SOLUTIONS

Friday, Mar 14, 2008 [redacted]

EXAM II – MATH 135, SPRING 2008

READ EACH PROBLEM CAREFULLY! To get full credit, you must show all work!
The exam has 7 problems on 3 pages! Turn in all pages!
NO GRAPHING CALCULATORS ALLOWED!

- **Problem 1**

- Find the derivative of the following functions

(a) $f(x) = e^{\sin(3x)}$, $f'(x) = e^{\sin(3x)} \cos(3x) \cdot 3 = 3e^{\sin(3x)} \cos(3x)$.

(b) $g(x) = \ln(2x+1)$, $g'(x) = \frac{1}{2x+1} \cdot 2 = \frac{2}{2x+1}$

- **Problem 2**

Find the equation of the tangent line to the curve

$$y = x^2 e^{x-1}$$

at the point where $x = 1$.

$$\frac{dy}{dx} = 2xe^{x-1} + x^2 e^{x-1} = (2x + x^2)e^{x-1}$$

At $x = 1 \Rightarrow y = 1^2 e^{1-1} = 1$

$$m = \frac{dy}{dx}(1) = (2 \cdot 1 + 1^2)e^{1-1} = 3$$

$\Rightarrow$ Equation of tangent line

$y - 1 = 3(x-1) \Rightarrow y - 1 = 3x - 3 \Rightarrow \boxed{y = 3x - 2}$

- **Problem 3**

(a) Find the derivative of the function

$$f(x) = \frac{\sqrt{x}}{(x+1)^2} \quad , \quad \text{Domain: } x \geq 0$$

(b) Determine the points (if any) where the tangent line to the graph $y = f(x)$ is horizontal.

(a) $f'(x) = \dfrac{\frac{1}{2\sqrt{x}}(x+1)^2 - \sqrt{x}\cdot 2(x+1)}{(x+1)^4} = \dfrac{\frac{1}{2\sqrt{x}}(x+1)[x+1-2x]}{(x+1)^4} =$

$= \dfrac{(x+1)(1-x)}{2\sqrt{x}(x+1)^4} = \dfrac{1-x}{2\sqrt{x}(x+1)^3}$.

(b) Tangent line is horizontal when $f'(x)=0 \Rightarrow \dfrac{1-x}{2\sqrt{x}(x+1)^3}=0 \Rightarrow \boxed{x=1}$

- **Problem 4**

Using implicit differentiation, find $\frac{dy}{dx}$, given that $x^2 + 2y^2 = 3$.

$2x + 4y\dfrac{dy}{dx} = 0 \Rightarrow 4y\dfrac{dy}{dx} = -2x \Rightarrow \dfrac{dy}{dx} = -\dfrac{2x}{4y}$

or $\boxed{\dfrac{dy}{dx} = -\dfrac{x}{2y}}$

- **Problem 5**

(a) Find the linear approximation of $f(x) = \sqrt{1+3x}$ near $x = 0$.
(b) Write down the differential dy, where $y = \sqrt{1+3x}$.

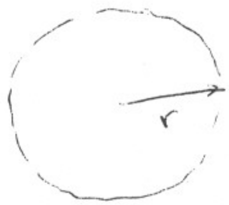
(a) $f(x) \simeq f(0) + f'(0)(x-0)$. near $x=0$. $f(0)=1$

$f'(x) = \frac{1}{2}(1+3x)^{-1/2}\cdot 3 = \dfrac{3}{2\sqrt{1+3x}} \Rightarrow f'(0) = \dfrac{3}{2\sqrt{1+3\cdot 0}} = \dfrac{3}{2}$

$\Rightarrow \boxed{f(x) \simeq 1 + \frac{3}{2}x}$ near $x=0$

(b) $dy = \dfrac{dy}{dx}dx = \boxed{\dfrac{3}{2\sqrt{1+3x}}dx}$.

- **Problem 6**
If a (spherical) snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which its diameter decreases when the diameter is 10cm. [Surface area of a sphere is $S = 4\pi r^2$, where r is the radius.]



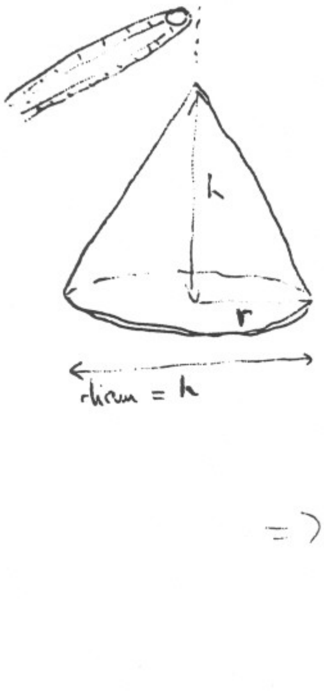
diameter $\tilde{d} = 2r \Rightarrow r = \frac{\tilde{d}}{2}$

surface area $S = 4\pi r^2 = 4\pi\left(\frac{\tilde{d}}{2}\right)^2 = \pi \tilde{d}^2 \Rightarrow S = \pi \tilde{d}^2$

$\Rightarrow \frac{dS}{d\tilde{d}} = 2\pi \tilde{d}$

$$\frac{dS}{dt} = \frac{dS}{d\tilde{d}} \frac{d\tilde{d}}{dt} = 2\pi \tilde{d} \frac{d\tilde{d}}{dt} \Rightarrow 1 = 2\pi \cdot 10 \frac{d\tilde{d}}{dt} \Rightarrow \frac{d\tilde{d}}{dt} = \boxed{\frac{1}{20\pi} \ cm/min}$$

- **Problem 7**
Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and it forms a pile in the shape of a cone whose base diameter and height are always equal (see picture). How fast is the height of the pile increasing when the pile is 10 ft high? [Volume of a cone is $V = \frac{1}{3}\pi r^2 h$.]



$r = \frac{1}{2} h$

$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

$\Rightarrow \frac{dV}{dh} = \frac{1}{12}\pi \, 3h^2 = \frac{1}{4}\pi h^2$

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$$\Rightarrow 30 = \frac{1}{4}\pi \cdot 10^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{120}{100\pi} = \boxed{\frac{12}{10\pi} \ ft/min}$$