SOLUTIONS TO SAMPLE EXAM 3

MATH 135 - SPRING 2008

(1) $m(t) = m_0 e^{kt}$; Half-life (t_h): $\frac{1}{2} m_0 = m_0 e^{kt_h} \Rightarrow$

$\Rightarrow \frac{1}{2} = e^{kt_h} \Rightarrow kt_h = \ln \frac{1}{2} = -\ln 2$

$k = \frac{-\ln 2}{t_h}$

If $t_h = 5.24$ (years) $\Rightarrow k = -\frac{\ln 2}{5.24}$.

(a) If $\left.\begin{array}{l} m_0 = 100 \text{ (mg)} \\ t = 20 \text{ (years)} \end{array}\right] \Rightarrow m(20) = 100\, e^{k \cdot 20} =$

$= 100\, e^{-\frac{\ln 2}{5.24} \cdot 20}$

$= \boxed{7.1 \text{ (mg)}}$

(b) If $m(t) = 1$ mg $\Rightarrow 1 = 100 e^{kt} \Rightarrow \frac{1}{100} = e^{kt}$

$\Rightarrow kt = \ln \frac{1}{100} = -\ln 100 \Rightarrow t = \frac{-\ln 100}{k} = \frac{-\ln 100}{-\ln 2}(5.24)$

$= \boxed{34.81 \text{ (years)}}$

(2)

(a) $\lim_{x \to \infty} \frac{\ln 3x}{x} \overset{l'H}{=} \lim_{x \to \infty} \frac{\frac{1}{3x} \cdot 3}{1} = \lim_{x \to \infty} \frac{1}{x} = 0.$

(b) $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2} \overset{l'H}{=} \lim_{x \to 0} \frac{2e^{2x} - 2}{2x} = \lim_{x \to 0} \frac{e^{2x} - 1}{x} \overset{l'H}{=}$

$\overset{l'H}{=} \lim_{x \to 0} \frac{2e^{2x}}{1} = \lim_{x \to 0} 2e^{2x} = 2e^0 = \boxed{2}.$

(c) $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1}\frac{\ln x-(x-1)}{(x-1)\ln x}$

$$\overset{l'H}{=}\lim_{x\to 1}\frac{\frac{1}{x}-1}{\ln x+\frac{x-1}{x}}=\lim_{x\to 1}\frac{\frac{1-x}{x}}{\frac{x\ln x+x-1}{x}}=$$

$$=\lim_{x\to 1}\frac{1-x}{x\ln x+x-1}\overset{l'H}{=}\lim_{x\to 1}\frac{-1}{\ln x+x\cdot\frac{1}{x}+1}=$$

$$=\lim_{x\to 1}\frac{-1}{\ln x+2}=\boxed{-\frac{1}{2}}.$$

(d) $\lim_{x\to 0^+}\frac{1-\cos x}{x^3}\overset{l'H}{=}\lim_{x\to 0^+}\frac{\sin x}{3x^2}=\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\frac{1}{3x}$

$$=\lim_{x\to 0^+}\frac{\sin x}{x}\lim_{x\to 0^+}\frac{1}{3x}=1\cdot\lim_{x\to 0^+}\frac{1}{3x}=+\infty.$$

Also $\lim_{x\to 0^-}\frac{1-\cos x}{x^3}=-\infty$ so $\lim_{x\to 0}\frac{1-\cos x}{x^3}$ $\boxed{DNE}$.

(e) $\lim_{x\to 2}\frac{\sqrt{5+2x}-3}{x-2}\overset{l'H}{=}\lim_{x\to 2}\frac{\frac{1}{2\sqrt{5+2x}}\cdot 2-0}{1}=\lim_{x\to 2}\frac{1}{\sqrt{5+2x}}=$

$$=\frac{1}{\sqrt{5+2\cdot 2}}=\boxed{\frac{1}{3}}.$$

③ (a) $f(x) = x^4 - 2x^2 + 3$, $x \in [-2, 3]$

$$f'(x) = 4x^3 - 4x \Rightarrow \text{critical points}$$
$$= 4x(x^2-1) \qquad (f'(x)=0)$$
$$= 4x(x-1)(x+1) \quad \text{are } \boxed{x=0}, \boxed{x=1}, \boxed{x=-1}$$

Since all are in the interval $[-2, 3]$:

$$f''(x) = 12x^2 - 4 \quad , \quad \text{(second deriv. test} \Rightarrow \begin{cases} f''(0) = -4 < 0 \Rightarrow x=0 \text{ local max} \\ f''(1) = 12-4 > 0 \Rightarrow x=1 \text{ local min} \\ f''(-1) = 12-4>0 \Rightarrow x=-1 \text{ local min} \end{cases}$$

interior critical points
$$\begin{cases} f(0) = 3 \\ f(1) = 1^4 - 2\cdot 1^2 + 3 = 2 \\ f(-1) = (-1)^4 - 2(-1)^2 + 3 = 2 \end{cases} \rightarrow \boxed{\text{absolute min value} = 2}$$

end points
$$\begin{cases} f(-2) = (-2)^4 - 2(-2)^2 + 3 = 16 - 8 + 3 = 11 \\ f(3) = 3^4 - 2\cdot 3^2 + 3 = 81 - 18 + 3 = 66 \end{cases} \leftarrow \boxed{\text{absolute maximum value} = 66}$$

(b), (c) similar

3

(b) $f(x) = \frac{\ln x}{x}, \quad x \in [1, e]$

$$f'(x) = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$f'(x) = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$

$\Rightarrow$ no interior critical point, hence no local extreme. inside $[1, e]$

$f(1) = \frac{\ln 1}{1} = 0$

$f(e) = \frac{\ln e}{e} = \frac{1}{e}$

$\Rightarrow$ absolute max ~~value~~ $= \frac{1}{e}$

absolute min value $= 0$

$\Rightarrow x = \frac{1}{e}$ is a local max point.

(c) $f(x) = x\sqrt{4 - x^2}, \quad x \in [-1, 2]$

$f(-1) = -\sqrt{3}$

$f(2) = 0$

$f(-\sqrt{2}) = -2 \leftarrow \boxed{\text{abs min}}$

$f(\sqrt{2}) = 2 \leftarrow \boxed{\text{abs max}}$

$$f'(x) = 1 \cdot \sqrt{4 - x^2} + x \frac{1}{2\sqrt{4 - x^2}} \cdot (-2x) =$$

$$= \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = \frac{(4 - x^2) - x^2}{\sqrt{4 - x^2}} = \frac{4 - 2x^2}{\sqrt{4 - x^2}}$$

$\Rightarrow f'(x) = 0$ when $4 - 2x^2 = 0 \Rightarrow 2(2 - x^2) = 0 \Rightarrow x^2 = 2$

$\Rightarrow \boxed{x = \pm\sqrt{2}}$

x	$-$[illegible]		$-\sqrt{2}$		$+\sqrt{2}$		$+2$
$f'(x)$		$- - -$	0	$+ +$	0	$- - -$	
$f(x)$		$\searrow$		$\nearrow$		$\searrow$	

$\Rightarrow x = -\sqrt{2}$ is local min

$x = +\sqrt{2}$ is local max.

(4) $f(x) = \sin x + \cos x \quad , \quad x \in [0, \pi]$

$f'(x) = \cos x - \sin x = 0 \Rightarrow \sin x = \cos x$

$\Rightarrow \tan x = \frac{\sin x}{\cos x} = 1 \Rightarrow x = \frac{\pi}{4}$

$\Rightarrow$ only on interior point which is also critical point

$f''(x) = -\sin x - \cos x \quad , \quad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} < 0$

$\Rightarrow x = \frac{\pi}{4}$ is local max

$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \boxed{\sqrt{2}}$

$f(0) = \sin 0 + \cos 0 = \boxed{1}$

$f(\pi) = \sin \pi + \cos \pi = \boxed{-1}$

$\Rightarrow$ Abs. max value $= \sqrt{2}$.

(5) $f(x) = x e^{-2x}$

$f'(x) = e^{-2x} + x(-2)e^{-2x} = (1-2x)e^{-2x}$

$f''(x) = (-2)e^{-2x} + (1-2x)(-2)e^{-2x} =$

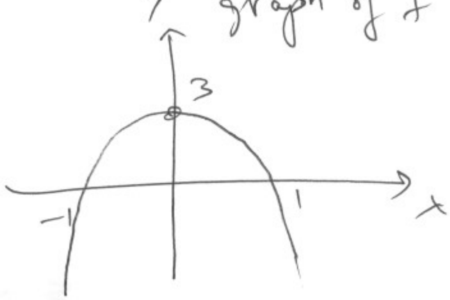
$= (-2)e^{-2x}\left(1 + (1-2x)\right) = -2e^{-2x}(2-2x)$

$= -4(1-x)e^{-2x}$

$f''(x) = 0 \Rightarrow \boxed{x=1}$. f'' changes sign at $x=1 \Rightarrow \boxed{x=1}$ inflection!

(6) $f(x) = 2 + 3x - x^3$

graph of f'



(a) $f'(x) = 3 - 3x^2 \Rightarrow$

$= 3(1-x^2)$

(b) Critical points for f are where $f'(x) = 0 \Rightarrow 1 - x^2 = 0$

$x^2 = 1$

$\boxed{x = \pm 1}$

$f''(x) = -6x$

$f''(1) = -6 < 0 \Rightarrow x = 1$ is local max

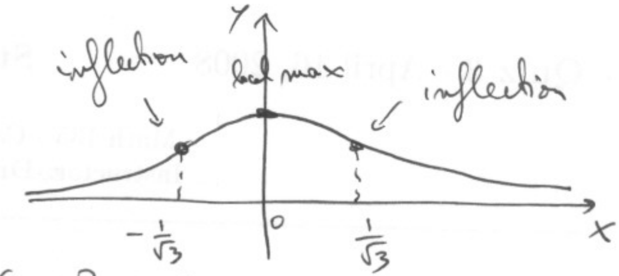
$f''(-1) = -6(-1) = 6 > 0 \Rightarrow x = -1$ is local min

(c) $f'' > 0$ on $(-\infty, 0)$ and $f'' < 0$ on $(0, \infty)$

$\Rightarrow$ f is concave up on $(-\infty, 0)$ and down on $(0, \infty)$

$x = 0$ is an inflection point.

(7) $f(x) = \dfrac{1}{1+x^2}$



(1) domain : all reals ($1+x^2 \neq 0$ for all x)

(2) symmetries : f is even (since $f(-x) = \dfrac{1}{1+(-x)^2} = \dfrac{1}{1+x^2} = f(x)$)

(3) critical points : $f'(x) = -\dfrac{2x}{(1+x^2)^2}$

$\Rightarrow$ $f'(x) = 0$ at $x = 0$ only $\Rightarrow$ $\boxed{x = 0}$ is the only critical point

(4) Also sign chart of $f'(x)$

x	$-\infty$			0			$+\infty$
$f'(x) = \dfrac{-2x}{(1+x^2)^2}$		+ + +		0		− − −	
$f(x) = \dfrac{1}{1+x^2}$		↗				↘	

$\Rightarrow$ f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$

(5) $f'(x) = -2 \dfrac{x}{(1+x^2)^2}$

$$\Rightarrow f''(x) = -2 \frac{1\cdot(1+x^2)^2 - x\cdot 2(1+x^2)\,2x}{(1+x^2)^4} = -2 \frac{(1+x^2)\left[1+x^2-4x^2\right]}{(1+x^2)^4}$$

$$= -2 \frac{(1-3x^2)}{(1+x^2)^3}.$$

$f''(x) = 0 \Rightarrow 1 - 3x^2 = 0 \Rightarrow 3x^2 = 1 \Rightarrow \boxed{x = \pm \frac{1}{\sqrt{3}}}$

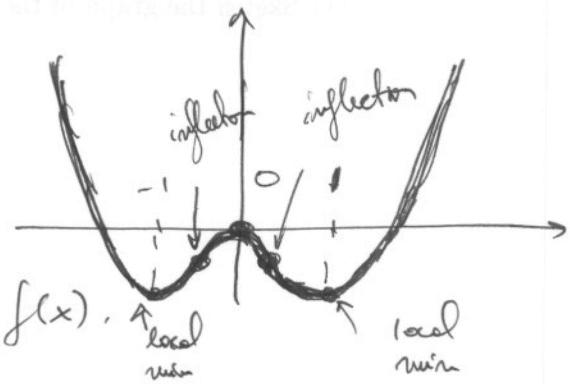
Since

x	$-\infty$		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$		$+\infty$
f''		+ + +	0	− −	0	+ + +	

$\Rightarrow$ both $\pm \frac{1}{\sqrt{3}}$ are inflection points

f is concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and on $\left(\frac{1}{\sqrt{3}}, +\infty\right)$

and concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



(8) $f(x) = x^4 - 2x^2$

(1) domain = all reals

(2) f is even function since $f(-x) = f(x)$.

(3) $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$

$\Rightarrow x = 0,\ x = \pm 1$ are critical points.

(4) Sign chart of f'

x	$-\infty$	-1		0		1	$+\infty$
$4x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$4x(x-1)(x+1)$	$-$	0	$+$	0	$-$	0	$+$
f	↘		↗		↘		↗

f is increasing on $(-1, 0)$ and on $(1, \infty)$

decreasing on $(-\infty, -1)$ and on $(0, 1)$.

(5) $f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$

$\Rightarrow f''(x) = 0$ as $x = \pm \frac{1}{\sqrt{3}}$

x	$-\infty$		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$		$+\infty$
f''		$+\ +\ +$	0	$-\ -$	0	$+\ +\ +$	

$\Rightarrow f$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and on $\left(\frac{1}{\sqrt{3}}, +\infty\right)$

and

concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

$\Rightarrow$ $x = \pm \frac{1}{\sqrt{3}}$ are both inflection points