, the population had grown to 180 deer. If this population grows exponentially, find a formula for the function.

Solution:

By defining our input variable to be t, years after 2009, the information listed can be written as two input-output pairs: (0,80) and (6,180). Notice that by choosing our input variable to be measured as years after the first year value provided, we have effectively "given" ourselves the initial value for the function: $a = 80$. This gives us an equation of the form

$f(t) = 80n^t$.

Substituting in our second input-output pair allows us to solve for n:
$180 = 80n^6$ Divide by 80
$n^6 = \frac{180}{80} = \frac{9}{4}$ Take the 6th root of both sides.
$n = \sqrt[6]{\frac{9}{4}} = 1.1447$

This gives us our equation for the population:

$$f(t) = 80(1.1447)^t$$

Recall that since $n = 1+r$, we can interpret this to mean that the population growth rate is $r = 0.1447$, and so the population is growing by about 14.47% each year.

In this example, you could also have used (9/4)^(1/6) to evaluate the 6th root if your calculator doesn't have an *n*th root button.

In the previous example, we chose to use the $f(x) = ab^x$ form of the exponential function rather than the $f(x) = a(1+r)^x$ form. This choice was entirely arbitrary – either form would be fine to use.

When finding equations, the value for *b* or *r* will usually have to be rounded to be written easily. To preserve accuracy, it is important to not over-round these values. Typically, you want to be sure to preserve at least 3 significant digits in the growth rate. For example, if your value for *b* was 1.00317643, you would want to round this no further than to 1.00318.

In the previous example, we were able to "give" ourselves the initial value by clever definition of our input variable. Next, we consider a situation where we can't do this.

Example 5

Find a formula for an exponential function passing through the points (-2,6) and (2,1).

Solution:
Since we don't have the initial value, we will take a general approach that will work for any function form with unknown parameters: we will substitute in both given input-output pairs in the function form $f(x) = an^x$ and solve for the unknown values, a and n.
Substituting in (-2, 6) gives $6 = an^{-2}$
Substituting in (2, 1) gives $1 = an^2$

We now solve these as a system of equations. To do so, we could try a substitution approach, solving one equation for a variable, then substituting that expression into the second equation.

Solving $6 = an^{-2}$ for *a*:

$$a = \frac{6}{n^{-2}} = 6n^2$$

In the second equation, $1 = ab^2$, we substitute the expression above for *a*:

$$1 = (6n^2)n^2$$
$$1 = 6n^4$$
$$\frac{1}{6} = n^4$$
$$n = \sqrt[4]{\frac{1}{6}} \approx 0.6389$$

Going back to the equation $a = 6n^2$ lets us find *a*:

$$a = 6n^2 = 6(0.6389)^2 = 2.4492$$

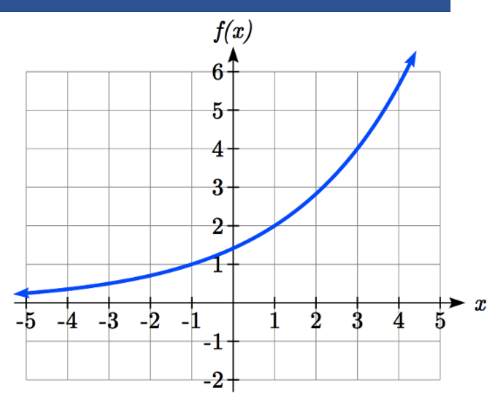
Putting this together gives the equation $f(x) = 2.4492(0.6389)^x$

Try it Now

Given the two points (1, 3) and (2, 4.5) find the equation of an exponential function that passes through these two points.

Example 6

Find an equation for the exponential function graphed.



Solution:

The initial value for the function is not clear in this graph, so we will instead work using two clearer points. There are three clear points: (-1, 1), (1, 2), and (3, 4). As we saw in the last example, two points are sufficient to find the equation for a standard exponential, so we will use the latter two points.

Substituting in (1,2) gives $2 = an^1$
Substituting in (3,4) gives $4 = an^3$

Solving the first equation for *a* gives $a = \frac{2}{n}$.

Substituting this expression for *a* into the second equation:

$$4 = an^3$$

$$4 = \frac{2}{n}n^3 = \frac{2n^3}{n}$$
Simplify the right-hand side
$$4 = 2n^2$$
$$2 = n^2$$
$$n = \pm\sqrt{2}$$

Since we restrict ourselves to positive values of n, we will use $n = \sqrt{2}$. We can then go back and find a:

$$a = \frac{2}{n} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

This gives us a final equation of $f(x) = \sqrt{2}(\sqrt{2})^x$.

Compound Interest

In the bank certificate of deposit (CD) example earlier in the section, we encountered compound interest. Typically bank accounts and other savings instruments in which earnings are reinvested, such as mutual funds and retirement accounts, utilize compound interest. The term *compounding* comes from the behavior that interest is earned not on the original value, but on the accumulated value of the account.

In the example from earlier, the interest was compounded monthly, so we took the annual interest rate, usually called the **nominal rate** or **annual percentage rate (APR)** and divided by 12, the number of compounds in a year, to find the monthly interest. The exponent was then measured in months.

Generalizing this, we can form a general formula for compound interest. If the APR is written in decimal form as *r*, and there are *k* compounding periods per year, then the interest per compounding period will be *r/k*. Likewise, if we are interested in the value after *t* years, then there will be *kt* compounding periods in that time.

Compound Interest Formula

Compound Interest can be calculated using the formula

$$A(t) = a\left(1 + \frac{r}{k}\right)^{kt}$$

Where

A(t) is the account value

t is measured in years

a is the starting amount of the account, often called the principal

r is the annual percentage rate (APR), also called the nominal rate

k is the number of compounding periods in one year

Example 7

If you invest \$3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution:

Since we are starting with \$3000, $a = 3000$
Our interest rate is 3%, so $r = 0.03$
Since we are compounding quarterly, we are compounding 4 times per year, so $k = 4$
We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{4(10)} = \$4045.05$$

The account will be worth \$4045.05 in 10 years.

Example 8

A 529 plan is a college savings plan in which a relative can invest money to pay for a child's later college tuition, and the account grows tax free. If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to \$40,000 over 18 years, and she believes the account will earn 6% compounded semi-annually (twice a year), how much will Lily need to invest in the account now?

Solution:

Since the account is earning 6%, $r = 0.06$
Since interest is compounded twice a year, $k = 2$

In this problem, we don't know how much we are starting with, so we will be solving for a, the initial amount needed. We do know we want the end amount to be \$40,000, so we will be looking for the value of a so that $A(18) = 40{,}000$.

$$40{,}000 = A(18) = a\left(1 + \frac{0.06}{2}\right)^{2(18)}$$
$$40{,}000 = a(2.8983)$$
$$a = \frac{40{,}000}{2.8983} \approx \$13{,}801$$

Lily will need to invest \$13,801 to have \$40,000 in 18 years.

Try it now

Recalculate example 2 from above with quarterly compounding.

Because of compounding throughout the year, with compound interest the actual increase in a year is *more* than the annual percentage rate. If \$1,000 were invested at 10%, the table below shows the value after 1 year at different compounding frequencies:

Frequency	Value after 1 year
Annually	\$1100
Semiannually	\$1102.50
Quarterly	\$1103.81
Monthly	\$1104.71
Daily	\$1105.16

If we were to compute the actual percentage increase for the daily compounding, there was an increase of \$105.16 from an original amount of \$1,000, for a percentage increase of $\frac{105.16}{1000} = 0.10516$= 10.516% increase. This quantity is called the **annual percentage yield (APY)**.

Notice that given any starting amount, the amount after 1 year would be $A(1) = a\left(1 + \frac{r}{k}\right)^k$. To find the total change, we would subtract the original amount, then to find the percentage change we would divide that by the original amount:

$$\frac{a\left(1 + \frac{r}{k}\right)^k - a}{a} = \left(1 + \frac{r}{k}\right)^k - 1$$

Annual Percentage Yield

The **annual percentage yield** is the actual percent a quantity increases in one year. It can be calculated as

$$APY = \left(1+\frac{r}{k}\right)^k - 1$$

This is equivalent to finding the value of $1 after 1 year, and subtracting the original dollar.

Example 9

Bank A offers an account paying 1.2% compounded quarterly. Bank B offers an account paying 1.1% compounded monthly. Which is offering a better rate?

We can compare these rates using the annual percentage yield – the actual percent increase in a year.

Bank A: $APY = \left(1+\frac{0.012}{4}\right)^4 - 1 = 0.012054 = 1.2054\%$

Bank B: $APY = \left(1+\frac{0.011}{12}\right)^{12} - 1 = 0.011056 = 1.1056\%$

Bank B's monthly compounding is not enough to catch up with Bank A's better APR. Bank A offers a better rate.

A Limit to Compounding

As we saw earlier, the amount we earn increases as we increase the compounding frequency. The table, though, shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to believe that although increasing the frequency of compounding will increase our result, there is an upper limit to this process.

To see this, let us examine the value of $1 invested at 100% interest for 1 year.

Frequency	**Value**
Annual	$2
Quarterly	$2.441406
Monthly	$2.613035
Daily	$2.714567
Hourly	$2.718127
Once per minute	$2.718279
Once per second	$2.718282

These values do indeed appear to be approaching an upper limit. This value ends up being so important that it gets represented by its own letter, much like how π represents a number.

Euler's Number: *e*

e is the letter used to represent the value that $\left(1+\frac{1}{k}\right)^k$ approaches as k gets big.

$$e \approx 2.718282$$

Because *e* is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of *e*, usually labeled e^x. Some computer software instead defines a function *exp(x)*, where *exp(x)* = e^x.

Because *e* arises when the time between compounds becomes very small, *e* allows us to define **continuous growth** and allows us to define a new toolkit function, $f(x) = e^x$.

Continuous Growth Formula

Continuous Growth can be calculated using the formula

$$f(x) = ae^{rx}$$

where

a is the starting amount

r is the continuous growth rate

This type of equation is commonly used when describing quantities that change more or less continuously, like chemical reactions, growth of large populations, and radioactive decay.

Example 10

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?

Solution:

Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative: r = -0.173

$f(3) = 100e^{-0.173(3)} \approx 59.512$mg of Radon-222 will remain.

Try it Now

Interpret the following: $S(t) = 20e^{0.12t}$ if *S(t)* represents the growth of a substance in grams, and time is measured in days.

Continuous growth is also often applied to compound interest, allowing us to talk about continuous compounding.

Example 11

If $1000 is invested in an account earning 10% compounded continuously, find the value after 1 year.

Solution:
Here, the continuous growth rate is 10%, so $r = 0.10$. We start with $1000, so $a = 1000$.
To find the value after 1 year,

$$f(1) = 1000e^{0.10(1)} \approx \$1105.17$$

Notice this is a $105.17 increase for the year. As a percent increase, this is $\frac{105.17}{1000} = 0.10517 = 10.517\%$ increase over the original $1000.

Notice that this value is slightly larger than the amount generated by daily compounding in the table computed earlier.

The continuous growth rate is like the nominal growth rate (or APR) – it reflects the growth rate before compounding takes effect. This is different than the annual growth rate used in the formula $f(x) = a(1 + r)^x$, which is like the annual percentage yield – it reflects the *actual* amount the output grows in a year.

While the continuous growth rate in the example above was 10%, the actual annual yield was 10.517%. This means we could write two different looking but equivalent formulas for this account's growth:

$f(t) = 1000e^{0.10t}$ using the 10% continuous growth rate
$f(t) = 1000(1.10517)^t$ using the 10.517% actual annual yield rate.

Important Topics of this Section

Percent growth
Exponential functions
- Finding formulas
- Interpreting equations
- Graphs

Exponential Growth & Decay
Compound interest
Annual Percent Yield
Continuous Growth

Section 3.1 Exercises

Conceptual Questions

1. Why do exponential functions have a horizontal asymptote?
2. How do the graphs of 2^x and $\left(\frac{1}{2}\right)^x$ differ? How are they similar?
3. What does each part of the formula $f(t) = a(1 + r)^t$ mean?
4. What does each part of the formula $A(t) = a\left(1 + \frac{r}{k}\right)^{kt}$ mean?
5. Where does the number e come from? Why is it important?

Practice Problems

For each table below, could the table represent a function that is linear, exponential, or neither?

1.

x	1	2	3	4
f(x)	70	40	10	-20

2.

x	1	2	3	4
g(x)	40	32	26	22

3.

x	1	2	3	4
h(x)	70	49	34.3	24.01

4.

x	1	2	3	4
k(x)	90	80	70	60

5.

x	1	2	3	4
m(x)	80	61	42.9	25.61

6.

x	1	2	3	4
n(x)	90	81	72.9	65.61

7. A population numbers 11,000 organisms initially and grows by 8.5% each year. Write an exponential model for the population.

8. A population is currently 6,000 and has been increasing by 1.2% each day. Write an exponential model for the population.

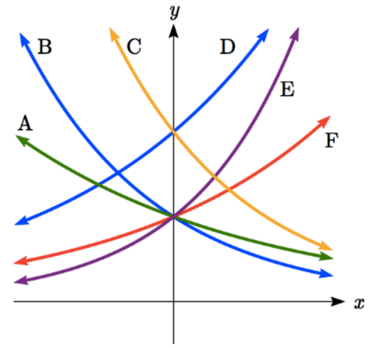
9. The fox population in a certain region has an annual growth rate of 9 percent per year. It is estimated that the population in the year 2010 was 23,900. Estimate the fox population in the year 2018.

10. The amount of area covered by blackberry bushes in a park has been growing by 12% each year. It is estimated that the area covered in 2009 was 4,500 square feet. Estimate the area that will be covered in 2020.

11. A vehicle purchased for $32,500 depreciates at a constant rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.

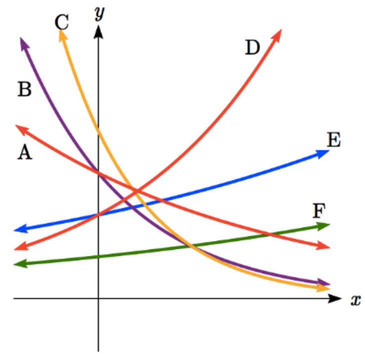
12. A business purchases $125,000 of office furniture which depreciates at a constant rate of 12% each year. Find the residual value of the furniture 6 years after purchase.

Match each function with one of the graphs below.



13. $f(x) = 2(0.69)^x$
14. $f(x) = 2(1.28)^x$
15. $f(x) = 2(0.81)^x$
16. $f(x) = 4(1.28)^x$
17. $f(x) = 2(1.59)^x$
18. $f(x) = 4(0.69)^x$

If all the graphs to the right have equations with form $\boldsymbol{f(x) = an^x}$,



19. Which graph has the largest value for *n*?
20. Which graph has the smallest value for *n*?
21. Which graph has the largest value for *a*?
22. Which graph has the smallest value for *a*?

Sketch a graph of each of the following transformations of $\boldsymbol{f(x) = 2^x}$*. Then find the domain and range, intercepts, end behavior, intervals of increasing/decreasing, and concavity.*

23. $f(x) = 2^{-x}$
24. $g(x) = -2^x$
25. $h(x) = 2^x + 3$
26. $f(x) = 2^x - 4$
27. $f(x) = 2^{x-2}$
28. $k(x) = 2^{x-3}$

Starting with the graph of $\boldsymbol{f(x) = 4^x}$*, find a formula for the function that results from*

29. Shifting $f(x)$ 4 units upwards
30. Shifting $f(x)$ 3 units downwards
31. Shifting $f(x)$ 2 units left
32. Shifting $f(x)$ 5 units right
33. Reflecting $f(x)$ about the *x*-axis
34. Reflecting $f(x)$ about the *y*-axis

Describe the long run behavior, as $\boldsymbol{x \to \infty}$ *and* $\boldsymbol{x \to -\infty}$ *of each function*

35. $f(x) = -5(4^x) - 1$
36. $f(x) = -2(3^x) + 2$
37. $f(x) = 3\left(\frac{1}{2}\right)^x - 2$
38. $f(x) = 4\left(\frac{1}{4}\right)^x + 1$
39. $f(x) = 3(4)^{-x} + 2$
40. $f(x) = -2(3)^{-x} - 1$

Graph each of the following functions using transformations. Then find the domain and range, intercepts, end behavior, intervals of increasing/decreasing, and concavity.

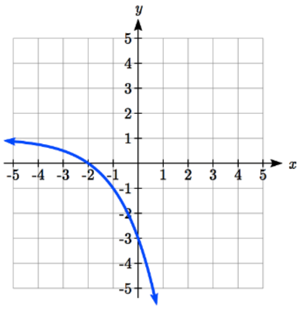
41. $f(x) = 5^x - 1$

42. $g(x) = -3^x$

43. $h(x) = 2\left(\frac{1}{3}\right)^x$

44. $k(x) = -\left(\frac{1}{5}\right)^x + 1$

Find a formula for each function graphed as a transformation of $\boldsymbol{f(x) = 2^x}$.



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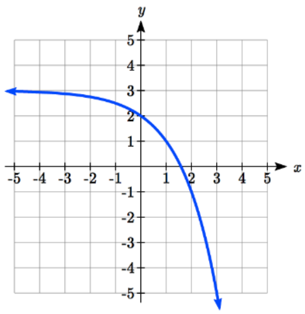

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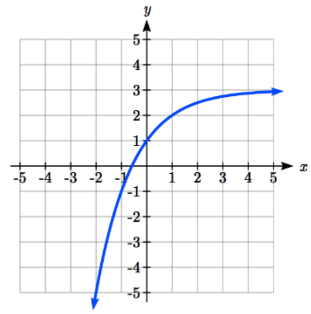

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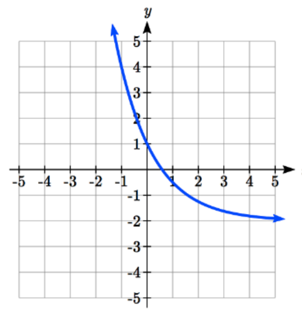
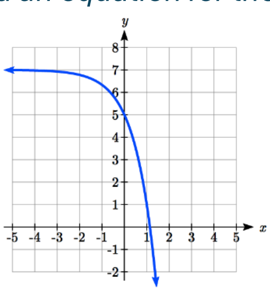


48.

Find an equation for the exponential function graphed.



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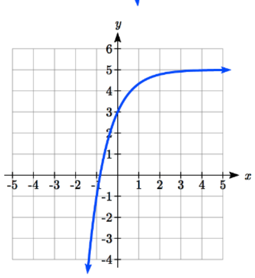

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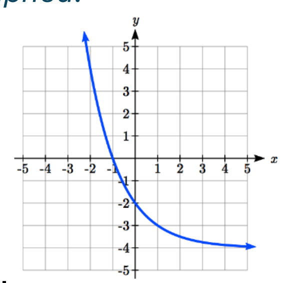
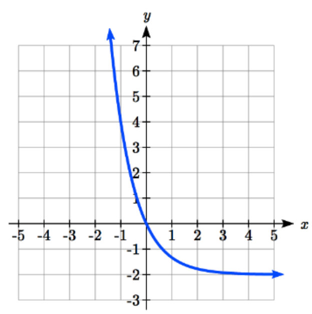


51.



52.

Find a formula for an exponential function passing through the two points.

53. $(0,\ 6),\ (3,\ 750)$
54. $(0,\ 3),\ (2,\ 75)$
55. $(0,\ 2000),\ (2,\ 20)$
56. $(0,\ 9000),\ (3,\ 72)$
57. $\left(-1, \frac{3}{2}\right),\ (3,\ 24)$
58. $\left(-1, \frac{2}{5}\right),\ (1,10)$
59. $(-2,6),\ (3,1)$
60. $(-3,4),\ (3,\ 2)$
61. $(3,\ 1),\ (5,\ 4)$
62. $(2,5),\ (6,\ 9)$

63. A radioactive substance decays exponentially. A scientist begins with 100 milligrams of a radioactive substance. After 35 hours, 50 mg of the substance remains. How many milligrams will remain after 54 hours?
64. A radioactive substance decays exponentially. A scientist begins with 110 milligrams of a radioactive substance. After 31 hours, 55 mg of the substance remains. How many milligrams will remain after 42 hours?
65. A house was valued at $110,000 in the year 1985. The value appreciated to $145,000 by the year 2005. What was the annual growth rate between 1985 and 2005? Assume that the house value continues to grow by the same percentage. What did the value equal in the year 2010?
66. An investment was valued at $11,000 in the year 1995. The value appreciated to $14,000 by the year 2008. What was the annual growth rate between 1995 and 2008? Assume that the value continues to grow by the same percentage. What did the value equal in the year 2012?

67. A car was valued at $38,000 in the year 2003. The value depreciated to $11,000 by the year 2009. Assume that the car value continues to drop by the same percentage. What was the value in the year 2013?

68. A car was valued at $24,000 in the year 2006. The value depreciated to $20,000 by the year 2009. Assume that the car value continues to drop by the same percentage. What was the value in the year 2014?

69. If $4,000 is invested in a bank account at an interest rate of 7 per cent per year, find the amount in the bank after 9 years if interest is compounded annually, quarterly, monthly, and continuously.
70. If $6,000 is invested in a bank account at an interest rate of 9 per cent per year, find the amount in the bank after 5 years if interest is compounded annually, quarterly, monthly, and continuously.

71. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 3% compounded quarterly.

72. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 5% compounded monthly.

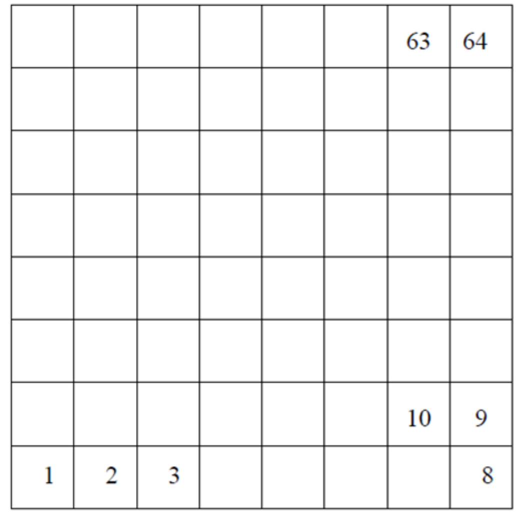
73. A population of bacteria is growing according to the equation $P(t) = 1600e^{0.21t}$, with *t* measured in years. Estimate when the population will exceed 7569.

74. A population of bacteria is growing according to the equation $P(t) = 1200e^{0.17\,t}$, with *t* measured in years. Estimate when the population will exceed 3443.

75. In 1968, the U.S. minimum wage was $1.60 per hour. In 1976, the minimum wage was $2.30 per hour. Assume the minimum wage grows according to an exponential model $w(t)$, where *t* represents the time in years after 1960. [UW]
    a. Find a formula for $w(t)$.
    b. What does the model predict for the minimum wage in 1960?
    c. If the minimum wage was $5.15 in 1996, is this above, below or equal to what the model predicts?

76. In 1989, research scientists published a model for predicting the cumulative number of AIDS cases (in thousands) reported in the United States: $a(t) = 155\left(\frac{t-1980}{10}\right)^3$, where *t* is the year. This paper was considered a "relief", since there was a fear the correct model would be of exponential type. Pick two data points predicted by the research model $a(t)$ to construct a new exponential model $b(t)$ for the number of cumulative AIDS cases. Discuss how the two models differ and explain the use of the word "relief." [UW]

77. You have a chess board as pictured, with squares numbered 1 through 64. You also have a huge change jar with an unlimited number of dimes. On the first square you place one dime. On the second square you stack 2 dimes. Then you continue, always doubling the number from the previous square. [UW]



a. How many dimes will you have stacked on the 10th square?
b. How many dimes will you have stacked on the nth square?
c. How many dimes will you have stacked on the 64th square?
d. Assuming a dime is 1 mm thick, how high will this last pile be?
e. The distance from the earth to the sun is approximately 150 million km. Relate the height of the last pile of dimes to this distance.

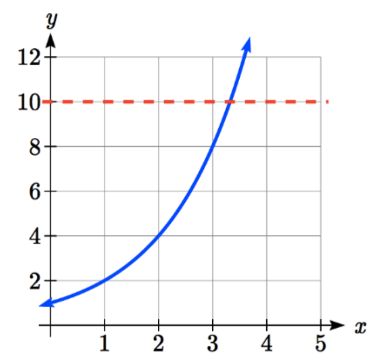
Section 3.2 Logarithmic Functions

A population of 50 flies is expected to double every week, leading to a function of the form $f(x) = 50(2)^x$, where *x* represents the number of weeks that have passed. When will this population reach 500? Trying to solve this problem leads to:

$500 = 50(2)^x$ Dividing both sides by 50 to isolate the exponential
$10 = 2^x$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that *x* must be some value between 3 and 4 since $g(x) = 2^x$ is increasing. We could use technology to create a table of values or graph to better estimate the solution.

From the graph, we could better estimate the solution to be around 3.3. This result is still fairly unsatisfactory, and since the exponential function is one-to-one, it would be great to have an inverse function. None of the functions we have already discussed would serve as an inverse function and so we must introduce a new function, named **log** as the inverse of an exponential function. Since exponential functions have different bases, we will define corresponding logarithms of different bases as well.



Logarithm
The logarithm (base n) function, written $\log_n(x)$, is the inverse of the exponential function (base n), n^x.

Since the logarithm and exponential are inverses, it follows that:

Properties of Logs: Inverse Properties
$\log_n(n^x) = x$ $n^{\log_n(x)} = x$

Recall from the definition of an inverse function that if $f(a) = c$, then $f^{-1}(c) = a$. Applying this to the exponential and logarithmic functions, we can convert between a logarithmic equation and its equivalent exponential.

Logarithm Equivalent to an Exponential
The statement $n^p = m$ is equivalent to the statement $\log_n(m) = p$.

Alternatively, we could show this by starting with the exponential function $m = n^p$, then taking the log base *n* of both sides, giving $\log_n(m) = \log_n(n^p)$. Using the inverse property of logs, we see that $\log_n(m) = p$.

Since log is a function, it is most correctly written as $\log_n(m)$, using parentheses to denote function evaluation, just as we would with *f(c)*. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_n m$.

Example 1

Write these exponential equations as logarithmic equations:

a) $2^3 = 8$
b) $5^2 = 25$
c) $10^{-4} = \frac{1}{10000}$

Solution:

a) $2^3 = 8$ is equivalent to $\log_2(8) = 3$
b) $5^2 = 25$ is equivalent to $\log_5(25) = 2$
c) $10^{-4} = \frac{1}{10000}$ is equivalent to $\log_{10}\left(\frac{1}{10000}\right) = -4$

Example 2

Write these logarithmic equations as exponential equations:

a) $\log_6(\sqrt{6}) = \frac{1}{2}$
b) $\log_3(9) = 2$

Solution:

a) $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to $6^{\frac{1}{2}} = \sqrt{6}$
b) $\log_3(9) = 2$ is equivalent to $3^2 = 9$

Try it Now

Write the exponential equation $4^2 = 16$ as a logarithmic equation.

Evaluating Logarithms

For many logarithm values, we'll need a calculator to find the value. Some logarithms can be evaluated using our knowledge of the relationship between logarithms and exponents.

Example 3

Evaluate $\log_3(27)$ exactly.

Solution:
We can temporarily write this as an equation:

$$\log_3(27) = x$$

Then rewrite in exponential form:

$$3^x = 27$$

So $\log_3(27)$ is asking "3 to what power makes 27"? We know $3^3 = 27$, so $\log_3(27) = 3$.

Example 4

Evaluate $\log_3\left(\frac{1}{27}\right)$ exactly.

Solution:
Again, we can restate this as

$$\log_3\left(\frac{1}{27}\right) = x$$

And then as

$$3^x = \frac{1}{27}$$

This time, we know $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$, so $\log_3\left(\frac{1}{27}\right) = -3$.

Example 5

Evaluate $\log_{27}(3)$ exactly.

Solution:
This time we have

$$\log_{27}(3) = x$$

So

$$27^x = 3$$

Well, we know $\sqrt[3]{27} = 27^{\frac{1}{3}} = 3$, so $\log_{27}(3) = \frac{1}{3}$.

Try it Now

Evaluate each logarithm exactly.

a) $\log_5 \frac{1}{25}$

b) $\log_4(16)$

c) $\log_{49}(7)$

The Common and Natural Logarithms

There are a couple of logarithms that are so important and ubiquitous, that they have their own special names and notation.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$.

The **natural log** is the logarithm with base e, and is typically written $\ln(x)$.

Example 6

Evaluate $\log(1000)$ using the definition of the common log.

Values of the common log

number	number as exponential	log(numbe
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

Solution:

To evaluate $\log(1000)$, we can let $x = \log(1000)$, then rewrite into exponential form using the common log base of 10: $10^x = 1000$.

From this, we might recognize that 1000 is the cube of 10, so $x = 3$.

We also can use the inverse property of logs to write $\log_{10}(10^3) = 3$.

Try it Now

Evaluate $\log(1000000)$.

Example 7

Evaluate $\ln(\sqrt{e})$.

Solution:

We can rewrite $\ln(\sqrt{e})$ as $\ln\left(e^{\frac{1}{2}}\right)$. Since ln is a log base e, we can use the inverse property for logs: $\ln\left(e^{\frac{1}{2}}\right) = \log_e\left(e^{\frac{1}{2}}\right) = \frac{1}{2}$.

Because these special logarithms are so common, most scientific calculators will have special buttons which allow you to evaluate them at any input value.

Example 8

Evaluate $\log(500)$ using your calculator or computer.

Solution:
Using a computer, we can evaluate $\log(500) \approx 2.69897$.

Solving Equations

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 9

Solve $\log_4(x) = 2$ for x.

Solution:
By rewriting this expression as an exponential,

$$4^2 = x$$

so

$$x = 16$$

Example 10

Solve $2^x = 10$ for x.

Solution:
By rewriting this expression as a logarithm, we get $x = \log_2(10)$.

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful – often we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of the common or natural log. Happily, this ends up not being a problem, as we'll see briefly.

To utilize the common or natural logarithm functions to evaluate expressions like $\log_2(10)$, we need to establish some additional properties.

Properties of Logs: Exponent Property

$$\log_n(M^p) = p \cdot \log_n(M)$$

To show why this is true, we offer a proof:

Since the logarithmic and exponential functions are inverses, $n^{\log_n(M)} = M$.

Raising both sides to the *p* power, we get $M^p = \left(n^{\log_n(M)}\right)^p$.

Utilizing the exponential rule that states $(x^p)^q = x^{p \cdot q}$, $M^p = \left(n^{\log_n(M)}\right)^p = n^{p \cdot \log_n(M)}$.

Taking the log of both sides, $\log_n(M^p) = \log_n\left(n^{p \cdot \log_n(M)}\right)$

Utilizing the inverse property on the right side yields the result: $\log_n(M^p) = p \cdot \log_n(M)$

Example 11

Rewrite $\log_3(25)$ using the exponent property for logs.

Solution:

Since $25 = 5^2$,

$$\log_3(25) = \log_3(5^2) = 2\log_3(5)$$

Example 12

Rewrite $4\ln(x)$ using the exponent property for logs.

Solution:

Using the property in reverse, $4\ln(x) = \ln(x^4)$.

Try it Now

Rewrite using the exponent property for logs: $\ln\left(\frac{1}{x^2}\right)$.

The exponent property allows us to find a method for changing the base of a logarithmic expression.

Properties of Logs: Change of Base

$$\log_n(M) = \frac{\log_r(M)}{\log_r(n)}$$

Proof:

Let $\log_n(M) = x$.
Rewriting as an exponential gives $n^x = M$.
Taking the log base *r* of both sides of this equation gives $\log_r(n^x) = \log_r(M)$,
Now utilizing the exponent property for logs on the left side, $x \cdot \log_r(n) = \log_r(M)$
Dividing, we obtain $x = \frac{\log_r(M)}{\log_r(n)}$. Replacing our original expression for *x*,

$$\log_n(M) = \frac{\log_r(M)}{\log_r(n)}$$

With this change of base formula, we can finally find a good decimal approximation to our question from the beginning of the section.

Example 13

Evaluate $\log_2(10)$ using the change of base formula.

Solution:
According to the change of base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base *e*:

$$\log_2(10) = \frac{\log_e(10)}{\log_e(2)} = \frac{\ln(10)}{\ln(2)}$$

Using our calculators to evaluate this,

$$\frac{\ln(10)}{\ln(2)} \approx \frac{2.30259}{0.69315} \approx 3.3219$$

This finally allows us to answer our original question – the population of flies we discussed at the beginning of the section will take 3.32 weeks to grow to 500.

Example 14

Evaluate $\log_5(100)$ using the change of base formula.

Solution:
We can rewrite this expression using any other base. If our calculators are able to evaluate the common logarithm, we could rewrite using the common log, base 10.

$$\log_5(100) = \frac{\log_{10}(100)}{\log_{10}(5)} \approx \frac{2}{0.69897} \approx 2.861$$

While we can solve the basic exponential equation $2^x = 10$ by rewriting in logarithmic form and then using the change of base formula to evaluate the logarithm, the proof of the change of base formula illuminates an alternative approach to solving exponential equations.

Solving exponential equations:

1. Isolate the exponential expressions when possible
2. Take the logarithm of both sides
3. Utilize the exponent property for logarithms to pull the variable out of the exponent
4. Use algebra to solve for the variable.

Example 15

Solve $2^x = 10$ for x.

Solution:

Using this alternative approach, rather than rewrite this exponential into logarithmic form, we will take the logarithm of both sides of the equation. Since we often wish to evaluate the result to a decimal answer, we will usually utilize either the common log or natural log. For this example, we'll use the natural log:

$$\begin{aligned} \ln(2^x) &= \ln(10) \\ x \cdot \ln(2) &= \ln(10) && \text{Exponent property for logs} \\ x &= \frac{\ln(10)}{\ln(2)} && \text{Divide by } \ln(2) \\ x &\approx 3.3219 \end{aligned}$$

Notice that this result matches the result we found using the change of base formula.

Example 16

In the first section, we predicted the population (in billions) of India t years after 2008 by using the function $f(t) = 1.14(1 + 0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?

Solution:

We need to solve for time t so that $f(t) = 2$.

$$\begin{aligned} 2 &= 1.14(1.0134)^t \\ \frac{2}{1.14} &= 1.0134^t && \text{Divide by 1.14 to isolate the exponential expression} \end{aligned}$$

$$\ln\left(\frac{2}{1.14}\right) = \ln(1.0134^t) \quad \text{Take the logarithm of both sides of the equation}$$

$$\ln\left(\frac{2}{1.14}\right) = t \cdot \ln(1.0134) \quad \text{Apply the exponent property on the right side}$$

$$t = \frac{\ln\left(\frac{2}{1.14}\right)}{\ln(1.0134)} \quad \text{Divide both sides by } \ln(1.0134)$$

$$t \approx 42.23 \text{ years}$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

Try it Now

Solve $5(0.93)^x = 10$.

Example 14

Solve $5(1.07)^{3t} = 2$.

Solution:
To start, we want to isolate the exponential part of the expression, the $(1.07)^{3t}$, so it is alone on one side of the equation. Then we can use the log to solve the equation. We can use any base log; this time we'll use the common log.

$$5(1.07)^{3t} = 2$$

$$(1.07)^{3t} = \frac{2}{5} \quad \text{Divide both sides by 5 to isolate the exponential}$$

$$\log((1.07)^{3t}) = \log\left(\frac{2}{5}\right) \quad \text{Take the log of both sides.}$$

$$3t \cdot \log(1.07) = \log\left(\frac{2}{5}\right) \quad \text{Use the exponent property for logs}$$

$$\frac{3t \cdot \log(1.07)}{3\log(1.07)} = \frac{\log\left(\frac{2}{5}\right)}{3\log(1.07)} \quad \text{Divide by on both sides}$$

$$t = \frac{\log\left(\frac{2}{5}\right)}{3\log(1.07)}$$

$$t \approx -4.5143$$

Note that when entering that expression on your calculator, be sure to put parentheses around the whole denominator to ensure the proper order of operations:

log(2/5)/(3*log(1.07))

In addition to solving exponential equations, logarithmic expressions are common in many physical situations.

Example 17

In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, $[H^+]$, measured in moles per liter, by the equation $pH = -\log([H^+])$.

a) If a liquid has concentration of 0.0001 moles per liter, determine the pH.
b) Determine the hydrogen ion concentration of a liquid with pH of 7.

Solution:

a) To answer the first question, we evaluate the expression $-\log(0.0001)$. While we could use our calculators for this, we do not really need them here, since we can use the inverse property of logs:

$$-\log(0.0001) = -\log(10^{-4}) = -(-4) = 4$$

b) To answer the second question, we need to solve the equation $7 = -\log([H^+])$. Begin by isolating the logarithm on one side of the equation by multiplying both sides by -1:

$$-7 = \log([H^+])$$

Rewriting into exponential form yields the answer:

$$[H^+] = 10^{-7} = 0.0000001 \text{ moles per liter}$$

Logarithms also provide us a mechanism for finding continuous growth models for exponential growth given two data points.

Example 18

A population grows from 100 to 130 in 2 weeks. Find the continuous growth rate.

Solution:

Measuring t in weeks, we are looking for an equation $P(t) = ae^{rt}$ so that $P(0) = 100$ and $P(2) = 130$. Using the first pair of values, $100 = ae^{r\cdot 0}$, so $a = 100$.

Using the second pair of values,

$$\begin{aligned}
130 &= 100e^{r\cdot 2} && \\
\frac{130}{100} &= e^{2r} && \text{Divide by 100} \\
\ln(1.3) &= \ln(e^{2r}) && \text{Take the natural log of both sides} \\
\ln(1.3) &= 2r && \text{Use the inverse property of logs} \\
r &= \frac{\ln(1.3)}{2} && \\
r &\approx 0.1312 &&
\end{aligned}$$

This population is growing at a continuous rate of 13.12% per week.

In general, we can relate the standard form of an exponential with the continuous growth form by noting (using *k* to represent the continuous growth rate to avoid the confusion of using *r* in two different ways in the same formula):

$$a(1+r)^x = ae^{kx}$$
$$(1+r)^x = e^{kx}$$
$$1+r = e^k$$

Converting Between Periodic to Continuous Growth Rate

In the equation $f(x) = a(1+r)^x$, r is the **periodic growth rate**, the percent growth each time period (weekly growth, annual growth, etc.).

In the equation $f(x) = ae^{kx}$, k is the **continuous growth rate**.

You can convert between these using: $1+r = e^k$.

Remember that the continuous growth rate *k* represents the nominal growth rate before accounting for the effects of continuous compounding, while *r* represents the actual percent increase in one time unit (one week, one year, etc.).

Example 19

A company's sales can be modeled by the function $S(t) = 5000e^{0.12t}$, with t measured in years. Find the annual growth rate.

Solution:
Noting that $1+r = e^k$, then $r = e^{0.12} - 1 = 0.1275$, so the annual growth rate is 12.75%. The sales function could also be written in the form

$$S(t) = 5000(1+0.1275)^t.$$

Important Topics of this Section

The Logarithmic function as the inverse of the exponential function

Writing logarithmic & exponential expressions

Properties of logs

- Inverse properties
- Exponential properties
- Change of base

Common log

Natural log

Solving exponential equations

Converting between periodic and continuous growth rate.

Section 3.2 Exercises

Conceptual Questions

1. What is a logarithm?
2. What is the inverse property of logarithms?
3. What is the exponential property of logarithms?
4. What is the change of base formula?
5. How can logarithms help us to solve exponential equations?

Practice Problems

Rewrite each equation in exponential form

1. $\log_4(q) = m$
2. $\log_3(t) = k$
3. $\log_a(b) = c$
4. $\log_p(z) = u$
5. $\log(v) = t$
6. $\log(r) = s$
7. $\ln(w) = n$
8. $\ln(x) = y$

Rewrite each equation in logarithmic form.

9. $4^x = y$
10. $5^y = x$
11. $c^d = k$
12. $n^z = L$
13. $10^a = b$
14. $10^p = v$
15. $e^k = h$
16. $e^y = x$

Evaluate each logarithm exactly.

17. $\log_8(64)$
18. $\log_{64}(4)$
19. $\log_5\left(\frac{1}{5}\right)$
20. $\log(10000)$
21. $\ln(e)$
22. $\log_3\left(\frac{1}{9}\right)$
23. $\log_2(8)$
24. $\log_8(2)$

Solve for x.

25. $\log_3(x) = 2$
26. $\log_4(x) = 3$
27. $\log_2(x) = -3$
28. $\log_5(x) = -01$
29. $\log(x) = 3$
30. $\log(x) = 5$
31. $\ln(x) = 2$
32. $\ln(x) = -2$

Simplify each expression using logarithm properties.

33. $\log_5(25)$
34. $\log_2(8)$
35. $\log_3\left(\frac{1}{27}\right)$
36. $\log_6\left(\frac{1}{36}\right)$
37. $\log_6(\sqrt{6})$
38. $\log_5(\sqrt[3]{5})$
39. $\log(10{,}000)$
40. $\log(100)$
41. $\log(0.001)$
42. $\log(0.00001)$
43. $\ln(e^{-2})$
44. $\ln(e^3)$

Evaluate using your calculator.

45. $\log(0.04)$
46. $\log(1045)$
47. $\ln(15)$
48. $\ln(0.02)$

Solve each equation for the variable.

49. $5^x = 14$
50. $3^x = 23$
51. $7^x = \frac{1}{15}$
52. $3^x = \frac{1}{4}$
53. $e^{5x} = 17$
54. $e^{3x} = 12$
55. $3^{4x-5} = 38$
56. $4^{2x-3} = 44$
57. $1000(1.03)^t = 5000$
58. $200(1.06)^t = 550$
59. $3(1.04)^{3t} = 8$
60. $2(1.08)^{4t} = 7$
61. $50e^{-0.12t} = 10$
62. $10e^{-0.03t} = 4$
63. $10 - 8\left(\frac{1}{2}\right)^x = 5$
64. $100 - 100\left(\frac{1}{4}\right)^x = 70$

Convert the equation into continuous growth form, $\boldsymbol{f(t) = ae^{kt}}$.

65. $f(t) = 300(0.91)^t$
66. $f(t) = 120(0.07)^t$
67. $f(t) = 10(1.04)^t$
68. $f(t) = 1400(1.12)^t$

Convert the equation into annual growth form, $\boldsymbol{f(t) = an^t}$.

69. $f(t) = 150e^{0.06t}$
70. $f(t) = 100e^{0.12t}$
71. $f(t) = 50e^{-0.012t}$
72. $f(t) = 80e^{-0.85t}$

73. The population of Kenya was 39.8 million in 2009 and has been growing by about 2.6% each year. If this trend continues, when will the population exceed 45 million?

74. The population of Algeria was 34.9 million in 2009 and has been growing by about 1.5% each year. If this trend continues, when will the population exceed 45 million?

75. The population of Seattle grew from 563,374 in 2000 to 608,660 in 2010. If the population continues to grow exponentially at the same rate, when will the population exceed 1 million people?

76. The median household income (adjusted for inflation) in Seattle grew from $42,948 in 1990 to $45,736 in 2000. If it continues to grow exponentially at the same rate, when will median income exceed $50,000?

77. A scientist begins with 100 mg of a radioactive substance. After 4 hours, it has decayed to 80 mg. How long after the process began will it take to decay to 15 mg?

78. A scientist begins with 100 mg of a radioactive substance. After 6 days, it has decayed to 60 mg. How long after the process began will it take to decay to 10 mg?

79. If $1000 is invested in an account earning 3% compounded monthly, how long will it take the account to grow in value to $1500?

80. If $1000 is invested in an account earning 2% compounded quarterly, how long will it take the account to grow in value to $1300?

Section 3.3 Graphs of Logarithmic Functions

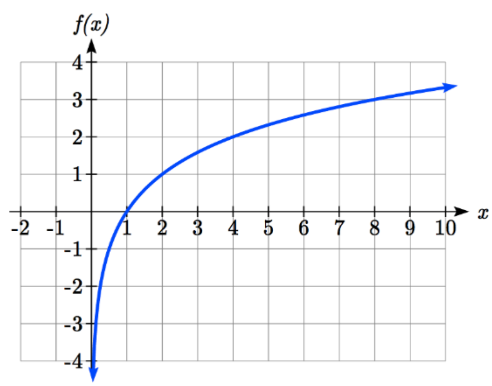
Recall that the exponential function $f(x) = 2^x$ produces this table of values

x	-3	-2	-1	0	1	2	3
f(x)	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Since the logarithmic function is an inverse of the exponential, and for an inverse we swap inputs and outputs, , $g(x) = \log_2(x)$, produces the table of values

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
g(x)	-3	-2	-1	0	1	2	3

In this second table, notice that
1) As the input increases, the output increases.
2) As input increases, the output increases more slowly.
3) Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs, so the domain of the log function is $(0, \infty)$.
4) Since the exponential function can accept all real numbers as inputs, the logarithm can output any real number, so the range is all real numbers or$(-\infty, \infty)$.



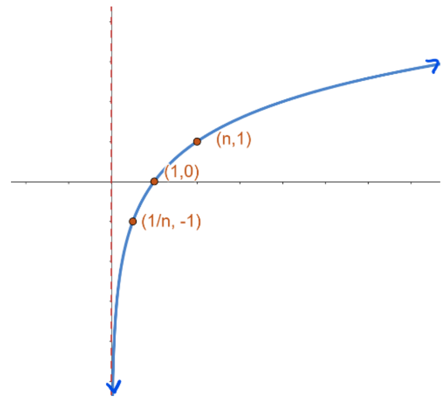
Sketching the graph, notice that as the input approaches zero from the right, the output of the function grows very large in the negative direction, indicating a vertical asymptote at *x* = 0.
In symbolic notation we write
As $x \to 0^+$, $g(x) \to -\infty$ and as $x \to \infty, g(x) \to \infty$.

Graphical Features of the Logarithm
Graphically, in the function $g(x) = \log_n(x)$ when $n > 1$
The graph has a horizontal intercept at $(1, 0)$
The graph has a vertical asymptote at $x = 0$
The graph is increasing and concave down
The domain of the function is $x > 0$, or $(0, \infty)$
The range of the function is all real numbers, or $(-\infty, \infty)$

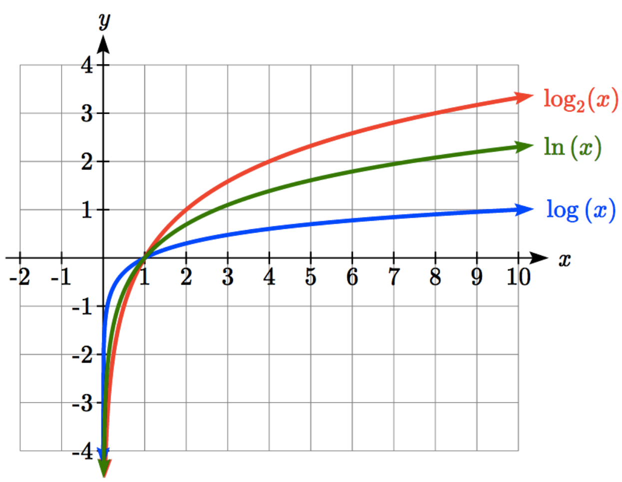
Parent graph and table for logarithms

To sketch a logarithm graph using transformations, we can use the following table and graph $g(x) = \log_n(x)$ when $n > 1$

x	$g(x)$
0	VA
$\frac{1}{n}$	-1
1	0
n	1



When sketching a general logarithm with base *n*, it can be helpful to remember that the graph will pass through the points (1, 0) and (*n*, 1).
To get a feeling for how the base affects the shape of the graph, examine the graphs below.



Notice that the larger the base, the slower the graph grows. For example, the common log graph, while it grows without bound, it does so very slowly. For example, to reach an output of 8, the input must be 100,000,000.

Another important observation made was the domain of the logarithm. Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of a composition involving a log.

Example 1

Find the domain of the function $f(x) = \log(5 - 2x)$.

Solution:
The logarithm is only defined with the input is positive, so this function will only be defined when $5 - 2x > 0$. Solving this inequality,

$$5 - 2x > 0$$
$$-2x > -5$$
$$x < \frac{5}{2}$$

The domain of this function is $x < 5/2$, or in interval notation, $\left(-\infty, \frac{5}{2}\right)$.

Try it Now

Find the domain of the function $f(x) = \log(x - 5) + 2$; before solving this as an inequality, consider how the function has been transformed.

Transformations of the Logarithmic Function

Our C-BAD transformations can work in the same way for logarithmic functions as for all other functions! We'll be graphing $h(x) = a\,g(bx + c) + d = a\log_n(bx + c) + d$.

Example 2

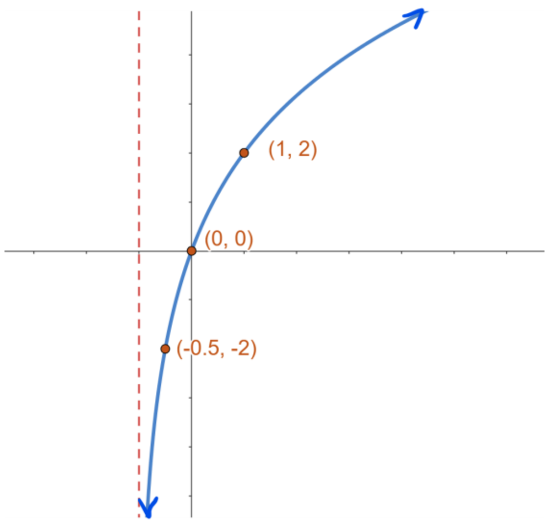
Graph $h(x) = 2\log_2(x + 1)$ using transformations. Then describe the features of the transformed graph.

Solution:
The logarithm part of the function tells us about our parent. Here, we'll be using $g(x) = \log_2(x)$, so the parent table is

x	$g(x) = \log_2(x)$
0	VA
½	-1
1	0
2	1

Then, if $h(x) = a\log_2(b\,x + c) + d$, we have $a = 2$ and $c = -1$. That's a shift left 1, and a vertical stretch of 2.

$X = x - 1$	$\leftarrow x$	$g(x) = \log_2(x) \rightarrow$	$h(X) = 2 \cdot g(x)$
-1	0	VA	VA
$-\frac{1}{2}$	½	-1	-2
0	1	0	0
1	2	1	2



Features of $h(x) = 2\log_2(x+1)$:
We can see from the graph that the x and y-intercept is $(0,0)$.
The domain is $(-1, \infty)$ and the range is $(-\infty, \infty)$.
The graph is increasing and concave down on its whole domain.
As $x \rightarrow -1^+, h(x) \rightarrow -\infty$ and as $x \rightarrow \infty, h(x) \rightarrow \infty$.

Example 3

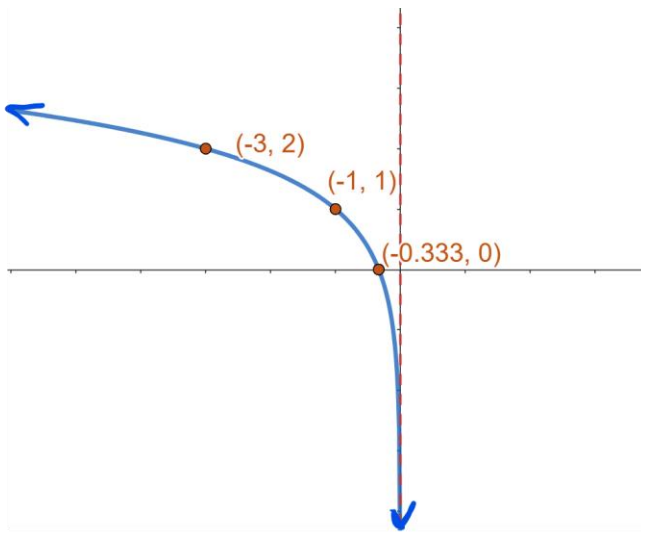
Graph $h(x) = \log_3(-x) + 1$ using transformations. Then describe the features of the transformed graph.

Solution:
The logarithm part of the function tells us about our parent. Here, we'll be using $g(x) = \log_3(x)$, so the parent table is

x	$g(x) = \log_3(x)$
0	VA
$1/3$	-1
1	0
3	1

Then, if $h(x) = a\log_3(b\,x + c) + d$, we have $b = -1$ and $d = 1$. That's a reflection over the y-axis and a shift up 1.

$X = -1 \cdot x$	$\leftarrow x$	$g(x) = \log_3(x) \rightarrow$	$h(X) = g(x) + 1$
0	0	VA	VA
$-\frac{1}{3}$	$1/3$	-1	0
-1	1	0	1
-3	3	1	2



Features of $h(x) = \log_3(-x) + 1$:
We can see from the graph that there is no y-intercept, and the x-intercept is at $-\frac{1}{3}$.
The domain is $(-\infty, 0)$ and the range is $(-\infty, \infty)$.
The graph is decreasing and concave down on its whole domain.
As $x \to 0^-, h(x) \to -\infty$ and as $x \to -\infty, h(x) \to \infty$.

Try it Now:
Graph $h(x) = -\log_4(x - 2) + 3$ using transformations. Then describe the features of the graph.

Transformations can be applied to a logarithmic function using the basic transformation techniques, but as with exponential functions, several transformations result in interesting relationships.

First recall the change of base property tells us that $\log_n(x) = \frac{\log_r(x)}{\log_r(n)} = \frac{1}{\log_r(n)} \log_r(x)$.

From this, we can see that $\log_r(n)$ is a vertical stretch or compression of the graph of the $\log_r(x)$ graph. This tells us that a vertical stretch or compression is equivalent to a change of base. For this reason, we typically represent all graphs of logarithmic functions in terms of the common or natural log functions.

Next, consider the effect of a horizontal compression on the graph of a logarithmic function. Considering $f(x) = \log(bx)$, we can use the sum property to see

$$f(x) = \log(bx) = \log(b) + \log(x)$$

Since log(*b)* is a constant, the effect of a horizontal compression is the same as the effect of a vertical shift.

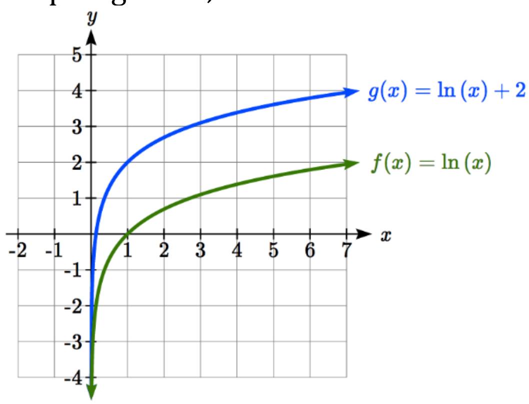
Example 4

Sketch $f(x) = \ln(x)$ and $g(x) = \ln(x) + 2$.

Solution:

We know the parent graph, $\ln(x)$, will have a vertical asymptote at $x = 0$ and pass through the points $(1, 0)$ and $(e, 1) \approx (2.718, 1)$. We know that in $g(x)$, $d = 2$, so $g(x)$ will be $f(x)$ shifted up 2.

Graphing these,



Note that this vertical shift could also be written as a horizontal compression, since

$$g(x) = \ln(x) + 2$$
$$= \ln(x) + \ln(e^2)$$
$$\ln(e^2 x).$$

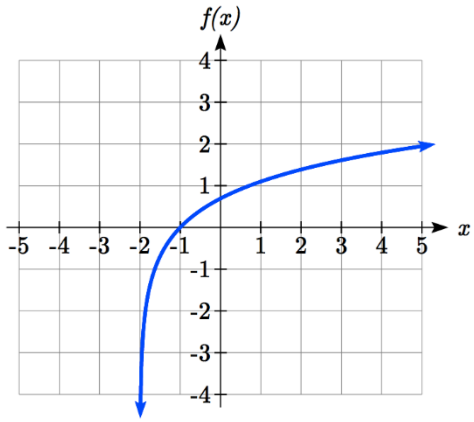
While a horizontal stretch or compression can be written as a vertical shift, a horizontal reflection is unique and separate from vertical shifting.

Finally, we will consider the effect of a horizontal shift on the graph of a logarithm.

Example 5

Sketch a graph of $f(x) = \ln(x + 2)$.

Solution:
This is a horizontal shift to the left by 2 units. Notice that none of our logarithm rules allow us rewrite this in another form, so the effect of this transformation is unique. Shifting the graph,



Notice that due to the horizontal shift, the vertical asymptote shifted to x = -2, and the domain shifted to $(-2, \infty)$.

Combining these transformations,

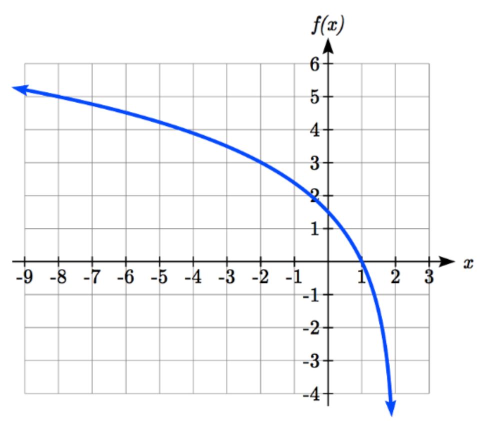
Example 6

Sketch a graph of $f(x) = 5\log(-x + 2)$.

This graph is that of the common logarithm, shifted left two units, horizontally reflected, and vertically stretched by a factor of 5.

The vertical asymptote will be shifted to x = 2, and the graph will have domain $(-\infty, 2)$. A rough sketch can be created by using the vertical asymptote along with a couple points on the graph, such as

$$f(1) = 5\log(-1 + 2) = 5\log(1) = 0$$
$$f(-8) = 5\log(-(-8) + 2) = 5\log(10) = 5$$



Try it Now

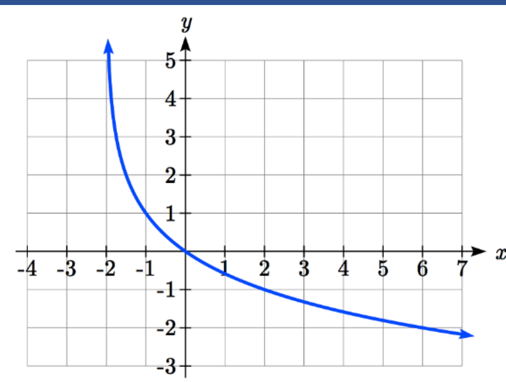
Sketch a graph of the function $f(x) = -3\log(x-2)+1$.

Transformations of Logs

Any transformed logarithmic function can be written in the form
$f(x) = a\log(x+c)+d$, or $f(x) = a\log\left(-(x-c)\right)+d$ if horizontally reflected,
where $x = c$ is the vertical asymptote.

Example 7

Find an equation for the logarithmic function graphed.



Solution:

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift (equivalent to horizontal stretch) or the vertical stretch (equivalent to a change of base). We know so far that the equation will have form

$$f(x) = -a\log(x+2)+d$$

It appears the graph passes through the points (–1, 1) and (2, –1). Substituting in (–1, 1),

$$1 = -a\log(-1+2)+d$$
$$1 = -a\log(1)+d$$
$$1 = d$$

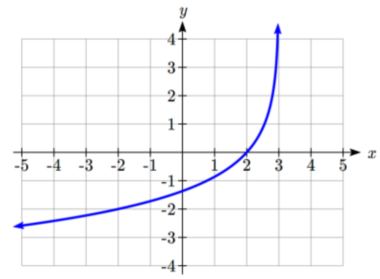
Next, substituting in (2, –1),

$$-1 = -a\log(2+2)+1$$
$$-2 = -a\log(4)$$
$$a = \frac{2}{\log(4)}$$

This gives us the equation $f(x) = -\frac{2}{\log(4)}\log(x+2)+1$.

This could also be written as $f(x) = -2\log_4(x+2)+1$.

Try it Now
Write an equation for the function graphed here.



Important Topics of this Section
Graph of the logarithmic function (domain and range)
Transformation of logarithmic functions
Creating graphs from equations
Creating equations from graphs

Section 3.3 Exercises

Conceptual Questions

1. How are the graphs of $\boldsymbol{f(x) = 2^x}$, $\boldsymbol{g(x) = \left(\frac{1}{2}\right)^x}$, $\boldsymbol{h(x) = log_2(x)}$, and $\boldsymbol{k(x) = log_{\frac{1}{2}}(x)}$ related? How are they similar? How are they different?
2. How can we find the domain of a logarithmic function?
3. True or false: a logarithmic function will always have an x-intercept.
4. True or false: a logarithmic function will always have a y-intercept.

Practice Problems

For each function, find the domain and the vertical asymptote.

1. $f(x) = \log(x - 5)$
2. $f(x) = \log(x + 2)$
3. $f(x) = \ln(3 - x)$
4. $f(x) = \ln(5 - x)$
5. $f(x) = \log(3x + 1)$
6. $f(x) = \log(2x + 5)$
7. $f(x) = 3\log(-x) + 2$
8. $f(x) = 2\log(-x) + 1$

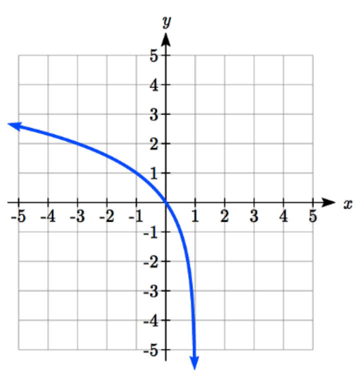
Sketch a graph of each pair of functions.

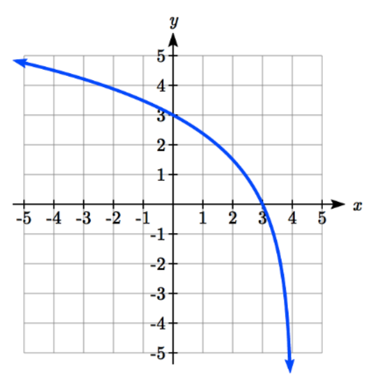
9. $f(x) = \log(x), g(x) = \ln(x)$
10. $f(x) = \log_2(x), g(x) = \log_4(x)$

Sketch each transformation. Then describe the features of the function.

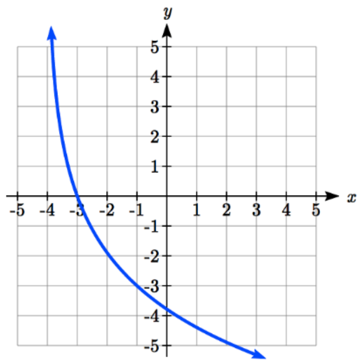
11. $f(x) = 2\log(x)$
12. $f(x) = 3\ln(x)$
13. $f(x) = \ln(-x)$
14. $f(x) = -\log(x)$
15. $f(x) = \log_2(x + 2)$
16. $f(x) = -\log_2(-x)$
17. $f(x) = \log_2(2x - 1)$
18. $f(x) = 2\log_3(x + 4)$
19. $f(x) = -2\log_3(3x) + 1$
20. $f(x) = 4\log_3(-x - 2)$

Find a formula for the transformed logarithm graph shown.

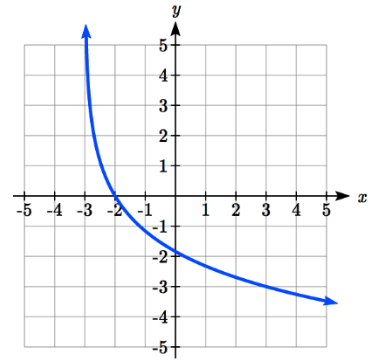
21. 

22. 

Find a formula for the transformed logarithm graph shown.



23.



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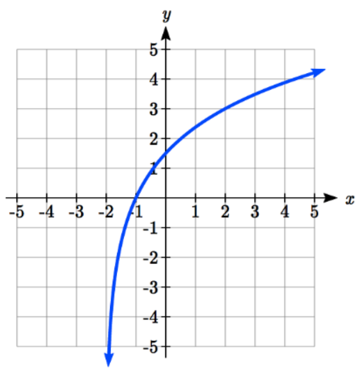

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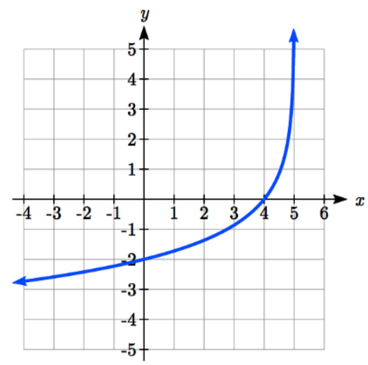

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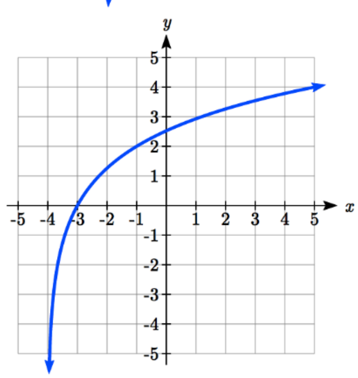

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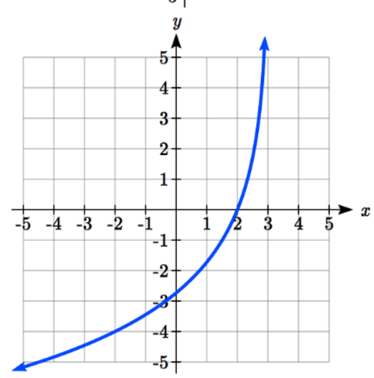


28.

Section 3.4 Logarithmic Properties

In the previous section, we derived two important properties of logarithms, which allowed us to solve some basic exponential and logarithmic equations.

Properties of Logs

Inverse Properties:

$$\log_n(n^x) = x$$
$$n^{\log_n(x)} = x$$

Exponential Property:

$$\log_n(M^p) = p \cdot \log_n(M)$$

Change of Base:

$$\log_n(M) = \frac{\log_r(M)}{\log_r(n)}$$

While these properties allow us to solve a large number of problems, they are not sufficient to solve all problems involving exponential and logarithmic equations.

Properties of Logs

Sum of Logs Property:

$$\log_n(M) + \log_n(Q) = \log_n(MQ)$$

Difference of Logs Property:

$$\log_n(M) - \log_n(Q) = \log_n\left(\frac{M}{Q}\right)$$

It's just as important to know what properties logarithms do *not* satisfy as to memorize the valid properties listed above. In particular, the logarithm is not a linear function, which means that it does not distribute: $\log(M + Q) \neq \log(M) + \log(Q)$.

To help in this process we offer a proof to help solidify our new rules and show how they follow from properties you've already seen.

Let $m = \log_n(M)$ and $q = \log_n(Q)$.
By definition of the logarithm, $n^m = M$ and $n^q = Q$.
Using these expressions, $MQ = n^m \cdot n^q$
Using exponent rules on the right, $MQ = n^{m+q}$
Taking the log of both sides, and utilizing the inverse property of logs,
$$\log_n(MQ) = \log_n(n^{m+q}) = m + q$$

Replacing *m* and *q* with their definition establishes the result
$$\log_n(MQ) = \log_n(M) + \log_n(Q)$$

The proof for the difference property is very similar.

With these properties, we can rewrite expressions involving multiple logs as a single log, or break an expression involving a single log into expressions involving multiple logs.

Example 1

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Solution:
Using the sum of logs property on the first two terms,
$$\log_3(5) + \log_3(8)$$
$$= \log_3(5 \cdot 8)$$
$$= \log_3(40)$$

This reduces our original expression to $\log_3(40) - \log_3(2)$.

Then using the difference of logs property,
$$\log_3(40) - \log_3(2)$$
$$= \log_3\left(\frac{40}{2}\right)$$
$$= \log_3(2)$$

Example 2

Evaluate $2\log(5) + \log(4)$ without a calculator by first rewriting as a single logarithm.

Solution:
On the first term, we can use the exponent property of logs to write
$$2\log(5)$$
$$= \log(5^2)$$
$$= \log(25)$$

With the expression reduced to a sum of two logs, $\log(25) + \log(4)$, we can utilize the sum of logs property

$$\log(25) + \log(4)$$
$$= \log(25 \cdot 4)$$
$$= \log(100)$$

Since $100 = 10^2$, we can evaluate this log without a calculator:

$$\log(100)$$
$$= \log_{10}(10^2)$$
$$= 2$$

Try it Now

Without a calculator evaluate by first rewriting as a single logarithm:

$$\log_2(8) + \log_2(4)$$

Example 3

Rewrite $\ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs

Solution:

First, noticing we have a quotient of two expressions, we can utilize the difference property of logs to write

$$\ln\left(\frac{x^4y}{7}\right)$$
$$= \ln(x^4y) - \ln(7)$$

Then seeing the product in the first term, we use the sum property

$$\ln(x^4y) - \ln(7)$$
$$= \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we could use the exponent property on the first term

$$= \ln(x^4) + \ln(y) - \ln(7)$$
$$= 4 \cdot \ln(x) + \ln(y) - \ln(7)$$

value	log(value)
1	0.0000000
2	0.3010300
3	0.4771213
4	0.6020600
5	0.6989700
6	0.7781513
7	0.8450980
8	0.9030900
9	0.9542425
10	1.0000000

Interestingly, solving exponential equations was not the reason logarithms were originally developed. Historically, until the advent of calculators and computers, the power of logarithms was that these log properties reduced multiplication, division, roots, or powers to be evaluated using addition, subtraction, division and multiplication, respectively, which are much easier to compute without a calculator. Large books were published listing the logarithms of numbers, such as in the table to the right. To find the product of two numbers, the sum of log property was used. Suppose for example we didn't know the value of 2 times 3. Using the sum property of logs:

$$\log(2 \cdot 3) = \log(2) + \log(3)$$

Using the log table,

$$\begin{aligned} \log(2 \cdot 3) &= \log(2) + \log(3) \\ &= 0.3010300 + 0.4771213 \\ &= 0.7781513 \end{aligned}$$

We can then use the table again in reverse, looking for 0.7781513 as an output of the logarithm. From that we can determine:

$$\log(2 \cdot 3) = 0.7781513 = \log(6)$$

.

By using addition and the table of logs, we were able to determine $2 \cdot 3 = 6$.

Likewise, to compute a cube root like $\sqrt[3]{8}$

$$\begin{aligned} &\log(\sqrt[3]{8}) \\ &= \log\left(8^{\frac{1}{3}}\right) \\ &= \frac{1}{3} \cdot \log(8) \\ &= \frac{1}{3}(0.9030900) \\ &= 0.3010300 \\ &= \log(2) \end{aligned}$$

So. $\sqrt[3]{8} = 2$

Although these calculations are simple and insignificant, they illustrate the same idea that was used for hundreds of years as an efficient way to calculate the product, quotient, roots, and powers of large and complicated numbers, either using tables of logarithms or mechanical tools called slide rules.

These properties still have other practical applications for interpreting changes in exponential and logarithmic relationships.

Example 4

Recall that in chemistry, $pH = -\log([H^+])$. If the concentration of hydrogen ions in a liquid is doubled, what is the affect on pH?

Solution:
Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid, so $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is $pH = -\log(2C)$.

Using the sum property of logs,

$$\begin{aligned} pH &= -\log(2C) \\ &= -(\log(2) + \log(C)) \\ &= -\log(2) - \log(C) \end{aligned}$$

Since $P = -\log(C)$, the new pH is

$$\begin{aligned} pH &= P - \log(2) \\ &= P - 0.301 \end{aligned}$$

When the concentration of hydrogen ions is doubled, the pH decreases by 0.301.

Log properties in solving equations

The logarithm properties often arise when solving problems involving logarithms. First, we'll look at a simpler log equation.

Example 5

Solve $\log(2x - 6) = 3$.

Solution:
To solve for x, we need to get it out from inside the log function. There are two ways we can approach this.

Method 1: Rewrite as an exponential.

Recall that since the common log is base 10, $\log(A) = B$ can be rewritten as the exponential $10^B = A$. Likewise, $\log(2x - 6) = 3$ can be rewritten in exponential form as

$$10^3 = 2x - 6$$

Method 2: Exponentiate both sides.

If $A = B$, then $10^A = 10^B$. Using this idea, since $\log(2x - 6) = 3$, then $10^{\log(2x-6)} = 10^3$. Use the inverse property of logs to rewrite the left side and get $2x - 6 = 10^3$.

Using either method, we now need to solve $2x - 6 = 10^3$. Evaluate 10^3 to get

$$\begin{aligned} 2x - 6 &= 1000 \\ 2x &= 1006 \quad \text{Add 6 to both sides} \\ x &= 503 \quad \text{Divide 2 on both sides} \end{aligned}$$

Occasionally the solving process will result in extraneous solutions – answers that are outside the domain of the original equation. In this case, our answer looks fine.

Example 6

Solve $\log(50x + 25) - \log(x) = 2$.

Solution:

In order to rewrite in exponential form, we need a single logarithmic expression on the left side of the equation. Using the difference property of logs, we can rewrite the left side:

$$\log\left(\frac{50x + 25}{x}\right) = 2$$

Rewriting in exponential form reduces this to an algebraic equation:

$$\begin{aligned} \frac{50x + 25}{x} &= 10^2 \\ 50x + 25 &= 100x \quad \text{Multiply both sides by x} \\ 25 &= 50x \quad \text{Subtract 50x on both sides} \\ x &= \frac{25}{50} \quad \text{Divide by 50} \\ x &= \frac{1}{2} \end{aligned}$$

Checking this answer in the original equation, we can verify there are no domain issues, and this answer is correct.

Try it Now

Solve $\log(x^2 - 4) = 1 + \log(x + 2)$.

Example 7

Solve $\ln(x+2)+\ln(x+1)=\ln(4x+14)$.

Solution:

$$\ln(x+2)+\ln(x+1) = \ln(4x+14)$$

$$\ln\big((x+2)(x+1)\big) = \ln(4x+14) \quad \text{Use the sum of logs property on the right}$$

$$\ln(x^2+3x+2) = \ln(4x+14) \quad \text{Expand}$$

We have a log on both side of the equation this time. Rewriting in exponential form would be tricky, so instead we can exponentiate both sides.

$$e^{\ln(x^2+3x+2)} = e^{\ln(4x+14)}$$

$$x^2+3x+2 = 4x+14 \quad \text{Use the inverse property of logs}$$

Now we ask ourselves our solving questions again.
What kind of equation is this? Quadratic.
What are our solving strategies? Either factoring and the ZPP, or the quadratic formula.
Either way, we need to move all the terms to one side so the equation is equal to 0.

$$x^2-x-12 = 0 \quad \text{Subtract 4x and 14 from both sides}$$

$$(x-4)(x+3) = 0 \quad \text{Factor}$$

$$x=4 \qquad x=-3 \quad \text{Solve with ZPP}$$

Checking our answers, notice that evaluating the original equation at $x=-3$ would result in us evaluating $\ln(-1)$, which is undefined. That answer is outside the domain of the original equation, so it is an extraneous solution and we discard it. There is one solution: $x=4$.

More complex exponential equations can often be solved in more than one way. In the following example, we will solve the same problem in two ways – one using logarithm properties, and the other using exponential properties.

Example 8a

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year[13]. If these trends continue, when will the population of Kenya match that of Sudan?

Solution:
We start by writing an equation for each population in terms of t, the number of years after 2008.

$$K(t)=38.8(1+0.0264)^t$$

$$S(t)=41.3(1+0.0224)^t$$

---

[13] World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 24, 2010

To find when the populations will be equal, we can set the equations equal

$$38.8(1+0.0264)^t = 41.3(1+0.0224)^t$$

For our first approach, we take the log of both sides of the equation.

$$\log(38.8(1+0.0264)^t) = \log(41.3(1+0.0224)^t)$$

Utilizing the sum property of logs, we can rewrite each side,

$$\log(38.8) + \log(1.0264^t) = \log(41.3) + \log(1.0224^t)$$

Then utilizing the exponent property, we can pull the variables out of the exponent

$$\log(38.8) + t \cdot \log(1.0264) = \log(41.3) + t \cdot \log(1.0224)$$

Moving all the terms involving t to one side of the equation and the rest of the terms to the other side,

$$t \cdot \log(1.0264) - t \cdot \log(1.0224) = \log(41.3) - \log(38.8)$$

Factoring out the t on the left,

$$t(\log(1.0264) - \log(1.0224)) = \log(41.3) - \log(38.8)$$

Dividing to solve for t

$$t = \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)} \approx 15.991$$

In approximately 15.991 years the populations will be equal.

Example 8b

Solve the problem above by rewriting before taking the log.

Solution:

Starting at the equation

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

Divide to move the exponential terms to one side of the equation and the constants to the other side

$$\frac{1.0264^t}{1.0224^t} = \frac{41.3}{38.8}$$

Using exponent rules to group on the left,

$$\left(\frac{1.0264}{1.0224}\right)^t = \frac{41.3}{38.8}$$

Taking the log of both sides

$$\log\left(\left(\frac{1.0264}{1.0224}\right)^t\right) = \log\left(\frac{41.3}{38.8}\right)$$

Utilizing the exponent property on the left,

$$t \cdot \log\left(\frac{1.0264}{1.0224}\right) = \log\left(\frac{41.3}{38.8}\right)$$

Dividing gives

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} \approx 15.991$$

While the answer does not immediately appear identical to that produced using the previous method, note that by using the difference property of logs, the answer could be rewritten:

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)}$$
$$= \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)}$$

While both methods work equally well, it often requires fewer steps to utilize algebra before taking logs, rather than relying solely on log properties.

Try it Now

Tank A contains 10 liters of water, and 35% of the water evaporates each week. Tank B contains 30 liters of water, and 50% of the water evaporates each week. In how many weeks will the tanks contain the same amount of water?

Important Topics of this Section
Inverse
Exponential
Change of base
Sum of logs property
Difference of logs property
Solving equations using log rules

Section 3.4 Exercises

Conceptual Questions

1. What is the sum of logs property and why does it work?
2. What is the difference of logs property and why does it work?
3. What are some strategies for solving exponential equations?
4. What are some strategies for equations involving logarithms?

Practice Problems

Simplify to a single logarithm, using logarithm properties.

1. $\log_3(28) - \log_3(7)$
2. $\log_3(32) - \log_3(4)$
3. $-\log_3\left(\frac{1}{7}\right)$
4. $-\log_4\left(\frac{1}{5}\right)$
5. $\log_3\left(\frac{1}{10}\right) + \log_3(50)$
6. $\log_4(3) + \log_4(7)$
7. $\frac{1}{3}\log_7(8)$
8. $\frac{1}{2}\log_5(36)$
9. $\log(2x^4) + \log(3x^5)$
10. $\ln(4x^2) + \ln(3x^3)$
11. $\ln(6x^9) - \ln(3x^2)$
12. $\log(12x^4) - \log(4x)$
13. $2\log(x) + 3\log(x+1)$
14. $3\log(x) + 2\log(x^2)$
15. $\log(x) - \frac{1}{2}\log(y) + 3\log(z)$
16. $2\log(x) + \frac{1}{3}\log(y) - \log(z)$

Use logarithm properties to expand each expression.

17. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$
18. $\log\left(\frac{a^2b^3}{c^5}\right)$
19. $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$
20. $\ln\left(\frac{a^{-2}b^3}{c^{-5}}\right)$
21. $\log\left(\sqrt{x^3y^{-4}}\right)$
22. $\log\left(\sqrt{x^{-3}y^2}\right)$
23. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$
24. $\ln\left(\frac{x}{\sqrt{1-x^2}}\right)$
25. $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$
26. $\log\left(x^3y^4\sqrt[7]{x^3y^9}\right)$

Solve each equation for the variable.

27. $4^{4x-7} = 3^{9x-6}$
28. $2^{2x-5} = 7^{3x-7}$
29. $17(1.14)^x = 19(1.16)^x$
30. $20(1.07)^x = 8(1.13)^x$
31. $5e^{0.12t} = 10e^{0.08t}$
32. $3e^{0.09t} = e^{014t}$
33. $\log_2(7x+6) = 3$
34. $\log_3(2x+4) = 2$
35. $2\ln(3x) + 3 = 1$
36. $4\ln(5x) + 5 = 2$
37. $\log(x^3) = 2$
38. $\log(x^5) = 3$

39. $\log(x) + \log(x+3) = 3$

40. $\log(x+4) + \log(x) = 9$

Solve each equation for the variable.

41. $\log(x+4) - \log(x+3) = 1$

42. $\log(x+5) - \log(x+2) = 2$

43. $\log_6(x^2) - \log_6(x+1) = 1$

44. $\log_3(x^2) - \log_3(x+2) = 5$

45. $\log(x+12) = \log(x) + \log(12)$

46. $\log(x+15) = \log(x) + \log(15)$

47. $\ln(x) + \ln(x-3) = \ln(7x)$

48. $\ln(x) + \ln(x-6) = \ln(6x)$

Section 3.5 Exponential and Logarithmic Models

While we have explored some basic applications of exponential and logarithmic functions, in this section we explore some important applications in more depth.

Radioactive Decay

In an earlier section, we discussed radioactive decay – the idea that radioactive isotopes change over time. One of the common terms associated with radioactive decay is half-life.

Half Life

The **half-life** of a radioactive isotope is the time it takes for half the substance to decay.

Given the basic exponential growth/decay equation $h(t) = an^t$, half-life can be found by solving for when half the original amount remains; by solving $\frac{1}{2}a = a(n)^t$, or more simply $\frac{1}{2} = n^t$. Notice how the initial amount is irrelevant when solving for half-life.

Example 1

Bismuth-210 is an isotope that decays by about 13% each day. What is the half-life of Bismuth-210?

Solution:

We were not given a starting quantity, so we could either make up a value or use an unknown constant to represent the starting amount. To show that starting quantity does not affect the result, let us denote the initial quantity by the constant *a*. Then the decay of Bismuth-210 can be described by the equation $Q(d) = a(0.87)^d$.

To find the half-life, we want to determine when the remaining quantity is half the original: $\frac{1}{2}a$. Solving,

$$\begin{aligned}
\frac{1}{2}a &= a(0.87)^d \\
\frac{1}{2} &= 0.87^d && \text{Divide by } a \\
\log\left(\frac{1}{2}\right) &= \log(0.87^d) && \text{Take the log of both sides} \\
\log\left(\frac{1}{2}\right) &= d \cdot \log(0.87) && \text{Use the exponent property of logs} \\
d &= \frac{\log\left(\frac{1}{2}\right)}{\log(0.87)} && \text{Divide to solve for } d \\
d &\approx 4.977 \text{ days}
\end{aligned}$$

This tells us that the half-life of Bismuth-210 is approximately 5 days.

Example 2

Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, how much will remain after 30 years? 60 years? 90 years?

Solution:

Since the half-life is 30 years, after 30 years, half the original amount, 100mg, will remain.

After 60 years, another 30 years have passed, so during that second 30 years, another half of the substance will decay, leaving 50mg.

After 90 years, another 30 years have passed, so another half of the substance will decay, leaving 25mg.

Example 3

Cesium-137 has a half-life of about 30 years. Find the annual decay rate.

Solution:

Since we are looking for an annual decay rate, we will use an equation of the form $Q(t) = a(1 + r)^t$. We know that after 30 years, half the original amount will remain. Using this information

$$\frac{1}{2}a = a(1+r)^{30}$$

$$\frac{1}{2} = (1+r)^{30} \qquad \text{Dividing by } a$$

$$\sqrt[30]{\frac{1}{2}} = 1+r \qquad \text{Taking the 30th root of both sides}$$

$$r = \sqrt[30]{\frac{1}{2}} - 1 \qquad \text{Subtracting one from both sides}$$

$$r \approx -0.02284$$

This tells us cesium-137 is decaying at an annual rate of 2.284% per year.

Try it Now

Chlorine-36 is eliminated from the body with a biological half-life of 10 days[14]. Find the daily decay rate.

Example 4

Carbon-14 is a radioactive isotope that is present in organic materials, and is commonly used for dating historical artifacts. Carbon-14 has a half-life of 5730 years. If a bone fragment is found that contains 20% of its original carbon-14, how old is the bone?

Solution:

To find how old the bone is, we first will need to find an equation for the decay of the carbon-14. We could either use a continuous or annual decay formula, but opt to use the continuous decay formula since it is more common in scientific texts. The half life tells us that after 5730 years, half the original substance remains. Solving for the rate,

$$\frac{1}{2}a = ae^{r5730}$$

$$\frac{1}{2} = e^{5730r} \qquad \text{Dividing by } a$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{5730r}\right) \qquad \text{Taking the natural log of both sides}$$

$$\ln\left(\frac{1}{2}\right) = 5730r \qquad \text{Use the inverse property of logs on the right side}$$

$$r = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -0.000121$$

[14] http://www.ead.anl.gov/pub/doc/chlorine.pdf

Now we know the decay will follow the equation $Q(t) = ae^{-0.000121t}$. To find how old the bone fragment is that contains 20% of the original amount, we solve for t so that $Q(t) = 0.20a$.

$$0.20a = ae^{-0.000121t}$$
$$0.20 = e^{-0.000121t}$$
$$\ln(0.20) = \ln(e^{-0.000121t})$$
$$\ln(0.20) = -0.000121t$$
$$t = \frac{\ln(0.20)}{-0.000121} \approx 13301 \text{ years}$$

The bone fragment is about 13,300 years old.

Try it Now

In Example 2, we learned that Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Doubling Time

For decaying quantities, we asked how long it takes for half the substance to decay. For growing quantities we might ask how long it takes for the quantity to double.

Doubling Time

The **doubling time** of a growing quantity is the time it takes for the quantity to double.

Given the basic exponential growth equation $h(t) = an^t$, doubling time can be found by solving for when the original quantity has doubled; by solving $2a = a(n)^t$, or more simply $2 = n^t$. Like with decay, the initial amount is irrelevant when solving for doubling time.

Example 5

Cancer cells sometimes increase exponentially. If a cancerous growth contained 300 cells last month and 360 cells this month, how long will it take for the number of cancer cells to double?

Solution:
Defining t to be time in months, with $t = 0$ corresponding to this month, we are given two pieces of data: this month, (0, 360), and last month, (-1, 300).

From this data, we can find an equation for the growth. Using the form $C(t) = an^t$, we know immediately $a = 360$, giving $C(t) = 360n^t$. Substituting in (-1, 300),

$$300 = 360n^{-1}$$
$$300 = \frac{360}{n}$$
$$n = \frac{360}{300} = 1.2$$

This gives us the equation $C(t) = 360(1.2)^t$.

To find the doubling time, we look for the time when we will have twice the original amount, so when $C(t) = 2a$.

$$2a = a(1.2)^t$$
$$2 = (1.2)^t$$
$$\log(2) = \log(1.2^t)$$
$$\log(2) = t \cdot \log(1.2)$$
$$t = \frac{\log(2)}{\log(1.2)} \approx 3.802$$

It will take about months for the number of cancer cells to double.

Example 6

Use of a new social networking website has been growing exponentially, with the number of new members doubling every 5 months. If the site currently has 120,000 users and this trend continues, how many users will the site have in 1 year?

Solution:
We can use the doubling time to find a function that models the number of site users, and then use that equation to answer the question. While we could use an arbitrary a as we have before for the initial amount, in this case, we know the initial amount was 120,000.

If we use a continuous growth equation, it would look like $N(t) = 120e^{rt}$, measured in thousands of users after t months. Based on the doubling time, there would be 240 thousand users after 5 months. This allows us to solve for the continuous growth rate:

$$240 = 120e^{r5}$$
$$2 = e^{r5}$$
$$\ln(2) = \ln(e^{5r})$$

$$\ln(2) = 5r$$
$$r = \frac{\ln(2)}{5} \approx 0.1386$$

Now that we have an equation, $N(t) = 120e^{0.1386t}$, we can predict the number of users after 12 months:

$$N(12) = 120e^{0.1386 \cdot 12}$$
$$\approx 633.140 \text{ thousand users}$$

So after 1 year, we would expect the site to have around 633,140 users.

Try it Now

If tuition at a college is increasing by 6.6% each year, how many years will it take for tuition to double?

Newton's Law of Cooling

When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off towards the surrounding air temperature. This "leveling off" will correspond to a horizontal asymptote in the graph of the temperature function. Unless the room temperature is zero, this will correspond to a vertical shift of the generic exponential decay function.

Newton's Law of Cooling

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$T(t) = ae^{kt} + T_S$

Where

t is time

a is a constant determined by the initial temperature of the object

k is a constant, the continuous rate of cooling of the object

While an equation of the form $T(t) = ab^t + T_S$ could be used, the continuous growth form is more common.

Example 7

A cheesecake is taken out of the oven with an ideal internal temperature of 165 degrees Fahrenheit, and is placed into a 35 degree refrigerator. After 10 minutes, the cheesecake has cooled to 150 degrees. If you must wait until the cheesecake has cooled to 70 degrees before you eat it, how long will you have to wait?

Solution:
Since the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially towards 35, following the equation

$$T(t) = ae^{kt} + 35$$

We know the initial temperature was 165, so $T(0) = 165$. Substituting in these values,

$$165 = ae^{k\cdot 0} + 35$$
$$165 = a + 35$$
$$a = 130$$

We were given another pair of data, $T(10) = 150$, which we can use to solve for k

$$150 = 130e^{k\cdot 10} + 35$$
$$115 = 130e^{10k}$$
$$\frac{115}{130} = e^{10k}$$
$$\ln\left(\frac{115}{130}\right) = 10k$$
$$k = \frac{\ln\left(\frac{115}{130}\right)}{10} = -0.0123$$

Together this gives us the equation for cooling: $T(t) = 130e^{(-0.0123t)} + 35$.

Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$70 = 130e^{-0.0123t} + 35$$
$$35 = 130e^{-0.0123t}$$
$$\frac{35}{130} = e^{-0.0123t}$$
$$\ln\left(\frac{35}{130}\right) = -0.0123t$$
$$t = \frac{\ln\left(\frac{35}{130}\right)}{-0.0123} \approx 106.68$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool. Of course, if you like your cheesecake served chilled, you'd have to wait a bit longer.

Try it Now

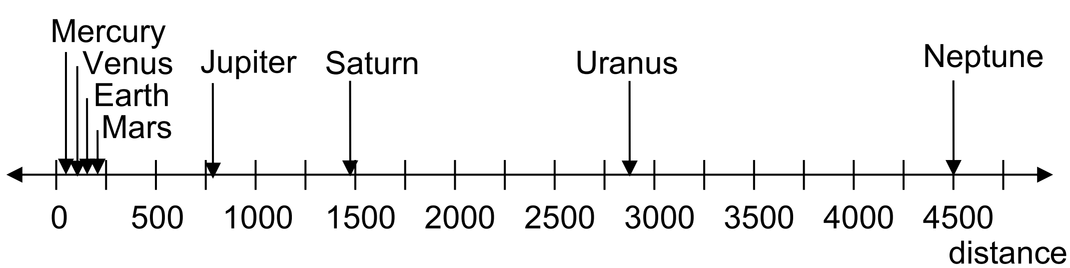
A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Logarithmic Scales

For quantities that vary greatly in magnitude, a standard scale of measurement is not always effective, and utilizing logarithms can make the values more manageable. For example, if the average distances from the sun to the major bodies in our solar system are listed, you see they vary greatly.

Planet	**Distance (millions of km)**
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	779
Saturn	1430
Uranus	2880
Neptune	4500

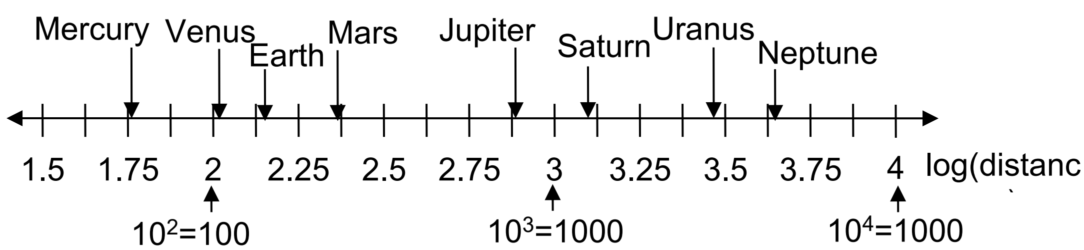
Placed on a linear scale – one with equally spaced values – these values get bunched up.



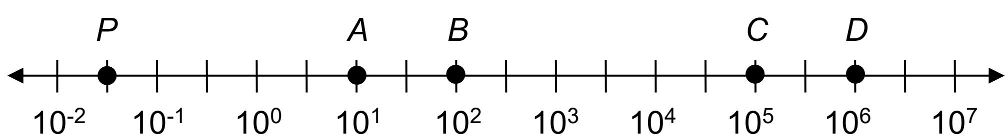
However, computing the logarithm of each value and plotting these new values on a number line results in a more manageable graph, and makes the relative distances more apparent.[15]

[15] It is interesting to note the large gap between Mars and Jupiter on the log number line. The asteroid belt is located there, which scientists believe is a planet that never formed because of the effects of the gravity of Jupiter.

Planet	**Distance (millions of km)**	**log(distance)**
Mercury	58	1.76
Venus	108	2.03
Earth	150	2.18
Mars	228	2.36
Jupiter	779	2.89
Saturn	1430	3.16
Uranus	2880	3.46
Neptune	4500	3.65



Sometimes, as shown above, the scale on a logarithmic number line will show the log values, but more commonly the original values are listed as powers of 10, as shown below.



Example 8

Estimate the value of point P on the log scale above

Solution:

The point P appears to be half way between -2 and -1 in log value, so if V is the value of this point,

$$\log(V) \approx -1.5$$

Rewriting in exponential form,

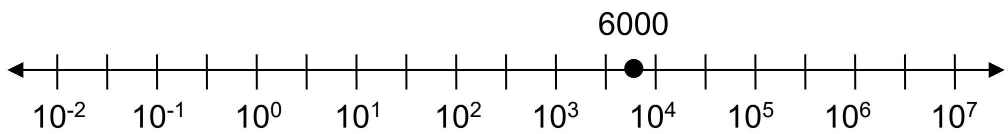
$$V \approx 10^{-1.5} = 0.0316$$

Example 9

Place the number 6000 on a logarithmic scale.

Solution:

Since $\log(6000) \approx 3.8$, this point would belong on the log scale about here:



Try it Now

Plot the data in the table below on a logarithmic scale[16].

Source of Sound/Noise	Approximate Sound Pressure in μPa (micro Pascals)
Launching of the Space Shuttle	2,000,000,000
Full Symphony Orchestra	2,000,000
Diesel Freight Train at High Speed at 25 m	200,000
Normal Conversation	20,000
Soft Whispering at 2 m in Library	2,000

Notice that on the log scale above Example 8, the visual distance on the scale between points *A* and *B* and between *C* and *D* is the same. When looking at the values these points correspond to, notice *B* is ten times the value of *A*, and *D* is ten times the value of *C*. A visual *linear* difference between points corresponds to a *relative* (ratio) change between the corresponding values.

Logarithms are useful for showing these relative changes. For example, comparing \$1,000,000 to \$10,000, the first is 100 times larger than the second.

$$\frac{1{,}000{,}000}{10{,}000} = 100 = 10^2$$

Likewise, comparing \$1000 to \$10, the first is 100 times larger than the second.

$$\frac{1{,}000}{10} = 100 = 10^2$$

When one quantity is roughly ten times larger than another, we say it is one **order of magnitude** larger. In both cases described above, the first number was two orders of magnitude larger than the second.

[16] From http://www.epd.gov.hk/epd/noise_education/web/ENG_EPD_HTML/m1/intro_5.html, retrieved Oct 2, 2010

Notice that the order of magnitude can be found as the common logarithm of the ratio of the quantities. On the log scale above, *B* is one order of magnitude larger than *A*, and *D* is one order of magnitude larger than *C*.

Orders of Magnitude

Given two values A and B, to determine how many **orders of magnitude** A is greater than B,

Difference in orders of magnitude $= \log\left(\frac{A}{B}\right)$

Example 10

On the log scale above Example 8, how many orders of magnitude larger is C than B?

Solution:
The value B corresponds to $10^2 = 100$
The value C corresponds to $10^5 = 100{,}000$

The relative change is $\frac{100{,}000}{100} = 1000 = \frac{10^5}{10^2} = 10^3$. The log of this value is 3.

C is three orders of magnitude greater than B, which can be seen on the log scale by the visual difference between the points on the scale.

Try it Now

Using the table from the previous Try it Now, what is the difference of order of magnitude between the softest sound a human can hear and the launching of the space shuttle?

Earthquakes

An example of a logarithmic scale is the Moment Magnitude Scale (MMS) used for earthquakes. This scale is commonly and mistakenly called the Richter Scale, which was a very similar scale succeeded by the MMS.

Moment Magnitude Scale

For an earthquake with seismic moment S, a measurement of earth movement, the MMS value, or magnitude of the earthquake, is

$M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$

Where $S_0 = 10^{16}$ is a baseline measure for the seismic moment.

Example 11

If one earthquake has a MMS magnitude of 6.0, and another has a magnitude of 8.0, how much more powerful (in terms of earth movement) is the second earthquake?

Solution:

Since the first earthquake has magnitude 6.0, we can find the amount of earth movement for that quake, which we'll denote S_1. The value of S_0 is not particularly relevant, so we will not replace it with its value.

$$6.0 = \frac{2}{3}\log\left(\frac{S_1}{S_0}\right)$$

$$6.0\left(\frac{3}{2}\right) = \log\left(\frac{S_1}{S_0}\right)$$

$$9 = \log\left(\frac{S_1}{S_0}\right)$$

$$\frac{S_1}{S_0} = 10^9$$

$$S_1 = 10^9 S_0$$

This tells us the first earthquake has about 10^9 times more earth movement than the baseline measure.

Doing the same with the second earthquake, S_2, with a magnitude of 8.0,

$$8.0 = \frac{2}{3}\log\left(\frac{S_2}{S_0}\right)$$

$$S_2 = 10^{12} S_0$$

Comparing the earth movement of the second earthquake to the first,

$$\frac{S_2}{S_1} = \frac{10^{12} S_0}{10^9 S_0} = 10^3 = 1000$$

The second value's earth movement is 1000 times as large as the first earthquake.

Example 12

One earthquake has magnitude of 3.0. If a second earthquake has twice as much earth movement as the first earthquake, find the magnitude of the second quake.

Since the first quake has magnitude 3.0,

$$3.0 = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$

Solving for S,

$$3.0\left(\frac{3}{2}\right) = \log\left(\frac{S}{S_0}\right)$$

$$4.5 = \log\left(\frac{S}{S_0}\right)$$

$$10^{4.5} = \frac{S}{S_0}$$

$$S = 10^{4.5}S_0$$

Since the second earthquake has twice as much earth movement, for the second quake,

$$S = 2 \cdot 10^{4.5}S_0$$

Finding the magnitude,

$$M = \frac{2}{3}\log\left(\frac{2 \cdot 10^{4.5}S_0}{S_0}\right)$$

$$M = \frac{2}{3}\log\left(2 \cdot 10^{4.5}\right) \approx 3.201$$

The second earthquake with twice as much earth movement will have a magnitude of about 3.2.

In fact, using log properties, we could show that whenever the earth movement doubles, the magnitude will increase by about 0.201:

$$M = \frac{2}{3}\log\left(\frac{2S}{S_0}\right) = \frac{2}{3}\log\left(2 \cdot \frac{S}{S_0}\right)$$

$$M = \frac{2}{3}\left(\log(2) + \log\left(\frac{S}{S_0}\right)\right)$$

$$M = \frac{2}{3}\log(2) + \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$

$$M = 0.201 + \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$

This illustrates the most important feature of a log scale: that *multiplying* the quantity being considered will *add* to the scale value, and vice versa.

Important Topics of this Section
Radioactive decay
Half life
Doubling time
Newton's law of cooling
Logarithmic Scales
Orders of Magnitude
Moment Magnitude scale

Section 3.5 Exercises

Conceptual Questions

1. Given a half-life, how can we find k, the continuous growth rate?
2. How are the equations $f(t) = ae^{kt}$ and $f(t) = a(1+r)^k$ related?
3. How does a logarithmic scale work?

Practice Problems

1. You go to the doctor and he injects you with 13 milligrams of radioactive dye. After 12 minutes, 4.75 milligrams of dye remain in your system. To leave the doctor's office, you must pass through a radiation detector without sounding the alarm. If the detector will sound the alarm whenever more than 2 milligrams of the dye are in your system, how long will your visit to the doctor take, assuming you were given the dye as soon as you arrived and the amount of dye decays exponentially?

2. You take 200 milligrams of a headache medicine, and after 4 hours, 120 milligrams remain in your system. If the effects of the medicine wear off when less than 80 milligrams remain, when will you need to take a second dose, assuming the amount of medicine in your system decays exponentially?

3. The half-life of Radium-226 is 1590 years. If a sample initially contains 200 mg, how many milligrams will remain after 1000 years?

4. The half-life of Fermium-253 is 3 days. If a sample initially contains 100 mg, how many milligrams will remain after 1 week?

5. The half-life of Erbium-165 is 10.4 hours. After 24 hours a sample still contains 2 mg. What was the initial mass of the sample, and how much will remain after another 3 days?

6. The half-life of Nobelium-259 is 58 minutes. After 3 hours a sample still contains10 mg. What was the initial mass of the sample, and how much will remain after another 8 hours?

7. A scientist begins with 250 grams of a radioactive substance. After 225 minutes, the sample has decayed to 32 grams. Find the half-life of this substance.

8. A scientist begins with 20 grams of a radioactive substance. After 7 days, the sample has decayed to 17 grams. Find the half-life of this substance.

9. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. How long ago was the artifact made? (The half-life of carbon-14 is 5730 years.)

10. A wooden artifact from an archeological dig contains 15 percent of the carbon-14 that is present in living trees. How long ago was the artifact made? (The half-life of carbon-14 is 5730 years.)

11. A bacteria culture initially contains 1500 bacteria and doubles in size every half hour. Find the size of the population after: a) 2 hours b) 100 minutes

12. A bacteria culture initially contains 2000 bacteria and doubles in size every half hour. Find the size of the population after: a) 3 hours b) 80 minutes

13. The count of bacteria in a culture was 800 after 10 minutes and 1800 after 40 minutes.
    a. What was the initial size of the culture?
    b. Find the doubling time.
    c. Find the population after 105 minutes.
    d. When will the population reach 11000?

14. The count of bacteria in a culture was 600 after 20 minutes and 2000 after 35 minutes.
    a. What was the initial size of the culture?
    b. Find the doubling time.
    c. Find the population after 170 minutes.
    d. When will the population reach 12000?

15. Find the time required for an investment to double in value if invested in an account paying 3% compounded quarterly.

16. Find the time required for an investment to double in value if invested in an account paying 4% compounded monthly

17. The number of crystals that have formed after *t* hours is given by $n(t) = 20e^{0.013t}$ How long does it take the number of crystals to double?

18. The number of building permits in Pasco *t* years after 1992 roughly followed the equation $n(t) = 400e^{0.143t}$. What is the doubling time?

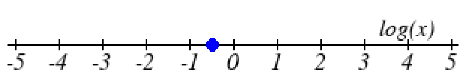
19. A turkey is pulled from the oven when the internal temperature is 165° Fahrenheit, and is allowed to cool in a 75° room. If the temperature of the turkey is 145° after half an hour,
    a. What will the temperature be after 50 minutes?
    b. How long will it take the turkey to cool to 110°?

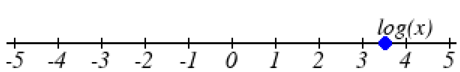
20. A cup of coffee is poured at 190° Fahrenheit, and is allowed to cool in a 70° room. If the temperature of the coffee is 170° after half an hour,
    a. What will the temperature be after 70 minutes?
    b. How long will it take the coffee to cool to 120°?

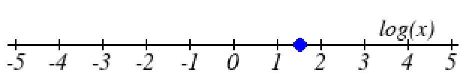
21. The population of fish in a farm-stocked lake after *t* years could be modeled by the equation $P(t) = \frac{1000}{1+9e^{-0.6t}}$.
    a. Sketch a graph of this equation.
    b. What is the initial population of fish?
    c. What will the population be after 2 years?
    d. How long will it take for the population to reach 900?

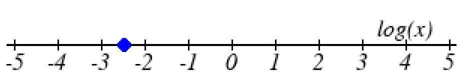
22. The number of people in a town who have heard a rumor after *t* days can be modeled by the equation $N(t) = \frac{500}{1+49e^{-0.7t}}$.
    a. Sketch a graph of this equation.
    b. How many people started the rumor?
    c. How many people have heard the rumor after 3 days?
    d. How long will it take until 300 people have heard the rumor?

Find the value of the number shown on each logarithmic scale

23. 

24. 

25. 

26. 

Plot each set of approximate values on a logarithmic scale.

27. Intensity of sounds: Whisper: $10^{-10}\ W/m^2$, Vacuum: $10^{-4}\ W/m^2$, Jet: $10^2 W/m^2$

28. Mass: Amoeba: $10^{-5} g$, Human: $10^5 g$, Statue of Liberty: $10^8 g$

29. The 1906 San Francisco earthquake had a magnitude of 7.9 on the MMS scale. Later there was an earthquake with magnitude 4.7 that caused only minor damage. How many times more intense was the San Francisco earthquake than the second one?

30. The 1906 San Francisco earthquake had a magnitude of 7.9 on the MMS scale. Later there was an earthquake with magnitude 6.5 that caused less damage. How many times more intense was the San Francisco earthquake than the second one?

31. One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake.

32. One earthquake has magnitude 4.8 on the MMS scale. If a second earthquake has 1200 times as much energy as the first, find the magnitude of the second quake.

33. A colony of yeast cells is estimated to contain 10^6 cells at time $t = 0$. After collecting experimental data in the lab, you decide that the total population of cells at time *t* hours is given by the function $f(t) = 10^6 e^{0.495105t}$. [UW]
    a. How many cells are present after one hour?
    b. How long does it take of the population to double?.
    c. Cherie, another member of your lab, looks at your notebook and says: "That formula is wrong, my calculations predict the formula for the number of yeast cells is given by the function. $f(t) = 10^6(2.042727)^{0.693147t}$." Should you be worried by Cherie's remark?
    d. Anja, a third member of your lab working with the same yeast cells, took these two measurements: 7.246×10^6 cells after 4 hours; 16.504×10^6 cells after 6 hours. Should you be worried by Anja's results? If Anja's measurements are correct, does your model over estimate or under estimate the number of yeast cells at time *t*?

34. As light from the surface penetrates water, its intensity is diminished. In the clear waters of the Caribbean, the intensity is decreased by 15 percent for every 3 meters of depth. Thus, the intensity will have the form of a general exponential function. [UW]

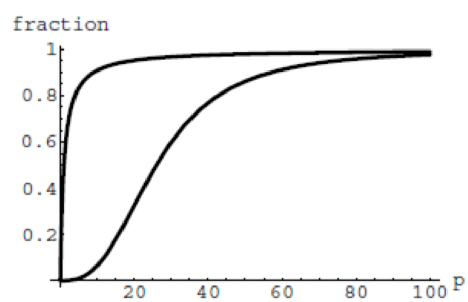
    a. If the intensity of light at the water's surface is I_0, find a formula for $I(d)$, the intensity of light at a depth of *d* meters. Your formula should depend on I_0 and *d*.

    b. At what depth will the light intensity be decreased to 1% of its surface intensity?

35. Myoglobin and hemoglobin are oxygen-carrying molecules in the human body. Hemoglobin is found inside red blood cells, which flow from the lungs to the muscles through the bloodstream. Myoglobin is found in muscle cells. The function $Y = M(p) = \frac{p}{1+p}$ calculates the fraction of myoglobin saturated with oxygen at a given pressure *p* Torrs. For example, at a pressure of 1 Torr, *M*(1) = 0.5, which means half of the myoglobin (i.e. 50%) is oxygen saturated. (Note: More precisely, you need to use something called the "partial pressure", but the distinction is not important for this problem.) Likewise, the function $Y = H(p) = \frac{p^{2.8}}{26^{2.5}+p^{2.8}}$ calculates the fraction of hemoglobin saturated with oxygen at a given pressure *p*. [UW]

    a. The graphs of $M(p)$ and $H(p)$ are given here on the domain $0 \leq p \leq 100$; which is which?

    b. If the pressure in the lungs is 100 Torrs, what is the level of oxygen saturation of the hemoglobin in the lungs?



    c. The pressure in an active muscle is 20 Torrs. What is the level of oxygen saturation of myoglobin in an active muscle? What is the level of hemoglobin in an active muscle?

    d. Define the efficiency of oxygen transport at a given pressure *p* to be $M(p) - H(p)$. What is the oxygen transport efficiency at 20 Torrs? At 40 Torrs? At 60 Torrs? Sketch the graph of $M(p) - H(p)$.; are there conditions under which transport efficiency is maximized (explain)?

36. The length of some fish are modeled by a von Bertalanffy growth function. For Pacific halibut, this function has the form $L(t) = 200(1 - 0.957e^{-0.18t})$ where $L(t)$ is the length (in centimeters) of a fish *t* years old. [UW]
    a. What is the length of a newborn halibut at birth?
    b. Use the formula to estimate the length of a 6–year–old halibut.
    c. At what age would you expect the halibut to be 120 cm long?
    d. What is the practical (physical) significance of the number 200 in the formula for $L(t)$?

37. A cancer cell lacks normal biological growth regulation and can divide continuously. Suppose a single mouse skin cell is cancerous and its mitotic cell cycle (the time for the cell to divide once) is 20 hours. The number of cells at time *t* grows according to an exponential model. [UW]
    a. Find a formula $C(t)$ for the number of cancerous skin cells after *t* hours.
    b. Assume a typical mouse skin cell is spherical of radius 50×10^{-4} cm. Find the combined volume of all cancerous skin cells after *t* hours. When will the volume of cancerous cells be 1 cm^3?

38. A ship embarked on a long voyage. At the start of the voyage, there were 500 ants in the cargo hold of the ship. One week into the voyage, there were 800 ants. Suppose the population of ants is an exponential function of time. [UW]
    a. How long did it take the population to double?
    b. How long did it take the population to triple?
    c. When were there be 10,000 ants on board?
    d. There also was an exponentially growing population of anteaters on board. At the start of the voyage there were 17 anteaters, and the population of anteaters doubled every 2.8 weeks. How long into the voyage were there 200 ants per anteater?

39. The populations of termites and spiders in a certain house are growing exponentially. The house contains 100 termites the day you move in. After 4 days, the house contains 200 termites. Three days after moving in, there are two times as many termites as spiders. Eight days after moving in, there were four times as many termites as spiders. How long (in days) does it take the population of spiders to triple? [UW]

Appendix: Selected Solutions

Section 0.1

1. a, c, e, h, i
3. Student answers may vary
5. Student answers may vary
7. $-5ab^2 + 14a^{2b} - 2b + 3a$
9. $12x$
11. $-2x^2 - 12x - 11$
13. $3x + 7$
15. $9 - 8x$
17. 14
19. -66
21. 20
23. -44
25. -2
27. c
29. b
31. $x = -3$
33. $x = 2$
35. $b = -4$
37. $x = 0$
39. $x = 0, 5$
41. $x = -2, 1$
43. $x = 0, 5, -7$

Section 0.2

1. $\frac{u^{12}}{9v^{12}}$
3. $12n^5$
5. vu^8
7. $\frac{n^{12}}{m^{84}}$
9. $20x^3y^8$
11. $x^{17}y^{12}$
13. $\frac{n^9}{m^{17}}$
15. p^{15}
17. $8x^{11}y^2$
19. $\frac{2y^5}{x^5}$
21. $x^{15}y^{16}$
23. y^{12}
25. $3x^2$
27. $\frac{4u}{v^2}$
29. $\frac{v^2}{4u^7}$
31. $8u^8v^5$
33. $x^{39}y^{108}$

Section 0.3

1. $(7x - 1)(x + 7)$
3. $(2b + 13)(b - 19)$
5. $(10v - 9)(9v - 10)$
7. $(2n + 5)(n - 3)$
9. $(5y + 14)(5y - 14)$
11. $(2m + 3)(2m - 3)$
13. $(18x + 11)(18x - 11)$
15. $(4a - 1)^2$
17. $(11x - 4)^2$
19. $(m - 10)^2$
21. $(6q + 5)^2$

Section 0.4

1. $x = -\frac{11}{4}$
3. $x = -\frac{3}{2}, \frac{3}{4}$
5. $x = -\frac{11}{2}$
7. $x = \frac{2}{7}$
9. $x = -\frac{5}{2}$
11. $x = -7, 3$
13. $x = 6$
15. $x = 7$
17. $x = -2 \pm \sqrt{2}$
19. $x = 3$
21. $x = 12$
23. $x = -\frac{1}{2} \pm \frac{\sqrt{17}}{2}$
25. $x = -5, \frac{1}{2}$
27. $x = -\frac{5}{2}, -\frac{1}{3}$
29. $x = 3, 7$

Section 1.1

1. (a) $f(40) = 13$, because the input 40 (in thousands of people) gives the output 13 (in tons of garbage)
3. (a) In 1995 (5 years after 1990) there were 30 ducks in the lake.
   (b) In 2000 (10 years after 1990) there were 40 ducks in the lake.
5. Graphs (a) (b) (d) and (e) represent *y* as a function of *x* because for every value of *x* there is only one value for *y*. Graphs (c) and (f) are not functions because they contain points that have more than one output for a given input, or values for *x* that have 2 or more values for *y*.
7. Tables (a) and (b) represent *y* as a function of *x* because for every value of *x* there is only one value for *y*. Table (c) is not a function because for the input x=10, there are two different outputs for *y*.
9. Tables (a) (b) and (d) represent *y* as a function of *x* because for every value of *x* there is only one value for *y*. Table (c) is not a function because for the input *x*=3, there are two different outputs for *y*.
11. Table (b) represents *y* as a function of *x* and is one-to-one because there is a unique output for every input, and a unique input for every output. Table (a) is not one-to-one because two different inputs give the same output, and table (c) is not a function because there are two different outputs for the same input x=8.
13. Graphs (b) (c) (e) and (f) are one-to-one functions because there is a unique input for every output. Graph (a) is not a function, and graph (d) is not one-to-one because it contains points which have the same output for two different inputs.
15. (a) $f(1) = 1$ (b) $f(3) = 1$
17. (a) $g(2) = 4$ (b) $g(-3) = 2$
19. (a) $f(3) = 53$ (b) $f(2) = 1$
21. $f(-2) = 4 - 2(-2) = 4 + 4 = 8, f(-1) = 6, f(0) = 4, f(1) = 4 - 2(1) = 4 - 2 = 2, f(2) = 0$, $f(a+h) = 4 - 2a - 2h$, $f(a+h) - f(a) = -2h$
23. $f(-2) = 8(-2)^2 - 7(-2) + 3 = 8(4) + 14 + 3 = 32 + 14 + 3 = 49, f(-1) = 18, f(0) = 3, f(1) = 8(1)^2 - 7(1) + 3 = 8 - 7 + 3 = 4, f(2) = 21$, $f(a+h) = 8a^2 + 16ah + 8h^2 - 7a - 7h + 3$, $f(a+h) - f(a) = 16ah + 8h^2 - 7h$
25. $f(-2) = -(-2)^3 + 2(-2) = -(-8) - 4 = 8 - 4 = 4, f(-1) = -(-1)^3 + 2(-1) = -(-1) - 2 = -1, f(0) = 0, f(1) = -(1)^3 + 2(1) = 1, f(2) = -4$, $f(a+h) = -a^3 - 3a^2h - 3ah^2 + 2a -$

$h^3 + 2h, f(a+h) - f(a) = -3a^2h - 3ah^2 + 2h$

27. $f(-2) = 3 + \sqrt{(-2)+3} = 3 + \sqrt{1} = 3 + 1 = 4, f(-1) = \sqrt{2} + 3 \approx 4.41, f(0) = \sqrt{3} + 3 \approx 4.73, f(1) = 3 + \sqrt{(1)+3} = 3 + \sqrt{4} = 3 + 2 = 5, f(2) = \sqrt{5} + 3 \approx 5.23, f(a+h) = 3 + \sqrt{a+h+3}, f(a+h) - f(a) = \sqrt{a+h+3} - \sqrt{a+3}$

29. . $f(-2) = \big((-2)-2\big)\big((-2)+3\big) = (-4)(1) = -4, f(-1) = -6, f(0) = -6, f(1) = \big((1)-2\big)\big((1)+3\big) = (-1)(4) = -4, f(2) = 0, f(a+h) = a^2 + 2ah + a + h^2 + h - 6, f(a+h) - f(a) = 2ah + h^2 + h$

31. $f(-2) = \frac{(-2)-3}{(-2)+1} = \frac{-5}{-1} = 5, f(-1) = \text{undefined}, f(0) = -3, f(1) = -1, f(2) = -\frac{1}{3}, f(a+h) = \frac{a+h-3}{a+h+1}, f(a+h) - f(a) = \frac{4h}{(a+1)(a+h+1)}$

33. $f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}, f(-1) = \frac{1}{2}, f(0) = 1, f(1) = 2, f(2) = 4, f(a+h) = 2^{a+h}$, $f(a+h) - f(a) = 2^{a+h} - 2^a$

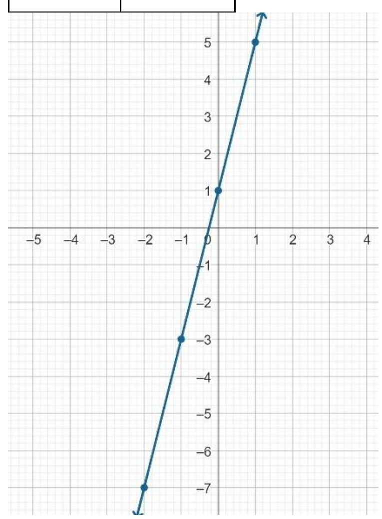
35. Using $f(x) = x^2 + 8x - 4$: $f(-1) = (-1)^2 + 8(-1) - 4 = 1 - 8 - 4 = -11; f(1) = 1^2 + 8(1) - 4 = 1 + 8 - 4 = 5.$

(a) $f(-1) + f(1) = -11 + 5 = -6$ (b) $f(-1) - f(1) = -11 - 5 = -16$

37. Using $f(t) = 3t + 5$:
(a) $f(0) = 3(0) + 5 = 5$
(b) $3t + 5 = 0$
$t = -\frac{5}{3}$

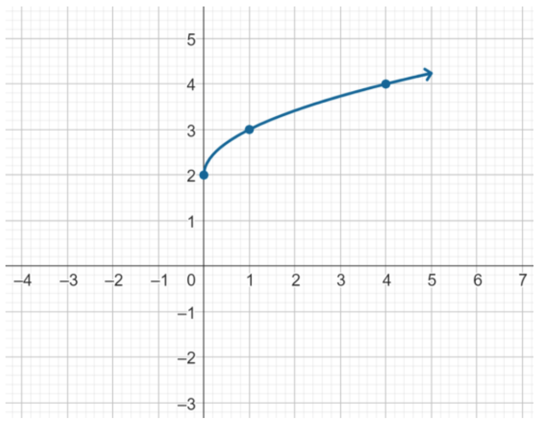
39.

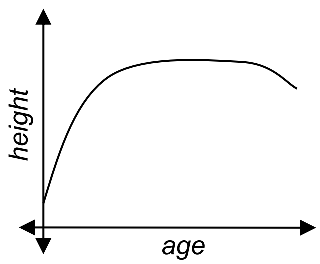
x	$f(x)$
-2	-7
-1	-3
0	1
1	5



41.

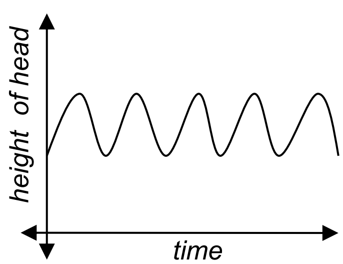
x	$f(x)$
0	2
1	3
4	4





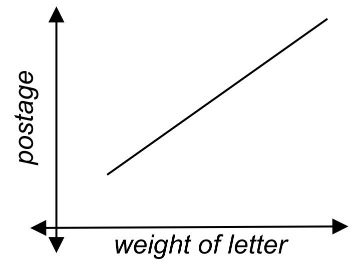
43. (a)

Graph (a)
At the beginning, as age increases, height increases. At some point, height stops increasing (as a person stops growing) and height stays the same as age increases. Then, when a person has aged, their height decreases slightly.



(b)

Graph (b)
As time elapses, the height of a person's head while jumping on a pogo stick as observed from a fixed point will go up and down in a periodic manner.



(c)

Graph (c)
The graph does not pass through the origin because you cannot mail a letter with zero postage or a letter with zero weight. The graph begins at the minimum postage and weight, and as the weight increases, the postage increases.

45. (a) t (b) $x = a$

(c) $f(b) = 0$ so $z = 0$. Then $f(z) = f(0) = r$.

(d) $L = (c, t), K = (a, p)$

Section 1.2

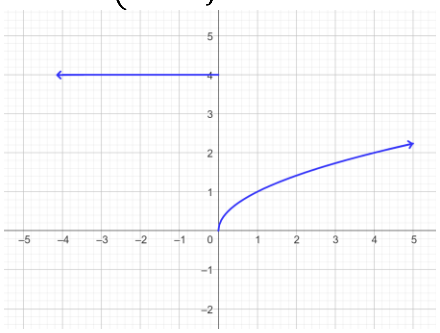
1. The domain is $[-5, 3)$; the range is $[0, 2]$
3. The domain is $(2, 8]$; the range is $[6,8)$
5. The domain is $[-4,4]$; the range is $[0, 2]$
7. The domain is $(-\infty, 1]$, the range is $[0, \infty)$
9. The domain is $\left[-6, -\frac{1}{6}\right] \cup [\frac{1}{6}, 6]$; the range is $\left[-6, -\frac{1}{6}\right] \cup \left[\frac{1}{6}, 6\right]$
11. The domain is $[0, 4]$, the range is $[-3, 0]$
13. $f(-1) = -4; f(0) = 6; f(2) = 20; f(4) = 34$

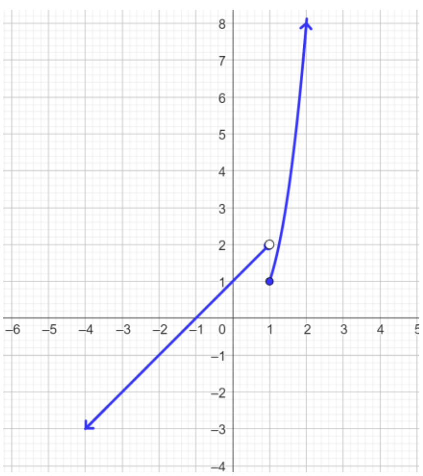
15. $f(-1) = -1, f(0) = -2, f(2) = 7, f(4) = 5$

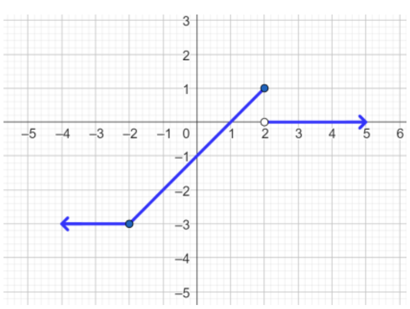
17. $f(-1) = 0,\ f(0) = 3,\ f(2) = 3,\ f(4) = 16$

19. $f(x) = \begin{cases} 2 & if \quad -6 \le x \le -1 \\ -2 & if \quad -1 < x \le 2 \\ -4 & if \quad 2 < x \le 4 \end{cases}$

21. $f(x) = \begin{cases} -x & if \quad x < 0 \\ x & if \ 0 \le x \le 2 \\ 3 & if \quad x > 2 \end{cases}$

23. 

25. 

27. 

Section 1.3

1. (a) x-intercepts: -2, 2; y-intercept: -4, (b) As $x \to \pm\infty, f(x) \to \infty$, (c) even
3. (a) y-intercept: -1, (b) As $x \to \pm\infty, f(x) \to\ 0$, (c) neither
5. (a)x-intercept: -2, y-intercept: 1, (b) As $x \to \infty, f(x) \to \infty$, as $x \to -3^+, f(x) \to -\infty$, (c) neither
7. Even
9. Odd
11. Even
13. x: 2, y: 6
15. x: $-\frac{11}{4}$, y: 22
17. x: none; y: 8
19. $(-\infty, -3 \cup (3, \infty)$
21. $(0, \infty)$
23. $(-\infty, 1) \cup (1, 2]$

Section 1.4

1. (a) $\frac{249-243}{2002-2001} = \frac{6}{1} =$ 6 million dollars per year
   (b) $\frac{249-243}{2004-2001} = \frac{6}{3} =$ 2 million dollars per year
3. The inputs $x = 1$ and $x = 4$ produce the points on the graph: (4,4) and (1,5). The average rate of change between these two points is $\frac{5-4}{1-4} = \frac{1}{-3} = -\frac{1}{3}$.
5. The inputs $x = 1$ and $x = 5$ when put into the function $f(x)$ produce the points (1,1) and (5,25). The average rate of change between these two points is $\frac{25-1}{5-1} = \frac{24}{4} = 6$.
7. The inputs $x = -3$ and $x = 3$ when put into the function $g(x)$

produce the points (-3, -82) and (3,80). The average rate of change between these two points is $\frac{80-(-82)}{3-(-3)}=\frac{162}{6}=\ 27.$

9. The inputs $t=-1$ and $t=3$ when put into the function $k(t)$ produce the points (-1,2) and (3,54.14$\overline{8}$). The average rate of change between these two points is $\frac{54.148-2}{3-(-1)}=\frac{52.148}{4}\approx 13.$

11. The inputs $x=1$ and $x=b$ when put into the function $f(x)$ produce the points (1,-3) and $(b, 4b^2-7)$. $Explanation$: $f(1)=4(1)^2-7=-3, f(b)=4(b)^2-7.$ The average rate of change between these two points is $\frac{(4b^2-7)-(-3)}{b-1}=\frac{4b^2-7+3}{b-1}=\frac{4b^2-4}{b-1}=\frac{4(b^2-1)}{b-1}=\frac{4(b+1)(b-1)}{(b-1)}=4(b+1).$

13. The inputs $x=2$ and $x=2+h$ when put into the function $h(x)$ produce the points (2,10) and $(2+h, 3h+10)$. $Explanation$: $h(2)=3(2)+4=10, h(2+h)=3(2+h)+4=6+3h+4=3h+10.$ The average rate of change between these two points is $\frac{(3h+10)-10}{(2+h)-2}=\frac{3h}{h}=3.$

15. The inputs $t=9$ and $t=9+h$ when put into the function $a(t)$ produce the points $(9, \frac{1}{13})$ and $\left(9+h, \frac{1}{h+13}\right)$. $Explanation$: $a(9)=\frac{1}{9+4}=\frac{1}{13}, a(9+h)=\frac{1}{(9+h)+4}=\frac{1}{h+13}$. The average rate of change between these two points is $\frac{\frac{1}{h+13}-\frac{1}{13}}{(9+h)-9}=\frac{\frac{1}{h+13}-\frac{1}{13}}{h}=\left(\frac{1}{h+13}-\frac{1}{13}\right)\left(\frac{1}{h}\right)=\frac{1}{h(h+13)}-\frac{1}{13h}=\frac{1}{h^2+13h}-\frac{1}{13h}\left(\frac{\frac{h}{13}+1}{\frac{h}{13}+1}\right)$ $(to\ make\ a\ common\ denominator$ $\frac{1}{h^2+13h}-\left(\frac{\frac{h}{13}+1}{h^2+13h}\right)=\frac{1-\frac{h}{13}-1}{h^2+13h}=\frac{\frac{h}{13}}{h^2+13h}=\frac{h}{13}\left(\frac{1}{h^2+13h}\right)=\frac{h}{13(h^2+13h)}=\frac{h}{13h(h+13)}=\frac{h}{13(h+13)}.$

17. The inputs $x=1$ and $x=1+h$ when put into the function $j(x)$ produce the points (1,3) and $(1+h, 3(1+h)^3)$. The average rate of change between these two

points is $\frac{3(1+h)^3-3}{(1+h)-1} = \frac{3(1+h)^3-3}{h} = \frac{3(h^3+3h^2+3h+1)-3}{h} = \frac{3h^3+9h^2+9h+3-3}{h} = \frac{3h^3+9h^2+9h}{h} = 3h^2+9h+9 = 3(h^2+3h+3)$.

19. The inputs $x = x$ and $x = x + h$ when put into the function $f(x)$ produce the points $(x, 2x^2+1)$ and $(x+h, 2(x+h)^2+1)$. The average rate of change between these two points is $\frac{(2(x+h)^2+1)-(2x^2+1)}{(x+h)-x} = \frac{(2(x+h)^2+1)-(2x^2+1)}{h} = \frac{2(x+h)^2+1-2x^2-1}{h} = \frac{2(x+h)^2-2x^2}{h} = \frac{2(x^2+2hx+h^2)-2x^2}{h} = \frac{2x^2+4hx+2h^2-2x^2}{h} = \frac{4hx+2h^2}{h} = 4x+2h = 2(2x+h)$.

21. Increasing on $(1, \infty)$, decreasing on $(-\infty, 1)$

23. Increasing on $(-\infty, 1)$ and $(3, 4)$. Decreasing on $(1, 3)$ and $(4, \infty)$

25. The function is increasing because as x increases, $f(x)$ also increases, and it is concave up because the rate at which $f(x)$ is changing is also increasing.

27. The function is decreasing because as x increases, $h(x)$ decreases. It is concave down because the rate of change is becoming more negative and thus it is decreasing.

29. The function is decreasing because as x increases, $f(x)$ decreases. It is concave up because the rate at which $f(x)$ is changing is increasing (becoming less negative).

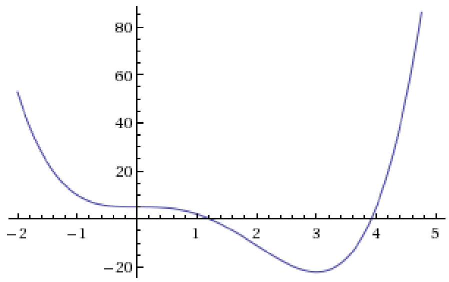
31. The function is increasing because as x increases, $h(x)$ also increases (becomes less negative). It is concave down because the rate at which $h(x)$ is changing is decreasing (adding larger and larger negative numbers).

33. The function is concave up on the interval $(-\infty, 1)$, and concave down on the interval $(1, \infty)$. This means that $x = 1$ is a point of inflection (where the graph changes concavity).

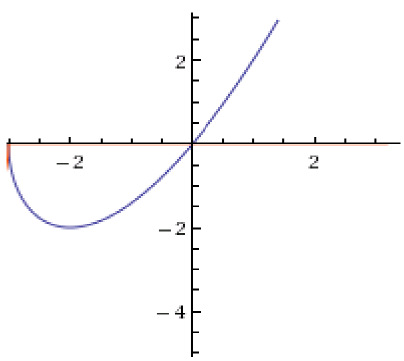
35. The function is concave down on all intervals except where there is an asymptote at $x \approx 3$.

37. From the graph, we can see that the function is decreasing on the interval $(-\infty, 3)$, and increasing on the interval $(3, \infty)$. This means that the function has a local minimum at $x = 3$. We can estimate that the function is concave down on the interval

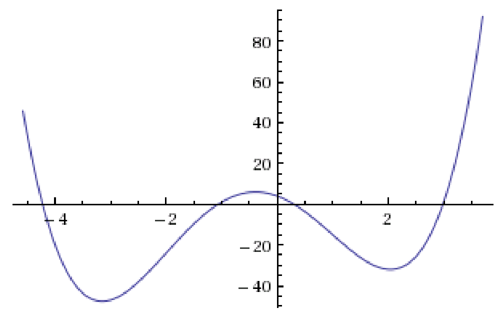
(0,2), and concave up on the intervals $(2, \infty)$and $(-\infty, 0)$. This means there are inflection points at $x = 2$ and $x = 0$.



39. From the graph, we can see that the function is decreasing on the interval $(-3, -2)$, and increasing on the interval $(-2, \infty)$. This means that the function has a local minimum at $x = -2$. The function is always concave up on its domain, (-3,∞). This means there are no points of inflection.



41. From the graph, we can see that the function is decreasing on the intervals $(-\infty, -3.15)$ *and* $(-0.38, 2.04)$, and increasing on the intervals $(-3.15, -0.38)$ *and* $(2.04, \infty)$. This means that the function has local minimums at $x = -3.15$ *and* $x = 2.04$ and a local maximum at $x = -0.38$. We can estimate that the function is concave down on the interval $(-2,1)$, and concave up on the intervals $(-\infty, -2)$ *and* $(1, \infty)$. This means there are inflection points at $x = -2$ *and* $x = 1$.



Section 1.5

1. $f(g(0)) = 4(7) + 8 = 26$, $g(f(0)) = 7 - (8)^2 = -57$
3. $f(g(0)) = \sqrt{(12) + 4} = 4$, $g(f(0)) = 12 - (2)^3 = 4$
5. $f(g(8)) = 4$
7. $g(f(5)) = 9$
9. $f(f(4)) = 4$
11. $g(g(2)) = 7$
13. $f(g(3)) = 0$
15. $g(f(1)) = 4$

17. $f(f(5)) = 3$

19. $g(g(2)) = 2$

21. $f(g(x)) = \frac{1}{\left(\frac{7}{x}+6\right)-6} = \frac{x}{7}$,

$g(f(x)) = \frac{7}{\left(\frac{1}{x-6}\right)} + 6 = 7x - 36$

23. $f(g(x)) = (\sqrt{x+2})^2 + 1 = x + 3$, $g(f(x)) = \sqrt{(x^2+1)+2} = \sqrt{(x^2+3)}$

25. $f(g(x)) = |5x+1|$, $g(f(x)) = 5|x| + 1$

27. $f(g(h(x))) = \left(\left(\sqrt{x}\right) - 6\right)^4 + 6$

29. B

31. (a) $N(T(t)) = 23(5t+1.5)^2 - 56(5t+1.5) + 1$

(b) 9429

33. $f(x) = x^2, g(x) = x + 2$

35. $f(x) = \frac{3}{x}, g(x) = x - 5$

37. $f(x) = 3 + x, g(x) = \sqrt{x-2}$

Section 1.6

1. 6
3. -4
5. $\frac{1}{2}$
7. (a) 3, (b) 2, (c) 2, (d) 3
9. (a) 0, (b) 7, (c) 1, (d) 3
11.

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

13. $f^{-1}(x) = x - 3$

15. $f^{-1}(x) = 2 - x$

17. $f^{-1}(x) = \frac{1}{11}(x - 7)$

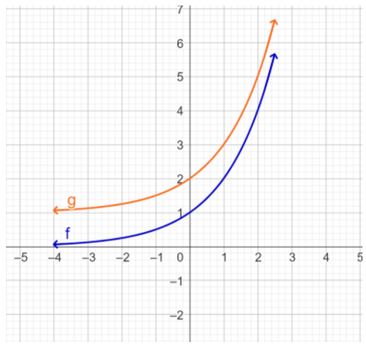
19. One-to-one on $[-7, \infty)$; inverse $f^{-1}(x) = \sqrt{x} - 7$

21. One-to-one on $[0, \infty)$; inverse $f^{-1}(x) = \sqrt{x+5}$

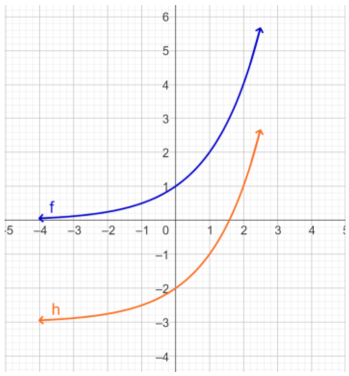
23. (a) x, (b) x, (c) they are inverses

Section 1.7

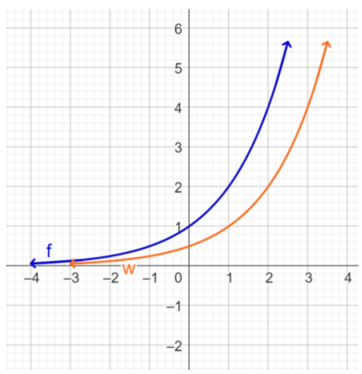
1. Shift right 49
3. Shift left 3
5. Shift up 5
7. Horizontally compress by 1/5
9. Vertically stretch by 6
11. Shift up 8
13. Shift down 7
15. Shift left 4 and down 1
17. Horizontal stretch by 3
19. Horizontal compress by ½
21. $f(x+2) + 1$
23. $f(x-3) - 4$
25. $-f(x)$
27. $f(-4x)$
29. $\frac{1}{3}f(x+2) - 3$
31. $f(2x-5) + 1$
33. $g(x) = f(x-1)$; $h(x) = f(x) + 1$



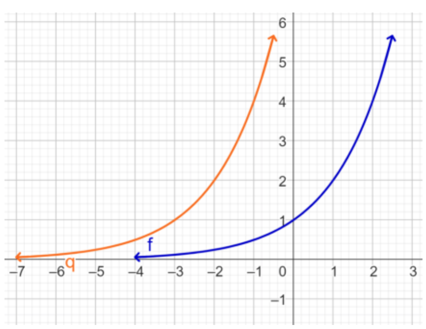
35. (a)



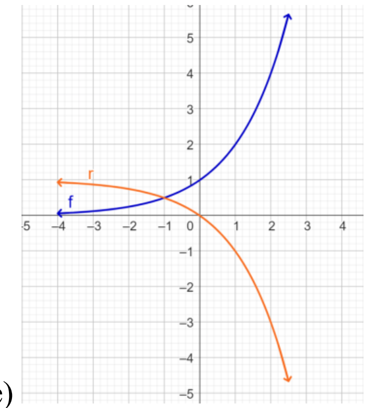
(b)



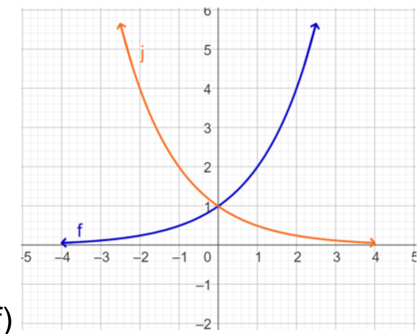
(c)



(d)

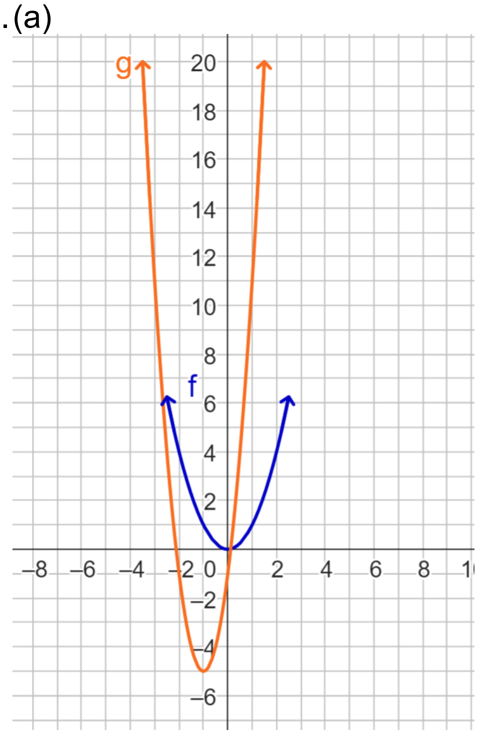


(e)

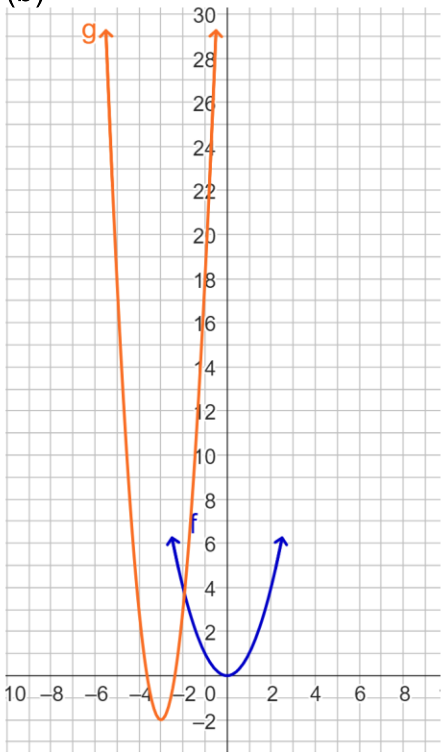


(f)

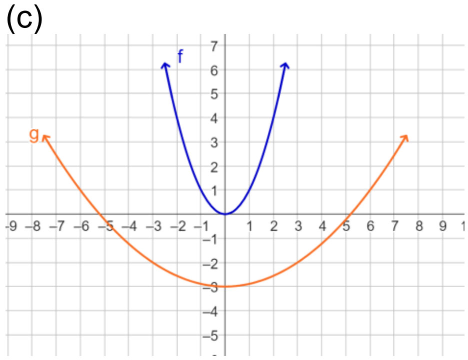
37. (a)



(b)



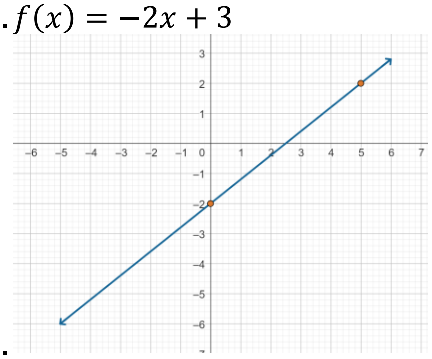
(c)



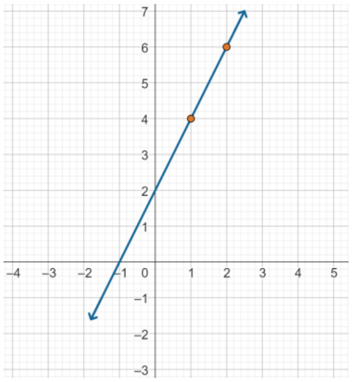
39. $g(x) = f(x-3) - 2$
41. $g(x) = f(x+3) - 1$
43. $g(x) = -f(x)$
45. $g(x) = -f(x+1) + 2$
47. $g(x) = f(-x) + 1$

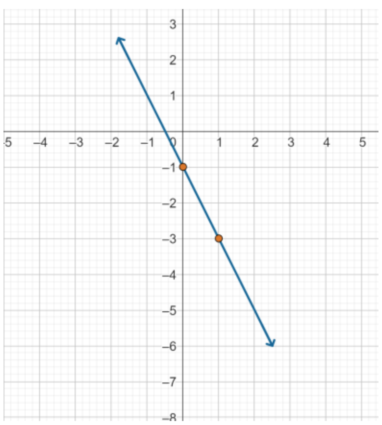
Section 2.1

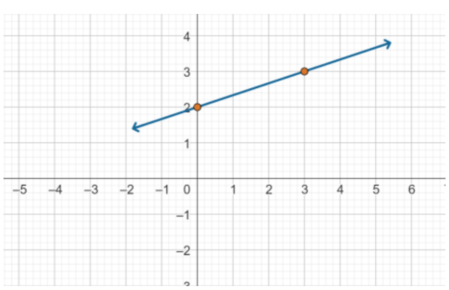
1. $P(t) = 1700t + 45000$
3. $S(t) = 10 + 2t$
5. $T(n) = 40 - 2n$
7. Increasing
9. Decreasing
11. Decreasing
13. Increasing
15. Decreasing
17. 3
19. $-\frac{1}{3}$
21. $\frac{4}{5}$
23. $\frac{2}{3}$
25. $S(t) = -\frac{1}{20}(x-2) + 1.4$
27. The population is decreasing at a rate of about 400 people per year.
29. Rate of change: plan costs 10 cents per minutes; initial value: base charge for the plan is $24
31. Terry starts at an elevation of 3000 feet and descends at a rate of 70 feet per second.
33. $f(x) = \frac{3}{5}(x+5) - 4 = \frac{3}{5}x - 1$
35. $f(x) = 3(x-2) + 4 = 3x - 2$
37. $f(x) = -\frac{1}{3}(x+1) + 4 = -\frac{1}{3}x + \frac{11}{3}$
39. $f(x) = -\frac{3}{2}x - 3$
41. $f(x) = \frac{2}{3}x + 1$
43. $f(x) = -2x + 3$

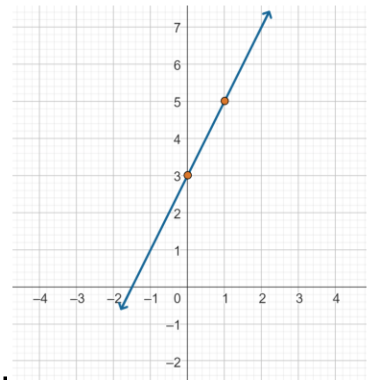


45. Rate of change: 4/5
Always increasing
Domain and Range $(-\infty, \infty)$
As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to -\infty$
x-intercept 5/2
y-\intercept -2

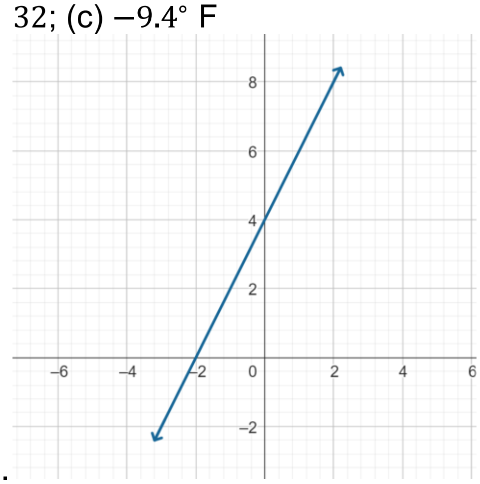
47. 
Rate of change: 2
Always increasing
Domain and Range $(-\infty, \infty)$
As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to -\infty$
x-intercept -1
y-intercept 2

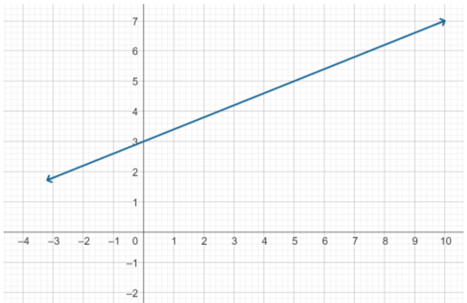
49. 
Rate of change: -2
Always decreasing
Domain and Range $(-\infty, \infty)$
As $x \to \infty, f(x) \to -\infty$ and as $x \to -\infty, f(x) \to \infty$
x-intercept -1/2
y-\intercept -1

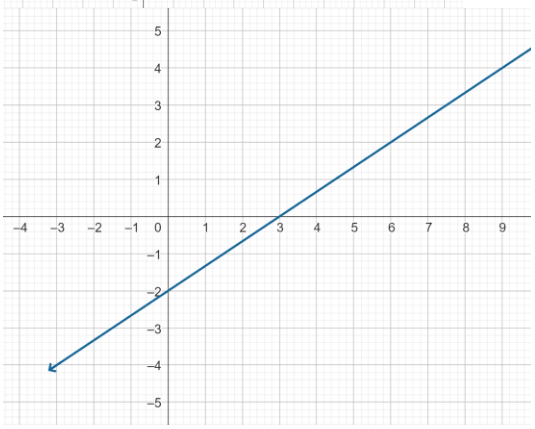
51. 
Rate of change: 1/3
Always increasing
Domain and Range $(-\infty, \infty)$
As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to -\infty$
x-intercept -6
y-intercept 2

53. 
Rate of change: 2
Always increasing
Domain and Range $(-\infty, \infty)$
As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to -\infty$
x-intercept -3/2
y-intercept 3

55. $p = -\frac{1}{250}n + 34$
57. $g(x) = -3x + 5$, h is not linear, $f(x) = 5x - 5$, $k(x) = 3(x - 5) + 13$
59. (a) $C = \frac{5}{9}(F - 32)$; (b) $F = \frac{9}{5}C + 32$; (c) $-9.4°$ F

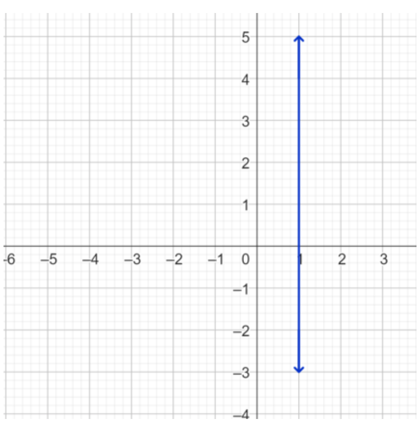
61. 

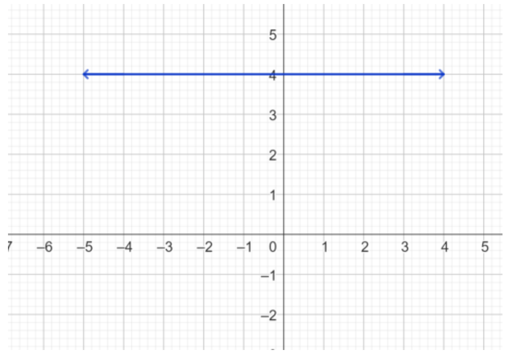
63. 

65. 

67. (a) $g(x) = \frac{1}{3}(x-1)+3$; (b) $\frac{1}{3}$; (c) 8/3
69. x: -2, y: 4
71. x: 1/5, y: 1
73. x: 5/7; y: 5/2
75. $f(x) = \begin{cases} -\frac{1}{2}x-2 & if -6 \le x \le -2 \\ -2 & if -2 < x \le 1 \\ \frac{3}{2}x - \frac{9}{2} & if\ 1 < x \le 6 \end{cases}$

Section 2.2

1. 

3. 

5. $f(x) = 3$
7. $x = -3$
9. Parallel
11. Neither
13. Perpendicular
15. $g(x) = -5(x-2) - 12$
17. $f(t) = \frac{1}{2}(t+4) - 1$
19. Intersect: $x = -1$; $(-\infty, -1)$
21. $x = \frac{6}{5}$
23. $x \le 4$; $(-\infty, 4]$
25. $x < \frac{8}{3}$; $\left(-\infty, \frac{8}{3}\right)$
27. $x \ge \frac{25}{18}$; $\left[\frac{25}{18}, \infty\right)$
29. $x < \frac{5}{4}$; $\left(-\infty, \frac{5}{4}\right)$
31. $x \le -\frac{1}{4}$; $\left(-\infty, -\frac{1}{4}\right]$
33. $f^{-1}(x) = 5x - 2$
35. $f^{-1}(x) = \frac{x-6}{2}$
37. Company A is better if you use less than 4.5 GB of data.

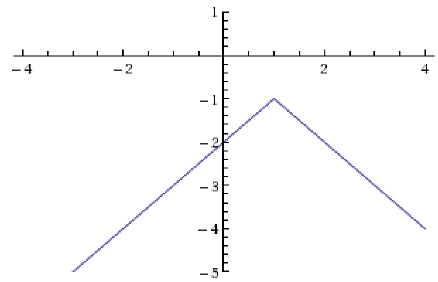
Section 2.3

1. (a) 696; (b) 4 years; (c) 174 students/year; (d) 305; (e) $P(t) = 174t + 305$; (f) 2219 people
3. (a) $f(x) = 0.15x + 10$; (b) The slope (0.15) is the price per minute of 15 cents, and the y-intercept (10) is the flat monthly fee of ten dollars. (c) 113.05
5. (a) $f(t) = 190t + 4170$; (b) 6640
7. (a) $R(t) = -2.1t + 16$; (b) 5.5; (c) 7.6 years

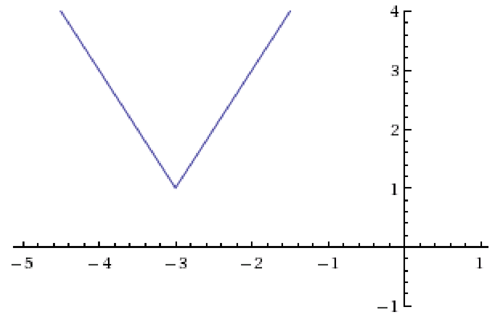
9. If you use less than 133 minutes, the first plan is cheaper, if you use more than 133 minutes, the second plan is cheaper.
11. If you sell less than \$42,857 of jewelry, option A produces a larger income, if you sell more than \$42,857 of jewelry, option B produces a larger income.
13. 19.98
15. 6 square units
17. $A = -\frac{b^2}{2m}$
19. (a) Mississippi home values increased at a rate of $\frac{71{,}400-25{,}200}{2000-1950} = 924 \text{ dollars/year}$, and the Hawaii home values increased at a rate of $\frac{272{,}700-74{,}400}{2000-1950} = 3{,}966 \text{ dollars/}$ year. So, Hawaii's home values increased faster. (b) 80,640 (c) 1934
21. 26.4 miles
23. 150 ft/min
25. 2.67 seconds
27. One pound of each

Section 2.4

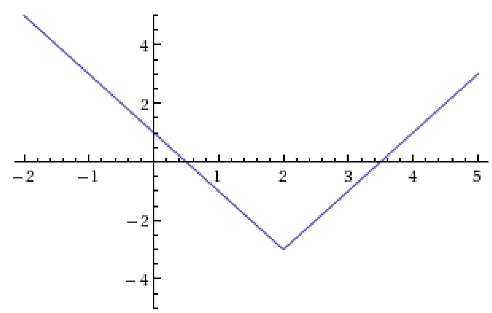
1. $g(x) = \frac{1}{2}|x+2| + 1$
3. $g(x) = -3.5|x-3| + 3$



5. Domain: $(-\infty, \infty)$; Range: $(-\infty, -1]$; y-intercept: -2, no x-intercept; Increasing on $(-\infty, 1)$, decreasing on $(1, \infty)$; Maximum of -1 at $x = 1$.



7. Domain: $(-\infty, \infty)$; Range: $[1, \infty)$; y-intercept: 7, no x-intercept; Increasing on $(-3, \infty)$, decreasing on $(-\infty, -3)$; Minimum of 1 at $x = -3$.



9. Domain: $(-\infty, \infty)$; Range: $[-3, \infty)$; y-intercept: 1, x-intercepts ½ and 7/2; Increasing on $(2, \infty)$, decreasing on $(-\infty, 2)$; Minimum of -3 at $x = 2$.
11. $x = \frac{13}{5}, -\frac{9}{5}$
13. $x = \frac{1}{2}, \frac{15}{2}$
15. $x = -\frac{1}{3}, -\frac{5}{3}$
17. Horizontal: $(4,0)$ and $(-6, 0)$; vertical $(0,8)$

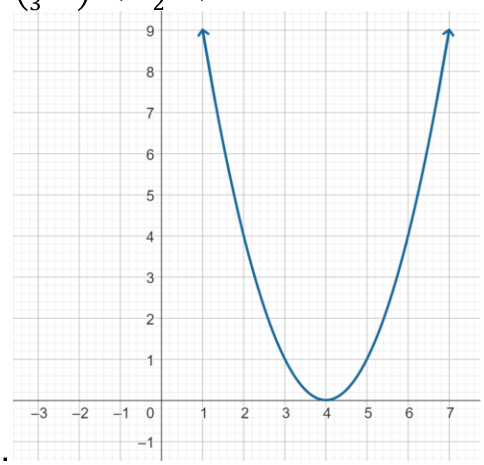
19. Horizontal: none; vertical: $(0, -7)$
21. $(-11, 1)$
23. $(-\infty - 1] \cup [5, \infty)$
25. $\left(-\frac{13}{3}, -\frac{5}{3}\right)$

Section 2.5

1. $(x - 12)^2 - 144$
3. $\left(x - \frac{1}{6}\right)^2 - \frac{1}{36}$
5. $\left(x + \frac{15}{2}\right)^2 - \frac{225}{4}$
7. $(x - 11)^2 - 121$
9. $5\left(x + \frac{1}{5}\right)^2 - \frac{1}{5}$
11. $4\left(x - \frac{7}{2}\right)^2 - 49$
13. $x = -3, 1$
15. $x = -1, 21$
17. No real solutions
19. $x = -\frac{5}{2} \pm \frac{\sqrt{33}}{2}$
21. $x = 1, 13$
23. $x = \frac{5}{2} \pm \frac{\sqrt{21}}{2}$
25. $x = 3 \pm \sqrt{23}$
27. $x = 1, 6$
29. $x = -2 \pm \sqrt{7}$
31. $x = -\frac{5}{3}, 3$
33. $x = \frac{2}{3}, 4$
35. $x = \frac{5}{6} \pm \frac{\sqrt{133}}{6}$
37. No real solutions

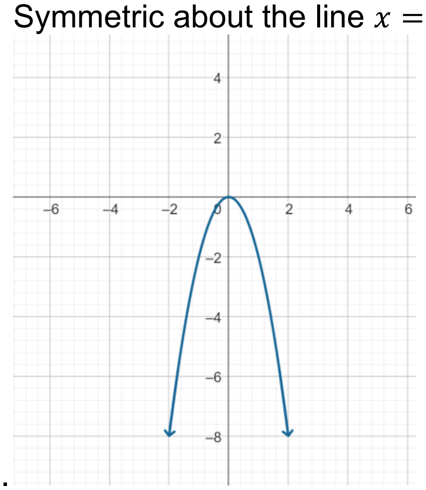
Section 2.6

1. As $x \to \pm\infty, f(x) \to \infty$
3. As $x \to \infty, f(x) \to \infty$, and as $x \to -\infty, f(x) \to -\infty$
5. As $x \to \pm\infty, f(x) \to -\infty$
7. As $x \to \infty, f(x) \to -\infty$, and as $x \to -\infty, f(x) \to \infty$
9. Degree 7, leading coefficient 4
11. Degree 2, leading coefficient -1
13. Degree 4, leading coefficient -2
15. Degree 3, leading coefficient 6
17. As $x \to \pm\infty, f(x) \to -\infty$
19. As $x \to \pm\infty, f(x) \to \infty$
21. 5 x-intercepts, 4 turning points
23. 3
25. 5
27. 3
29. 5
31. Vertical: (0, 12); horizontal: $(1, 0), (-2, 0), (3, 0)$
33. Vertical: (0,2); horizontal: $\left(\frac{1}{3}, 0\right), (-\frac{1}{2}, 0)$
35.

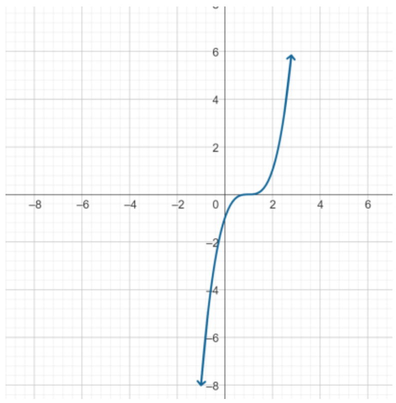


Domain: $(-\infty, \infty)$; Range: $[0, \infty)$; x-intercept 4, y-intercept 16; As $x \to \pm\infty, g(x) \to \infty$; increasing on $(4, \infty)$, decreasing on $(-\infty, 4)$. Symmetric about the line $x = 4$.

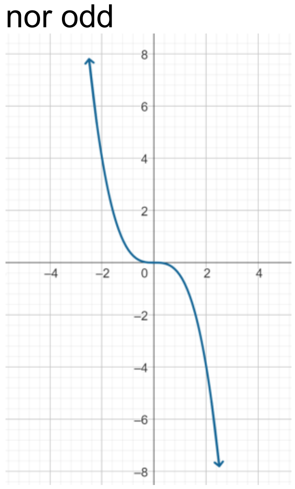
37.



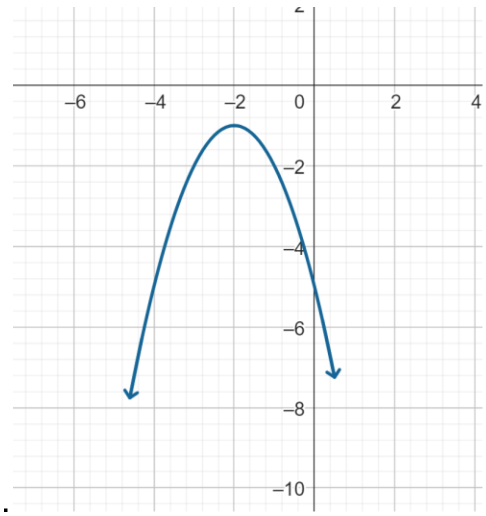
Domain: $(-\infty, \infty)$; Range: $(-\infty, 0]$; x-intercept 0, y-intercept 0; As $x \to \pm\infty, j(x) \to -\infty$; increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$. Symmetric about the y-axis (even).

39. 

Domain and range: $(-\infty, \infty)$; x-intercept 1, y-intercept -1; As $x \to \infty, g(x) \to \infty$ and as $x \to -\infty, g(x) \to -\infty$; increasing on $(-\infty, 1)$ and $(1, \infty)$, neither even nor odd

41. 

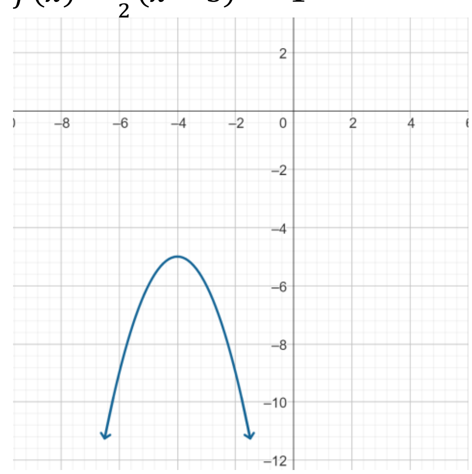
Domain and range: $(-\infty, \infty)$; x-intercept 0, y-intercept 0; As $x \to \infty, g(x) \to -\infty$ and as $x \to -\infty, g(x) \to \infty$; Decreasing on $(-\infty, 0)$ and $(0, \infty)$, origin symmetry (odd)

43. 

Domain: $(-\infty, \infty)$; Range: $(-\infty, -1]$; x-intercept none, y-intercept -5; As $x \to \pm\infty, g(x) \to -\infty$; increasing on $(-\infty, -2)$, decreasing on $(-2, \infty)$. Symmetric about the line x=-2 (neither even nor odd)

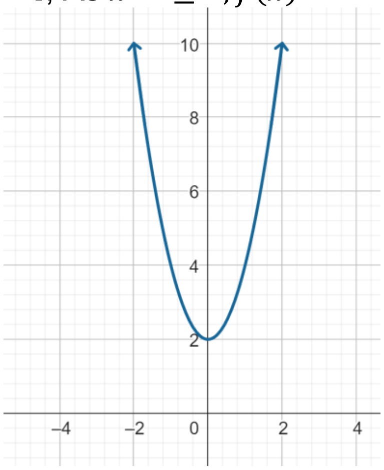
Section 2.7

1. $f(x) = (x-2)^2 - 3$
3. $f(x) = -2(x-2)^2 + 7$
5. $f(x) = \frac{1}{2}(x-3)^2 - 1$

7. 

y-intercept at -21, no x-intercepts; Domain: $(-\infty, \infty)$; Range: $(-\infty, -5]$; Increasing on $(-\infty, -4)$, decreasing on $(-4, \infty)$, concave down on

$(-\infty, \infty)$; Maximum of -5 at $x = -4$; As $x \to \pm\infty, f(x) \to -\infty$

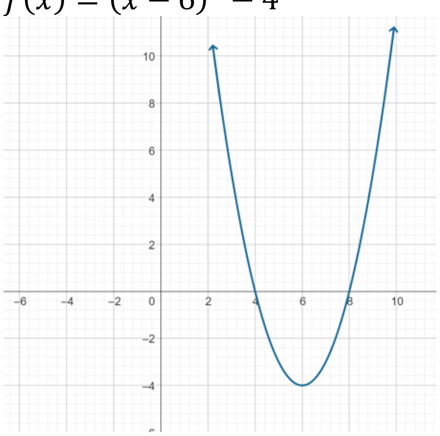
9. 

y-intercept at 2, no x-intercepts; Domain: $(-\infty, \infty)$; Range:$[2, \infty)$ Increasing on $(0, \infty)$ decreasing on $(-\infty, 0)$ concave up on $(-\infty, \infty)$; Minimum of 2 at $x = 0$; As $x \to \pm\infty, h(x) \to \infty$

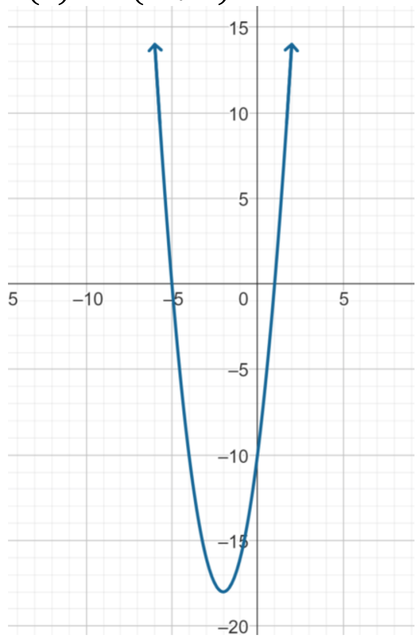
11. Vertex: $\left(-\frac{10}{4}, -\frac{1}{2}\right)$ x-intercepts: $(-3,0)(-2,0)$ y-intercept: $(0,12)$

13. Vertex: $\left(\frac{10}{4}, -\frac{29}{2}\right)$ x-intercepts: $(5,0)(-1,0)$ y-intercept: $(0,4)$

15. Vertex: $\left(\frac{3}{4}, 1.25\right)$ x-intercepts: $\pm\sqrt{5}$ y-intercept: $(0, -1)$

17. $f(x) = (x-6)^2 - 4$


y-intercept at 32, x-intercepts at 4 and 8; Domain: $(-\infty, \infty)$; Range:$[-4, \infty)$ Increasing on $(6, \infty)$ decreasing on $(-\infty, -4)$ concave up on $(-\infty, \infty)$; Minimum of -4 at $x = 6$; As $x \to \pm\infty, f(x) \to \infty$

19. $h(x) = 2(x+2)^2 - 18$


y-intercept at -10, x-intercepts at -5 and 1; Domain: $(-\infty, \infty)$; Range:$[-18, \infty)$ Increasing on $(-2, \infty)$ decreasing on $(-\infty, -2)$ concave up on $(-\infty, \infty)$; Minimum of -18 at $x = -2$; As $x \to \pm\infty, h(x) \to \infty$

21. $b = 32, c = -39$

23. $f(x) = -\frac{2}{3}x^2 - \frac{4}{3}x + 2$

25. $f(x) = \frac{3}{5}x^2 - \frac{21}{5x} + 6$

27. $y = -\frac{1}{4}x^2 + 2x - 4$

29. $y = -\frac{1}{9}x^2 - \frac{2}{3}x + 1$

31. Intersect at $x = -5, 1$; solution to inequality is $[-5, 1]$

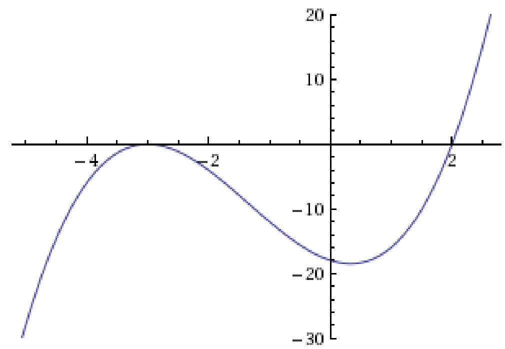
33. Intersect at $x = 0, 1$; solution to inequality is $(0,1)$

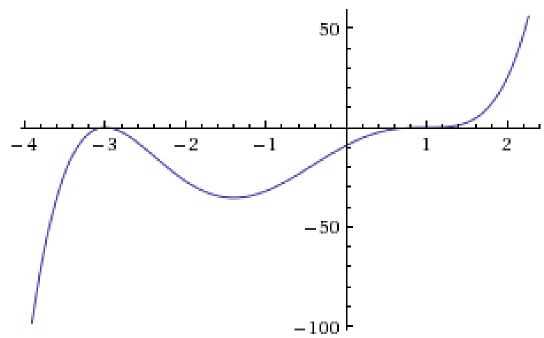
35. Intersect at $x = \frac{9}{2} \pm \frac{\sqrt{61}}{2}$; solution to inequality is $\left(-\infty, \frac{9}{2} - \frac{\sqrt{61}}{2}\right) \cup \left(\frac{9}{2} + \frac{\sqrt{61}}{2}, \infty\right)$

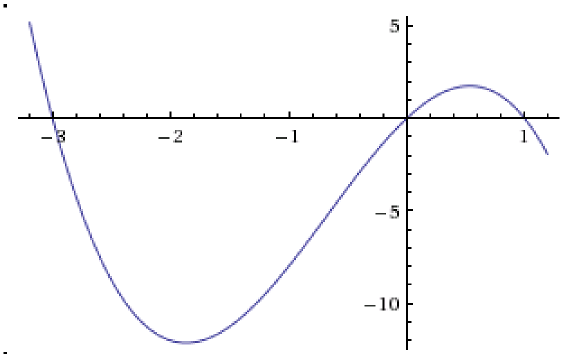
37. (a) 8 meters; (b) 120/49 meters; (c) about 5.2 seconds
39. (a) 6 feet; (b) 326 feet; (c) about 160.75 feet
41. The volume of the box can be expressed as $V = 6 * x * x \text{ or } V = 6x^2$. So if we want the volume to be 1000, we end up with the expression $6x^2 - 1000 = 0$. Solving this equation for x using the quadratic formula we get $= \pm 10\sqrt{\frac{5}{3}}$. Because we cannot have negative length, we are left with $x = 10\sqrt{\frac{5}{3}} \approx 12.90$. So the length of the side of our box is $12 + 12.90 = 24.90$. So, our piece of cardboard is $24.90 * 24.90 = 620$.
43. The vertical side should be 87.5 and the whole horizontal side should be 175 feet
45. $24.6344\ cm$
47. A ticket price of $10.70 would maximize revenue.
49. (a) 632809.375 ft; (b) 632812.5 ft; (c) 56125 ft; (d) 1.11 ft

Section 2.8

1. (a) C intercept at (0, 48)
   (b) t intercepts at (4,0), (-1,0), (6,0)
3. (a) C intercept at (0,0)
   (b) t intercepts at (2,0), (-1,0), (0,0)
5. $C(t) = 2t^4 - 8t^3 + 6t^2 = 2t^2(t^2 - 4t + 3) = 2t^2(t-1)(t-3)$.
   (a) C intercept at (0,0)
   (b) t intercepts at (0,0), (3,0) (1,0)
7. Zeros: $x \approx -1.65,\ x \approx 3.64,\ x \approx 5.$
9. (a) as $t \to \infty, h(t) \to \infty$.
   (b) as $t \to -\infty, h(t) \to -\infty$
11. (a) as $t \to \infty, p(t) \to -\infty$
    (b) as $t \to -\infty, p(t) \to -\infty$

13.


15.


17.


19. $(3, \infty)$
21. $(-2, 1) \cup (3, \infty)$
23. $(\frac{7}{2}, 6)$
25. $(-\infty, 1] \cup [4, \infty)$
27. $[3, \infty)$
29. $f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$
31. $f(x) = \frac{1}{3}(x-3)^2(x-1)^2(x+3)$
33. $f(x) = -15(x-1)^2(x-3)^3$
35. $f(x) = \frac{1}{2}(x+2)(x-1)(x-3)$.

37. $f(x) = -(x+1)^2(x-2)$

39. $f(x) = -\frac{1}{24}(x+3)(x+2)(x-2)(x-4)$

41. $f(x) = \frac{1}{24}(x+4)(x+2)(x-3)^2$

43. $f(x) = \frac{3}{32}(x+2)^2(x-3)^2$

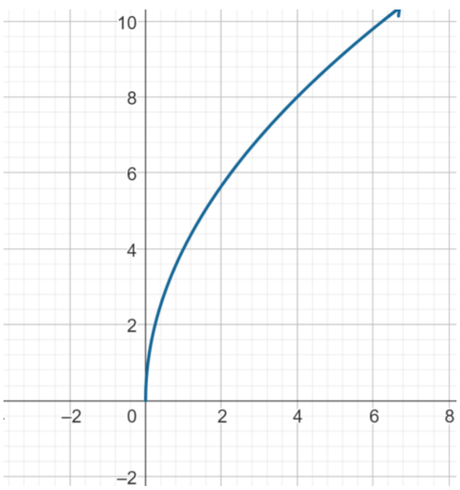
45. $f(x) = \frac{1}{6}(x+3)(x+2)(x-1)^3$

47. $f(x) = -\frac{1}{16}(x+3)(x+1)(x-2)^2(x-4)$

49. Base = 2.58, height = 3.34

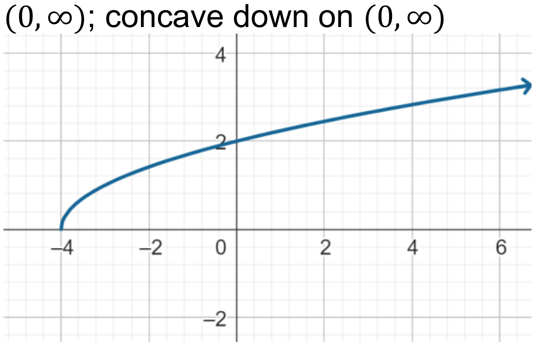
Section 2.9

1.



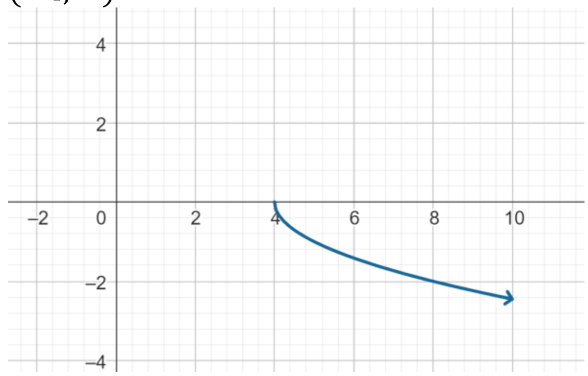
Domain: $[0,\infty)$; Range: $[0,\infty)$; x- and y-intercept at (0,0); As $x \to \infty, g(x) \to \infty$; Increasing on $(0,\infty)$; concave down on $(0,\infty)$

3.



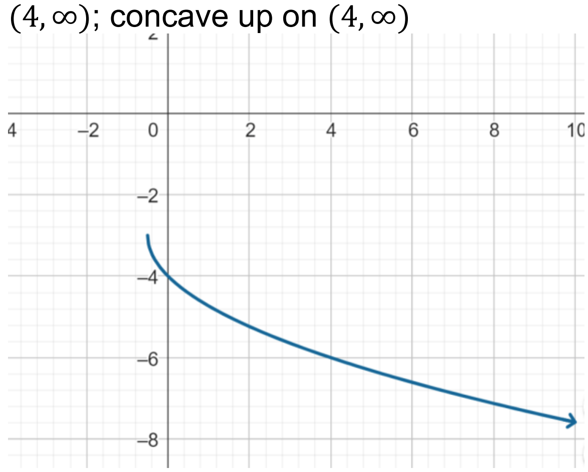
Domain: $[-4,\infty)$; Range: $[0,\infty)$; x-intercept -4; y-intercept 2; As $x \to \infty, j(x) \to \infty$; Increasing on $(-4,\infty)$; concave down on $(-4,\infty)$

5.



Domain: $[4,\infty)$; Range: $(-\infty,0]$; x-intercept 4, no y-intercept; As $x \to \infty, f(x) \to -\infty$; decreasing on $(4,\infty)$; concave up on $(4,\infty)$

7.



Domain: $[-\frac{1}{2},\infty)$; Range: $(-\infty,-3]$; no x-intercept; y-intercept -4; As $x \to \infty, h(x) \to -\infty$; decreasing on $(-\frac{1}{2},\infty)$; concave up on $(-\frac{1}{2},\infty)$

9. Restrict domain to $[4,\infty)$; $f^{-1}(x) = -4+\sqrt{x}$

11. Restrict domain to $(-\infty,0]$; $f^{-1}(x) = -\sqrt{12-x}$

13. Domain $(-\infty,\infty)$; $f^{-1}(x) = \sqrt[3]{\frac{x-1}{3}}$

15. $f^{-1}(x) = \frac{(x-9)^2}{4}+1$

17. $f^{-1}(x) = \left(\frac{x-9}{2}\right)^3$

19. $x = 27$

21. $x = 0$

23. $x = 7$
25. $x = 5$
27. $x = 0,1$
29. $x = 0$
31. $v \approx 65.57\, mph.$
33. $v \approx 34.07\, mph$
35. 14.14 feet
37. (a) $h = -2x^2 + 124x$ and $y = -4x$ (b) 1922 ft; (c) $\frac{62-\sqrt{3,844-2h}}{2}$; (d) The function given in (c) does not work when the function is going down. We would have had to choose the positive root in the numerator to give the right half of the parabola.

Section 2.10

1. $4x^2 + 3x - 1 = (x-3)(4x+15) + 44$
3. $5x^4 - 3x^3 + 2x^2 - 1 = (x^2+4)(5x^2 - 3x - 18) + (12x + 71)$
5. $9x^3 + 5 = (2x-3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$
7. $(3x^2 - 2x + 1) = (x-1)(3x+1) + 2$
9. $(3 - 4x - 2x^2) = (x+1)(-2x-2) + 5$
11. $(x^3 + 8) = (x+2)(x^2 - 2x + 4) + 0$
13. $(18x^2 - 15x - 25) = \left(x - \frac{5}{3}\right)(18x + 15) + 0$
15. $(2x^3 + x^2 + 2x + 1) = \left(x + \frac{1}{2}\right)(2x^2 + 2) + 0$
17. $(2x^3 - 3x + 1) = \left(x - \frac{1}{2}\right)\left(2x^2 + x - \frac{5}{2}\right) - \frac{1}{4}$
19. $(x^4 - 6x^2 + 9) = (x - \sqrt{3})(x^3 + \sqrt{3}\,x^2 - 3x - 3\sqrt{3}) + 0$
21. Dividing by $x - 1$ leaves $x^2 - 5x - 6 = (x-2)(x-3)$.
$x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$
23. Dividing by $x - \frac{2}{3}$ leaves $3x^2 + 6x + 3 = 3(x^2 + 2x + 1) = 3(x+1)^2$
$3x^3 + 4x^2 - x - 2 = 3\left(x - \frac{2}{3}\right)(x+1)^2$
25. Dividing by $x + 2$ leaves $x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$
$x^3 + 2x^2 - 3x - 6 = (x+2)(x + \sqrt{3})(x - \sqrt{3})$
27. Dividing by $x - \frac{1}{2}$ twice leaves $4x^2 - 24x + 36 = 4(x^2 - 6x + 9) = 4(x-3)^2$.
$4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4\left(x - \frac{1}{2}\right)^2 (x-3)^2$
29. All of the real zeros lie in the interval $[-7,7]$. Possible rational zeros are $\pm 1, \pm 2, \pm 3$
31. All of the real zeros lie in the interval $[-13,13]$

Possible rational zeros are ± 1, ± 2, ± 3, ± 4, ± 6, ± 12

33. All of the real zeros lie in the interval $[-8,8]$

Possible rational zeros are ± 1, ± 7

35. All of the real zeros lie in the interval $[-3,3]$

Possible rational zeros are $\pm\frac{1}{17}$, $\pm\frac{2}{17}$, $\pm\frac{5}{17}$, $\pm\frac{10}{17}$, ± 1, ± 2, ± 5, ± 10

37. All of the real zeros lie in the interval $\left[-\frac{14}{3}, \frac{14}{3}\right]$

Possible rational zeros are $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{5}{3}$, $\pm\frac{10}{3}$, ± 1, ± 2, ± 5, ± 10

39. Dividing by $x - 1$ leaves $x^2 - x - 6 = (x-3)(x+2)$, so
$x^3 - 2x^2 - 5x + 6 = (x-1)(x-3)(x+2)$. Real zeros are $x = -2$, $x = 1$, $x = 3$ (each has mult. 1)

41. Dividing by $x - 1$ leaves $x^3 + x^2 - 8x - 12$.
Dividing again by $x + 2$ leaves $x^2 - x - 6 = (x - 3)(x + 2)$, so $x^4 - 9x^2 - 4x + 12 = (x - 1)(x + 2)(x - 3)(x + 2)$. Real zeros are $x = -2$ (mult. 2), $x = 1$ (mult. 1), $x = 3$ (mult. 1)
43. Dividing by $x - 7$ leaves $x^2 + 1$, which is not further reducible.
Real zero is $x = 7$ (mult. 1)
45. Looking at a graph it's clear there are no integer roots. The first positive root looks near $\frac{1}{4}$ or $\frac{1}{3}$. From our list of possible rational zeros (see #7 above), the value near that is $\frac{5}{17}$.
Dividing by $x - \frac{5}{17}$ leaves $-17x^2 + 34 = -17(x^2 - 2) = -17\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$. So $-17x^3 + 5x^2 + 34x - 10 = -17\left(x - \frac{5}{17}\right)\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$.
The real zeros are $x = \frac{5}{17}$, $x = \pm\sqrt{2}$ (each has mult. 1)
47. Dividing by $x + 2$ leaves $3x^2 - 3x - 5$. This does not factor nicely, but we can use quadratic equation to solve for the roots. The real zeros are
$x = -2$, $x = \frac{3 \pm \sqrt{69}}{6}$ (each has mult. 1)
49. We can factor out the common factor x, giving $x(9x^2 - 5x - 1)$. The quadratic does not factor nicely, but we can use quadratic equation. The real zeros are
$x = 0$, $x = \frac{5 \pm \sqrt{61}}{18}$ (each has mult. 1)
51. We can factor this as $(x^2 + 5)(x^2 - 3)$. $x^2 + 5 = 0$ has no real solutions, but $x^2 - 3 = 0$ gives the real zeros: $x = \pm\sqrt{3}$ (each has mult. 1)
53. This can factor as $(3x^2 + 1)(x^2 - 5)$. $3x^2 + 1$ has no real solutions, but $x^2 - 5 = 0$ gives the real zeros: $x = \pm\sqrt{5}$ (each has mult. 1)
55. This can factor as $(x^3 - 5)(x^3 + 2)$. $x = \sqrt[3]{-2} = -\sqrt[3]{2}$, $x = \sqrt[3]{5}$ (each has mult. 1)
57. Dividing by $x - 2$ leaves $x^4 - 4 = (x^2 + 2)(x^2 - 2)$. $x^2 + 2$ gives no real solutions, but $x^2 - 2 = 0$ gives $x = \pm\sqrt{2}$. The real zeros are: $x = 2$, $x = \pm\sqrt{2}$ (each has mult. 1)
59. $x = -4$ (mult. 3), $x = 6$ (mult. 2)

Section 2.11

1. B
3. A
5. Vertical asymptote: $x = -4$, Horizontal asymptote: $y = 2$, Vertical intercept: $(0, -\frac{3}{4})$, Horizontal intercept: $(\frac{3}{2}, 0)$.
7. Vertical asymptote: $x = 2$, Horizontal asymptote: $y = 0$, Vertical intercept: $(0, 1)$, Horizontal intercept: none.
9. Vertical asymptotes: $= -4$, $\frac{4}{3}$, Horizontal asymptote: $y = 1$, Vertical intercept: $(0, \frac{5}{16})$, Horizontal intercepts: $\left(-\frac{1}{3}, 0\right)$, $(5, 0)$.
11. Vertical asymptote: there is no vertical asymptote, Horizontal asymptote: $y = 1$, Vertical intercept: $(0, 3)$, Horizontal intercept: $(-3, 0)$.
13. Vertical asymptote: $x = 3$, Horizontal asymptote: There is no horizontal asymptote, Vertical

intercept: $(0, \frac{1}{4})$, Horizontal intercepts: $(-1, 0), (\frac{1}{2}, 0)$.

15. Vertical asymptotes: *x* = 0, and *x* = 4, Horizontal asymptote: *y* = 0, Vertical intercept: none, Horizontal intercepts: $(-2, 0), (\frac{2}{3}, 0)$.
17. Vertical asymptotes: $x = -2$, 4, Horizontal asymptote: $y = 1$, Vertical intercept: $(0, -\frac{15}{16})$, Horizontal intercepts, (1, 0), (-3, 0), and (5, 0).
19. $x = 1, \frac{8}{3}$
21. $x = -\frac{2}{5}$
23. $x = \frac{1}{3}$
25. $x = 0, 4$
27. $y = \frac{50(x-2)(x+1)}{(x-5)(x+5)}$
29. $y = \frac{7(x-4)(x+6)}{(x+4)(x+5)}$
31. See problem 27 and/or problem 29. To get a double zero at $x = 2$, we need the numerator to be able to be broken down into two factors both of which are zero at $x = 2$. So, $(x-2)^2$ appears in the numerator.
33. $y = \frac{4(x-3)}{(x-4)(x+3)}$
35. $y = -\frac{27(x-2)}{(x-3)^2(x+3)}$
37. $y = \frac{(x-2)(x+3)}{3(x-1)}$
39. $y = \frac{-6(x-1)^2}{(x-2)^2(x+3)}$
41. $y = -\frac{2x(x-3)}{(x-4)(x+3)}$
43. $y = \frac{2(x-1)^3}{(x+1)(x-2)^2}$
45. $y = \frac{(x-2)(x-4)}{(x+1)(x-4)}$
47. $y = 3x - 2$
49. $y = \frac{1}{2}x + 1$
51. $y = -2x + 1$
53. (a) To get the percentage of water (non-acid) in the beaker we take $(n + 16)$ our total amount of water and divide by our total amount of solution $(n + 20)$. Then to get the percent of acid, we subtract the percent of water from 1, or $1 - \frac{(n+16)}{(n+20)}$.

    (b) Using our equation from part (a), we get $1 - \frac{(10+16)}{(10+20)} = 13.33\%$

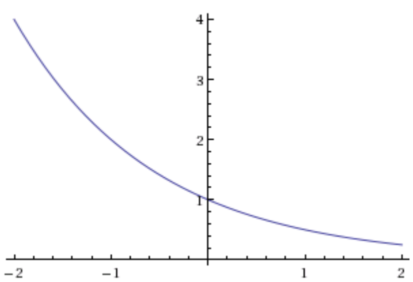
    (c) To get 4%, we use our equation from part (a) to solve: $4 = 1 - \frac{n+16}{n+20} \rightarrow\ -3 = \frac{(n+16)}{(n+20)} \rightarrow$ $-3n - 60 = n + 16 \rightarrow\ n = 19mL$.

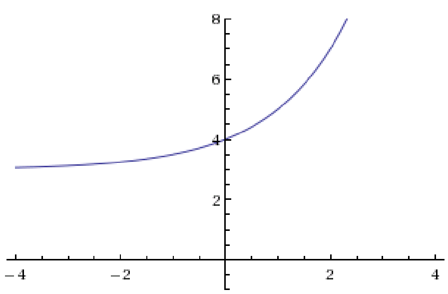
    (d) As $n \rightarrow \infty$, $\frac{n+16}{n+20} \rightarrow 1$, because the denominator and numerator have the same degree. So, as → ∞, $1 - \frac{n+16}{n+20} \rightarrow 0$. This means that our acid becomes insignificant compared to the water.
55. (a) $m(x) = \frac{0.2x-10.4}{x-10}$; (b) $x \approx 9.143, 9.916$; (c) The meter reading increases as Oscar gets closer to the magnet. The graph of the function for (a) has a vertical asymptote at $x = 10$, for which the graph shoots up when approaching from the left, so the magnet is 10 feet into the room. This is consistent with our answers for (b), in which x gets closer to 10 as the meter reading increases.

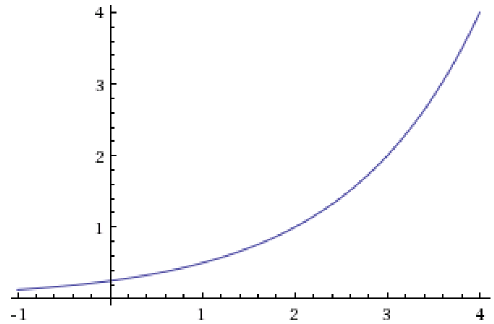
Section 3.1

1. Linear, because the average rate of change between any pair of points is constant.

3. Exponential, because the difference of consecutive inputs is constant and the ratio of consecutive outputs is constant.
5. Neither, because the average rate of change is not constant nor is the difference of consecutive inputs constant while the ratio of consecutive outputs is constant.
7. $f(x) = 11{,}000(1.085)^x$
9. $f(x) = 23{,}900(1.09)^x$; $f(8) = 47622$
11. $f(x) = 32{,}500(0.95)^x$; $f(12) = \$17{,}561.70$
13. B
15. A
17. E
19. The value of b affects the steepness of the slope, and graph D has the highest positive slope it has the largest value for b.
21. The value of a is your initial value, when your $x = 0$. Graph C has the largest value for a.

23.


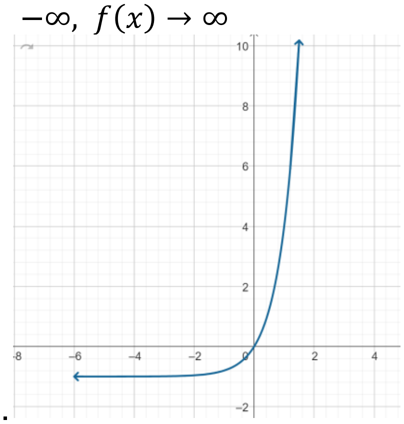
25.


27.


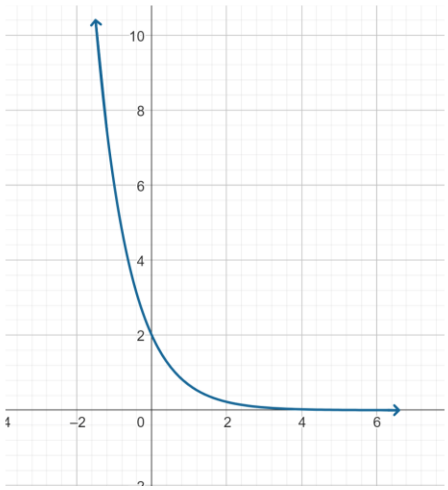
29. $f(x) = 4^x + 4$

31. $f(x) = 4^{x+2}$

33. $f(x) = -4^x$
35. as $x \to \infty, f(x) \to -\infty$. And as $x \to -\infty, f(x) = -1$.
37. as $x \to \infty,\ f(x) \to -2$ and as $x \to -\infty,\ f(x) \to +\infty$
39. as $x \to \infty,\ f(x) \to 2$ and as $x \to -\infty,\ f(x) \to \infty$

41.


Domain: $(-\infty, \infty)$; Range: $(-1, \infty)$; x- and y-intercept (0,0); As $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to -1$; Increasing and concave up on $(-\infty, \infty)$

43. 

Domain: $(-\infty, \infty)$; Range: $(0, \infty)$; y-intercept (0,2), no x-intercept; As $x \to \infty, f(x) \to 0$ and as $x \to -\infty, f(x) \to \infty$; Decreasing and concave up on $(-\infty, \infty)$

45. $f(x) = -2^{x+2} + 1$
47. $f(x) = -2^{-x} + 2$
49. $f(x) = -2(3)^x + 7$
51. $f(x) = 2\left(\frac{1}{2}\right)^x - 4$
53. $f(x) = 6(5)^x$
55. $f(x) = 2000(0.1)^x$
57. $f(x) = 3(2)^x$
59. $f(x) = 2.93(0.699)^x$
61. $f(x) = \frac{1}{8}(2)^x$
63. $33.58\, mg$
65. \$1,555,368.09 Annual growth rate: 1.39%
67. \$4,813.55
69. Annually: \$7353.84 Quarterly: \$47469.63 Monthly: \$7496.71 Continuously: \$7,501.44.
71. APY= .03034 $\approx 3.03\%$
73. $t = 7.4$ years
75. (a) $w(t) = 1.1130(1.0464)^t$; (b) \$1.11; (c) The actual minimum wage is less than the model predicted, using the equation you found in part a you can find w(36) which would correspond to the year 1996
77. (a) 512 dimes the first square would have 1 dime which is 2^0 the second would have 2 dimes which is 2^1 and so on, so the tenth square would have 2^9 or 512 dimes

(b) 2^{n-1} if n is the number of the square you are on the first square would have 1 dime which is 2^{1-1} the second would have 2 dimes which is 2^{2-1} the fifteenth square would have 16384 dimes which is 2^{15-1}

(c) 2^{63}, 2^{64-1}

(d) 9,223,372,036,854,775,808 mm

(e) There are 1 million millimeters in a kilometer, so the stack of dimes is about 9,223,372,036,855 km high, or about 9,223,372 million km. This is approximately 61,489 times greater than the distance of the earth to the sun.

Section 3.2

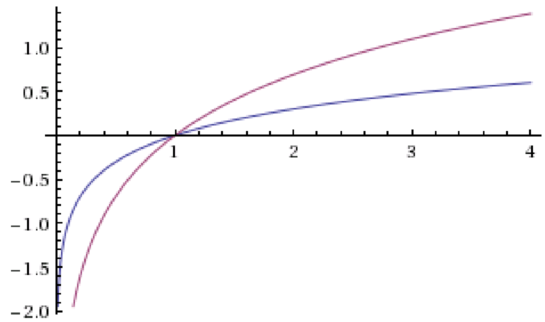
1. $4^m = q$
3. $a^c = b$
5. $10^t = v$
7. $e^n = w$
9. $\log_4 y = x$
11. $\log_c k = d$

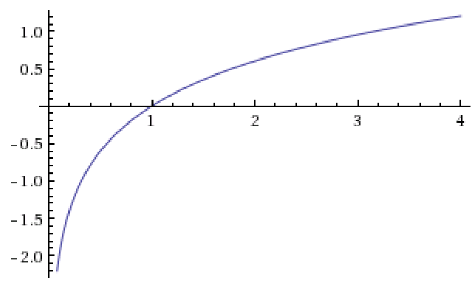
13. $\log b = a$
15. $\ln h = k$
17. 2
19. -1
21. 1
23. 3
25. $x = 9$
27. $x = \frac{1}{8}$
29. $x = 1000$
31. $x = e^2$ s
33. 2
35. -3
37. $\frac{1}{2}$
39. 4
41. -3
43. -2
45. $x = -1.398$ use calculator
47. $x = 2.708$ use calculator
49. $x \approx 1.639$
51. $x \approx -1.392$
53. $x \approx 0.567$
55. $x \approx 2.078$
57. $x \approx 54.449$
59. $x \approx 8.314$
61. $x \approx 13.412$
63. $x \approx .678$
65. $f(t) = 300e^{-.094t}$
67. $f(t) = 10e^{.0392t}$
69. $f(t) = 150(1.062)^t$
71. $f(t) = 50(.988)^t$
73. 4.78404 years
75. 74.2313 years
77. 34.0074 hrs
79. 13.5324 months

Section 3.3

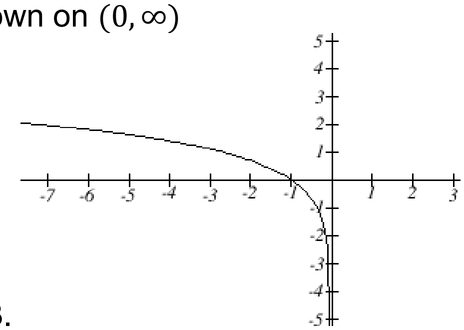
1. Domain: $x > 5$, vertical asymptote: $x = 5$.
3. Domain: $x < 3$, vertical asymptote: $x = 3$.
5. Domain: $x > -\frac{1}{3}$, vertical asymptote: $x = -\frac{1}{3}$.
7. Domain: $x < 0$, vertical asymptote: $x = 0$.

9.

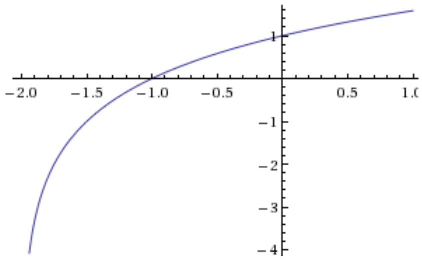




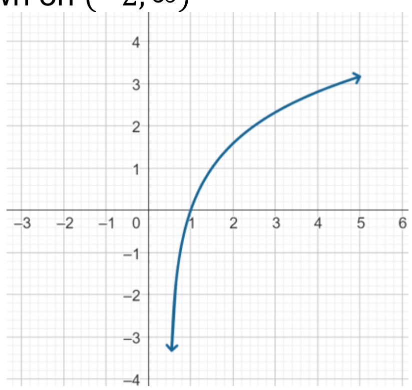
11.
Domain: $(0, \infty)$, Range: $(-\infty, \infty)$; Vertical asymptote $x = 0$; As $x \to 0^+, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$; x-intercept 1, no y-intercept; Increasing and concave down on $(0, \infty)$



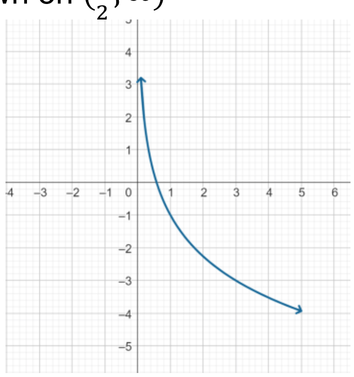
13.
Domain: $(-\infty, 0)$, Range: $(-\infty, \infty)$; Vertical asymptote $x = 0$; As $x \to 0^-, f(x) \to -\infty$, as $x \to -\infty, f(x) \to \infty$; x-intercept -1, no y-intercept; Decreasing and concave down on $(-\infty, 0)$

15. 

Domain: $(-2, \infty)$, Range: $(-\infty, \infty)$; Vertical asymptote $x = -2$; As $x \to -2^+, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$; x-intercept -1, y-intercept 1; Increasing and concave down on $(-2, \infty)$

17. 

Domain: $(\frac{1}{2}, \infty)$, Range: $(-\infty, \infty)$; Vertical asymptote $x = \frac{1}{2}$; As $x \to \frac{1}{2}^+, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$; x-intercept 1, no y-intercept; Increasing and concave down on $(\frac{1}{2}, \infty)$

19. 

Domain: $(0, \infty)$, Range: $(-\infty, \infty)$; Vertical asymptote $x = 0$; As $x \to 0^+, f(x) \to \infty$, as $x \to \infty, f(x) \to -\infty$; x-intercept $\sqrt{3}/3$, no y-intercpet; decreasing and concave up on $(0, \infty)$

17. $f(x) = 3.3219 \log(1 - x)$
19. $f(x) = -6.2877 \log(x + 4)$
21. $f(x) = 4.9829 \log(x + 2)$
23. $f(x) = -3.3219 \log(5 - x)$

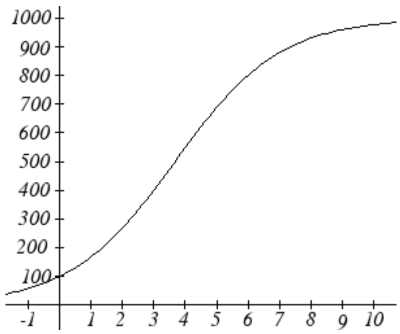
Section 3.4

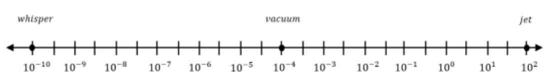
1. $\log_3 4$
3. $\log_3 7$
5. $\log_3 5$
7. $\log_7 2$
9. $\log(6x^9)$
11. $\ln(2x^7)$
13. $\log(x^2(x+1)^3)$
15. $\log\left(\frac{xz^3}{\sqrt{y}}\right)$
17. $15\log(x) + 13\log(y) - 19\log(z)$
19. $4\ln(b) - 2\ln(a) - 5\ln(c)$
21. $\frac{3}{2}\log(x) - 2\log(y)$
23. $\ln(y) + (\frac{1}{2}\ln(y) - \frac{1}{2}\ln(1-y))$
25. $2\log(x) + 3\log(y) + \frac{2}{3}\log(x) + \frac{5}{3}\log(y)$
27. $x \approx -.7167$
29. $x \approx -6.395$
31. $t \approx 17.329$
33. $x = \frac{2}{7}$
35. $x = \frac{1}{3e} \approx 0.1226$
37. $x = \sqrt[3]{100} \approx 4.642$
39. $x \approx 30.158$
41. $x = -\frac{26}{9} \approx -2.8889$
43. $x \approx -.872983$
45. $x = \frac{12}{11}$
47. $x = 10$

Section 3.5

1. 22.3 minutes
3. 129.3 mg
5. 0.01648 mg
7. 75.49 min
9. 422.169 years

11. (a) 23,700 bacteria; (b) 14,942 bacteria
13. (a) 611 bacteria. (b) 26.02 min. (c) 10,406 bacteria. (d) 107.057 min.
15. 23.19 years
17. 53.258 hours
19. a) 134.2°F (b) 1.879 hours, or 112.7 min.

21. 

(b) 100 fish; plug in $t = 0$
(c) 270 fish; plug in $t = 2$
(d) 7.34 years; let $P(t) = 900$ and solve for t.
23. 0.3162
25. 31.623
27.


29. $10^{4.8}$ or about 63,095.7 times greater.
31. 5.8167
33. (a) 1,640,671 bacteria
(b) 1.4 hours
(c) no, at small time values, the quantity is close enough, that you do not need to be worried. (d) You should not be worried, because both models are within an order of magnitude after 6 hours. Given the equation, you can plug in your time to solve for after 1 hour. You can then use the doubling formula to solve for t, by taking the natural log of both sides.
35. (a) $M(p)$ is the top graph $H(p)$ is the bottom graph
(b) 0.977507%
(c) $H(t) = 32.4\%, M(t) = 95.2\%$
(d) 20 torrs: 62.8%, 40 torrs: 20.6%, 60 torrs: 7.1%
37. (a) $C(t) = C_0 e^{.03466t}$ (b) $t = 433.87$ hours or $t \approx 18$ days
39. 31.5 days