SOLUTIONS TO PRACTICE EXAM I

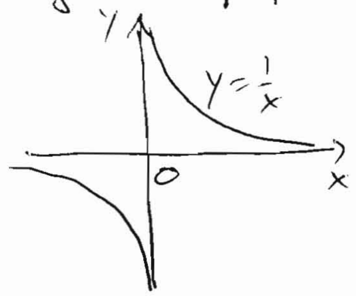
MATH 135-004 SPRING 2008

① $f(x) = \frac{1}{x+3}$. Domain of f : $x+3 \neq 0 \Rightarrow \boxed{x \neq -3}$

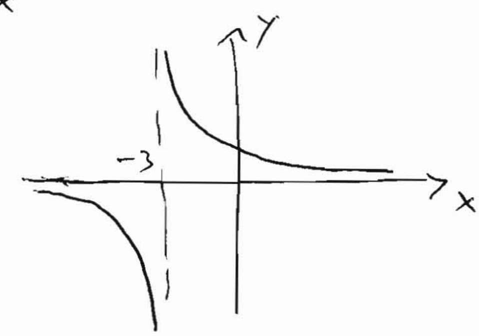
or

$(-\infty,-3) \cup (3,+\infty)$

We sketch the graph by transforming the graph of $y = \frac{1}{x}$



$\Rightarrow$



② Find the limit

$$\lim_{x \to 3} \frac{x^2-9}{x^2-2x-3} \qquad \left(\frac{0}{0} \text{ "undetermined" case}\right)$$

Factor both numerator and denominator:

$$\frac{x^2-9}{x^2-2x-3} = \frac{(x-3)(x+3)}{(x-3)(x+1)} = \frac{x+3}{x+1}, \quad x \neq 3$$

Hence $\lim_{x \to 3} \frac{x^2-9}{x^2-2x-3} = \lim_{x \to 3} \frac{x+3}{x+1} = \frac{3+3}{3+1} = \frac{6}{4} = \boxed{\frac{3}{2}}$

(3) $\lim_{x \to 2} (5-2x) = 1$

Fix $\varepsilon > 0$ (small). We need to find $\delta > 0$ such that

$|(5-2x)-1| < \varepsilon$ whenever $0 < |x-2| < \delta$

$|(5-2x)-1| < \varepsilon \iff |5-2x-1| < \varepsilon \iff |4-2x| < \varepsilon$

$\iff |2(x-2)| < \varepsilon \Rightarrow |x-2| < \varepsilon/2$

So it is enough to pick $\boxed{\delta = \varepsilon/2}$.

(4)

$$f(x) = \begin{cases} \sqrt{-x}, & x < 0 \\ 3-x, & 0 \le x < 3 \\ (x-3)^2, & x \ge 3 \end{cases}$$

The function f is continuous on $(-\infty, 0)$, $(0,3)$ and $(3,\infty)$ being expressed in terms of elementary functions.

At $x=0$: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0$
$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3-x = 3$
$\Rightarrow$ f is discontinuous at $x=0$

At $x=3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3-x) = 0$
$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x-3)^2 = 0$
$f(3) = (3-3)^2 = 0$
$\Rightarrow \lim_{x \to 3} f(x) = f(3)$ that is f is continuous at $x=3$

Hence, f is discontinuous <u>only</u> at $x=0$!

(5) $\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right) =$

$= \lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right)\dfrac{\sqrt{x^2+4x+1}+x}{\sqrt{x^2+4x+1}+x}$

$= \lim_{x\to\infty}\dfrac{(x^2+4x+1)-x^2}{\sqrt{x^2+4x+1}+x}$

$= \lim_{x\to\infty}\dfrac{4x+1}{\sqrt{x^2+4x+1}+x}$ (divide both num & den by x

$= \lim_{x\to\infty}\dfrac{4+\frac{1}{x}}{\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+1} =$

$= \dfrac{4+0}{\sqrt{1+0+0}+1} = \dfrac{4}{2} = \boxed{2}.$

(6) $f(x) = \dfrac{1}{\sqrt{x+2}}$. Domain of f : $x+2>0$

$\Rightarrow x>-2$

or $\boxed{(-2,+\infty)}$

$f'(x) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{\frac{1}{\sqrt{x+h+2}}-\frac{1}{\sqrt{x+2}}}{h}$

$= \lim_{h\to 0}\dfrac{\sqrt{x+2}-\sqrt{x+h+2}}{h(\sqrt{x+h+2})(\sqrt{x+2})}$ Multiply & divide by conjugate

$$= \lim_{h \to 0} \frac{\sqrt{x+2} - \sqrt{x+h+2}}{h\sqrt{x+h+2}\,\sqrt{x+2}} \cdot \frac{\sqrt{x+2} + \sqrt{x+h+2}}{\sqrt{x+2} + \sqrt{x+h+2}}$$

$$= \lim_{h \to 0} \frac{(x+2) - (x+h+2)}{h\sqrt{x+h+2}\,\sqrt{x+2}\left(\sqrt{x+2} + \sqrt{x+h+2}\right)}$$

$$= \lim_{h \to 0} \frac{-h}{h(- \ldots)}$$

$$= \lim_{h \to 0} \left(- \frac{1}{\sqrt{x+h+2}\,\sqrt{x+2}\left(\sqrt{x+2} + \sqrt{x+h+2}\right)} \right)$$

$$= - \frac{1}{\sqrt{x+2}\,\sqrt{x+2}\left(\sqrt{x+2} + \sqrt{x+2}\right)} = - \frac{1}{2(x+2)\sqrt{x+2}}.$$

So $\boxed{f'(x) = - \frac{1}{2(x+2)\sqrt{x+2}}}$

Domain of f' is $x+2 > 0 \Rightarrow x > -2$ or $\boxed{(-2, +\infty)}$

7. $y = x + \sqrt{x}$

$\rightarrow \frac{dy}{dx} = 1 + \frac{1}{2\sqrt{x}}$. At $x = 1 \Rightarrow \frac{dy}{dx}(1) = 1 + \frac{1}{2}$

$= \frac{3}{2}$

Eq. of tangent line at $P(1,2)$

$$(y - 2 = \frac{3}{2}(x-1)$$

$$\rightarrow y = 2 + \frac{3}{2}x - \frac{3}{2} \Rightarrow \boxed{y = \frac{3}{2}x + \frac{1}{2}}.$$

8. $g(t) = \frac{t}{1-t^2}$. Using Quotient Rule,

$$g'(t) = \frac{1(1-t^2) - t(-2t)}{(1-t^2)^2}$$

$$= \frac{1-t^2+2t^2}{(1-t^2)^2} = \frac{1+t^2}{(1-t^2)^2}$$

$g'(t) \neq 0$ since $1 + t^2 \neq 0$ for all t

$\Rightarrow$ g has no horizontal tangent lines.