SOLUTIONS

Friday, April 18, 2008 ~~STUDENT NAME:~~ __________

EXAM III – MATH 135, SPRING 2008

READ EACH PROBLEM CAREFULLY! To get full credit, you must show all work!
The exam has 7 problems on 4 pages! Turn in all pages! NO GRAPHING CALCULATORS ALLOWED!

- **Problem 1.**
  The half-live of Cesium-137 is 30 years. Suppose we have a 100-mg sample.
  (a) Find the mass that remains after t years.

  (b) How much of the sample remains after 100 years?

  (c) After how long will only 1 mg remain?

(a) $m(t) = m_0 e^{kt}$. To find k, use $t_h = 30$ half-life $\frac{1}{2} m_0 = m_0 e^{kt_h} \Rightarrow \frac{1}{2} = e^{kt_h} \Rightarrow$ $\Rightarrow k = -\frac{\ln 2}{t_h} = -\frac{\ln 2}{30}$

$\Rightarrow \boxed{m(t) = 100 e^{-\frac{\ln 2}{30} t}}$

(b) $m(100) = 100 e^{-\frac{\ln 2}{30}\cdot 100} \simeq \boxed{9.92 \text{ mg}}$

(c) $1 = 100 e^{-\frac{\ln 2}{30} t} \Rightarrow \frac{1}{100} = e^{-\frac{\ln 2}{30} t} \Rightarrow -\ln 100 = -\frac{\ln 2}{30} t \Rightarrow t = \frac{30 \ln 100}{\ln 2} \approx \boxed{199.32}$

- **Problem 2.**
  Use l'Hôpital's Rule to compute the following limits.

$$(a) \lim_{x\to\infty} \frac{x}{e^x} \overset{l'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = \boxed{0}$$

$$(b) \lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \boxed{0}$$ by l'Hopital since

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x\to 0^+} -\frac{1}{x}\frac{x^3}{2} = \lim_{x\to 0^+}\left(-\frac{1}{2}x^2\right) = 0.$$

$$(c) \lim_{x\to 0} \frac{1-\cos x}{\sin x} \overset{l'H}{=} \lim_{x\to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = \boxed{0}.$$

- **Problem 3.**
Find the local and absolute extreme values of the function on the interval specified.

$$f(x) = \frac{\ln x}{x^2}, \quad x \in [1, e]$$

$$f'(x) = \frac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{x^4} = \frac{x - 2x \ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$$

Critical points : $f'(x) = 0 \Rightarrow 1 - 2\ln x = 0 \Rightarrow 1 = 2\ln x \Rightarrow \frac{1}{2} = \ln x$

$\Rightarrow x = e^{\frac{1}{2}}$. At $x = e^{\frac{1}{2}}$, $f(e^{1/2}) = \frac{\ln e^{1/2}}{(e^{1/2})^2} = \frac{\frac{1}{2}}{e} = \boxed{\frac{1}{2e}}$

At the end points $f(1) = \frac{\ln 1}{1^2} = \boxed{0}$ and $f(e) = \frac{\ln e}{e^2} = \boxed{\frac{1}{e^2}} < \frac{1}{2e}$

$\Rightarrow$ Abs. max $= \frac{1}{2e}$

Abs min $= 0$

$x = e^{1/2}$ is local max

$x = e$ is local min.

- **Problem 4.**
Find the maximum value of the function

$$f(x) = x(1 - x), \quad x \in [0, 1]$$

$f(x) = x - x^2$

$f'(x) = 1 - 2x \Rightarrow f'(x) = 0$ at $x = \frac{1}{2}$

$\rightarrow f''(x) = -2 < 0 \rightarrow$

$\rightarrow x = \frac{1}{2}$ is a local max, $f(\frac{1}{2}) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$

At end points $x = 0, x = 1$, $f(0) = 0 = f(1)$

$\Rightarrow$ max is $\boxed{\frac{1}{4}}$.

- **Problem 5.**
Find all the inflection points (if any) for the function

$$f(x) = x(x + 1)^3.$$

$$f'(x) = 1 \cdot (x+1)^3 + x \cdot 3(x+1)^2 = (x+1)^2(x+1+3x) = (x+1)^2(4x+1).$$

$$f''(x) = 2(x+1)(4x+1) + (x+1)^2(4) = 2(x+1)[(4x+1) + 2(x+1)]$$

$$= 2(x+1)(6x+3) = 6(x+1)(2x+1)$$

$f''(x) = 0$ when $x = -1$ and $x = -\frac{1}{2}$. At both points, f'' changes sign $\Rightarrow$ both are inflection points!

x	$-\infty$			-1			$-\frac{1}{2}$			$+\infty$
f''	+	+	+	0	−	−	0	+	+	+

- **Problem 6.**
Sketch the graph of the function

$$f(x) = \frac{x}{1+x^2}$$

Determine: (1) the domain, (2) symmetries (if any), (3) the critical points, (4) intervals where the function increases or decreases, (5) concavity and inflection points, and (6) limits at infinity. Remember, a sign chart is always of great help!

(1) domain = all reals ($1+x^2 \neq 0$ for all x)

(2) symmetries : odd function $f(-x) = -f(x) \Rightarrow$ sym. about origin

(3) $f'(x) = \dfrac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} = \dfrac{1+x^2-2x^2}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2} \Rightarrow \boxed{x = \pm 1}$ are critical points

(4)

x	$-\infty$		-1		1		$+\infty$
$\frac{1-x^2}{(1+x^2)^2}$		$- - -$	0	$+ +$	0	$- - -$	

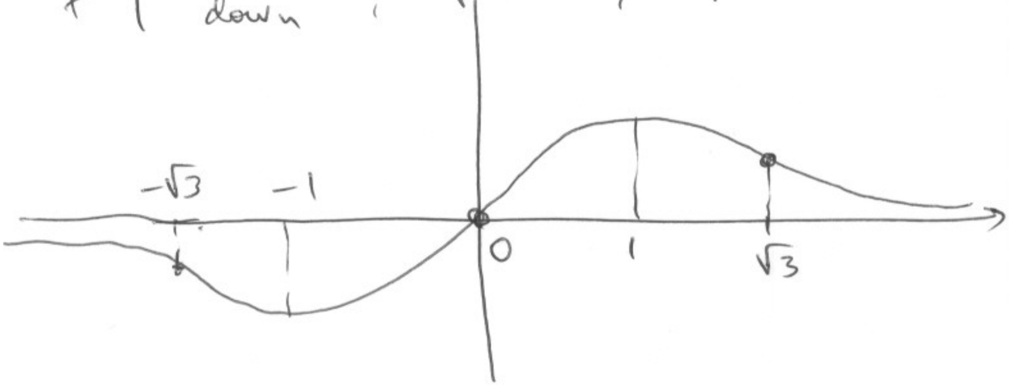
$\Rightarrow$ f is increasing on $(-1, 1)$
f is decreasing on $(1, \infty)$ and $(-\infty, -1)$

(5) $f''(x) = \dfrac{-2x\,(1+x^2)^2 - (1-x^2)\,2(1+x^2)\cdot 2x}{(1+x^2)^4} = \dfrac{-2x(1+x^2)\left[(1+x^2) + 2(1-x^2)\right]}{(1+x^2)^4}$

$$= \frac{-2x\,(3-x^2)}{(1+x^2)^3}$$

x	$-\infty$		$-\sqrt{3}$		0		$\sqrt{3}$		$+\infty$
$3-x^2$		$- - -$	0	$+ +$		$+$	0	$- -$	
f''		$-$	0	$+ +$	0	$- -$	0	$+ + +$	
f		concave down		up		down		up	

(6) $\displaystyle\lim_{x\to\pm\infty} \frac{x}{1+x^2} = 0.$



- **Problem 7.**
Consider the function
$$f(x) = (x^2 - 1)^2.$$
(a) Find the critical points for f. Find relative maxima and minima points.
(b) On which intervals(s) is f concave upwards? Are there any inflection points?

(a) $f'(x) = 2(x^2-1)\cdot 2x = 4x(x^2-1)$ -

$f'(x)=0 \Rightarrow \boxed{x=0}$ and $\boxed{x=\pm 1}$ are all critical points for f

Use the second derivative test:

$f''(x) = ?$ $\quad f'(x) = 4x^3 - 4x \Rightarrow f''(x) = 12x^2 - 4 = 4(3x^2-1)$

$f''(0) = -4 < 0 \Rightarrow x=0$ is a local max.

$f''(1) = 12-4 > 0 \Rightarrow x=1$ is local min.

$f''(-1) = 12-4 > 0 \Rightarrow x=-1$ is local min

(b) $f''(x) = 4(3x^2-1) \Rightarrow$

x	$-\infty$		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$		$+\infty$
f''		$+\ +$	0	$-\ -$	0	$+\ +\ +$	

$f''(x)=0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow f$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and on $\left(\frac{1}{\sqrt{3}}, +\infty\right)$

and concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$ are both inflection points!