SOLUTIONS

Friday, Feb 22, 2008 **SOLUTIONS** STUDENT NAME: ______________

EXAM I – MATH 135, SPRING 2008

READ EACH PROBLEM CAREFULLY! To get full credit, you must show all work!
The exam has 8 problems on 4 pages! Turn in all pages!
NO GRAPHING CALCULATORS ALLOWED!

- **Problem 1**
  Consider the function

$$f(x) = \frac{x^2 - 1}{x^2 - x - 2}$$

(a) Determine the domain of f. where denominator $\neq 0$

$x^2 - x - 2 \neq 0 \Rightarrow (x-2)(x+1) \neq 0 \Rightarrow \boxed{x \neq 2}$ and $\boxed{x \neq -1}$

Hence, domain is $\boxed{\text{all reals except } x = -1 \text{ and } x = 2}$

(b) Find **all vertical and horizontal asymptotes** of the graph $y = f(x)$. Explain!

Vertical asymptotes: where the function has infinite limits (at least one-sided)

$f(x) = \dfrac{(x-1)(x+1)}{(x-2)(x+1)} = \dfrac{x-1}{x-2} \Rightarrow \lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{x-1}{x-2} = \text{D.N.E}$

$\Rightarrow \boxed{x = 2}$ is vertical asymptote. (one sided limits are $\pm\infty$)

At $x = -1$ there is no vertical asymptote since $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to 2} \dfrac{x-1}{x-2} = \dfrac{-2}{-3} = \dfrac{2}{3}$ finite

Horizontal asymptote for $y = f(x)$ is $\boxed{y = 1}$ since $\lim\limits_{x \to \pm\infty} f(x) = 1$

- **Problem 2**
  Evaluate the limit

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} =$$

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \lim_{x \to 1} \frac{(\sqrt{x})^2 - 1^2}{(x-1)(\sqrt{x}+1)} = \lim_{x \to 1} \frac{x - 1}{(x-1)(\sqrt{x}+1)}$$

$$= \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{\sqrt{1} + 1} = \boxed{\frac{1}{2}}.$$

- **Problem 3**
  Using the precise $(\varepsilon - \delta)$ definition of the limit, prove that

$$\lim_{x \to -1} (3x+4) = 1$$

Fix $\varepsilon > 0$. We need to find δ such that

$$\text{if } 0 < |x-(-1)| < \delta \quad \text{then} \quad |(3x+4)-1| < \varepsilon$$

or

$$\text{if } 0 < |x+1| < \delta \quad \text{then} \quad |3x+3| < \varepsilon$$

$$|3(x+1)| < \varepsilon$$

$$|x+1| < \varepsilon/3$$

Hence we can pick $\boxed{\delta \le \frac{\varepsilon}{3}}$. This proves the statement!

- **Problem 4**
  Find the limit

$$\lim_{x \to \infty} \left(x - \sqrt{x^2 - 2x}\right)$$

$$x - \sqrt{x^2-2x} = \left(x - \sqrt{x^2-2x}\right) \frac{x+\sqrt{x^2-2x}}{x+\sqrt{x^2-2x}}$$

$$= \frac{x^2 - \left(\sqrt{x^2-2x}\right)^2}{x+\sqrt{x^2-2x}} = \frac{x^2-(x^2-2x)}{x+\sqrt{x^2-2x}}$$

$$= \frac{2x}{x+\sqrt{x^2-2x}}$$

$$\lim_{x\to\infty}\left(x-\sqrt{x^2-2x}\right) = \lim_{x\to\infty} \frac{2x}{x+\sqrt{x^2-2x}} = \lim_{x\to\infty} \frac{2}{1+\sqrt{1-\frac{2}{x}}}$$

(divide by x both num & denom)

$$= \frac{2}{1+\sqrt{1-0}} = \frac{2}{2} = \boxed{1}.$$

- **Problem 5**

Consider the function

$$f(x) = \begin{cases} x+2, & \text{if } x < 0 \\ x^2 - 1, & \text{if } 0 \le x < 2 \\ 3, & \text{if } x \ge 2 \end{cases}$$

Determine the points where f is discontinuous, if any. Give reasons!

Only possible discontinuities at $x=0$ and $x=2$.

At $x=0$: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(x+2) = 2$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(x^2-1) = -1$ $\Rightarrow \lim_{x\to 0} f(x)$ D.N.E

hence $\boxed{f \text{ is discontinuous at } x=0,}$

At $x=2$: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^2-1) = 2^2-1 = 3$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3 = 3$ $\Rightarrow \lim_{x\to 2} f(x) = 3 = f(2)$

$\Rightarrow f$ is continuous at $x=2$.

- **Problem 6**

Using the **limit definition of the derivative**, find $f'(x)$, where

$$f(x) = \sqrt{x+3}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \cdot \frac{(\sqrt{x+h+3} + \sqrt{x+3})}{(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$= \lim_{h\to 0} \frac{x+h+3 - (x+3)}{h(\sqrt{x+h+3} + \sqrt{x+3})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}} = \frac{1}{\sqrt{x+3} + \sqrt{x+3}} = \boxed{\frac{1}{2\sqrt{x+3}}}$$

- **Problem 7**
  Find the **equation of the tangent line** to the parabola $y = x^2 - 4x + 5$ at the point $P(1, 2)$.

Slope: $y'(x) = 2x - 4$. At $x = 1 \Rightarrow y'(1) = 2 \cdot 1 - 4 = -2$

Point-slope formula: $y - 2 = -2(x - 1)$

$y - y_1 = m(x - x_1)$

$y - 2 = -2x + 2 \Rightarrow \boxed{y = -2x + 4}$

or

$\boxed{2x + y = 4}$.

- **Problem 8**
  Differentiate the function

$$h(x) = \frac{x+1}{x^2+3}$$

and determine the points where the tangent line to the graph $y = h(x)$ is horizontal.

Using Quotient Rule:

$$h'(x) = \frac{1 \cdot (x^2+3) - (x+1)(2x)}{(x^2+3)^2}$$

$$= \frac{x^2 + 3 - 2x^2 - 2x}{(x^2+3)^2}$$

$$= \frac{-x^2 - 2x + 3}{(x^2+3)^2}$$

Horizontal tangents happen when $h'(x) = 0$

$\Rightarrow -x^2 - 2x + 3 = 0$ or $x^2 + 2x - 3 = 0$

$\Rightarrow (x+3)(x-1) = 0 \Rightarrow \boxed{x = -3}$ or $\boxed{x = 1}$