METHODS FOR SOLVING 1st order DIFFERENTIAL EQUATIONS and APPLICATIONS

Math 3400 - Spring 2016
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(1st order) Linear DE: If DE is of the (standard) form $\frac{dy}{dx} + p(x)y = f(x)$, then the integrating factor

$$\mu(x) = \mathbf{e}^{\int p(x)dx}$$

yields the general solution

$$y = \frac{1}{\mu(x)}(\int \mu(x)f(x)dx + C).$$

Separable DE: If DE is of the form $\frac{dy}{dx} = g(x)/f(y)$ then

$$\int f(y)dy = \int g(x)dx, \quad \text{or} \quad F(y) = G(x) + c.$$

(1st order) Homogeneous DE: If DE is of the form $\frac{dy}{dx} = F(\frac{y}{x})$ then the substitution $v = \frac{y}{x}$ makes it separable.

Bernoulli DE: If $\frac{dy}{dx} + p(x)y = q(x)y^n$ with $n \neq 0, 1$, then the substitution $v = y^{1-n}$ yields a linear DE in v.

Exact DE: If DE has differential form $M(x, y)dx + N(x, y)dy = 0$ with

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y) \qquad \text{(exact equation)}$$

then the solution is given implicitely by

$$F(x, y) = C$$

where F satisfies $F_x = M$ and $F_y = N$. More precisely, $F(x, y) = \int M(x, y)dx + g(y)$ and $g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx$.

(*) Integrating factor for non-exact equations: $M(x, y)dx + N(x, y)dy = 0$, with $M_y \neq N_x$. Then $\mu = \mu(x, y)$ can be found in one of the two cases:

- If $\frac{M_y - N_x}{N}$ depends only on x, one finds $\mu = \mu(x)$ by solving $\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$.
- If $\frac{N_x - M_y}{M}$ depends only on y, one finds $\mu = \mu(y)$ by solving $\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu$.

Reducible 2nd order DE:

• If the DE for $y = y(x)$ has the form $F(x, y', y'') = 0$ (with y explicitly missing), then the substitution $v = y'$ reduces it to a 1st order DE in $v = v(x)$.

• If the DE for $y = y(x)$ has the form $F(y, y', y'') = 0$ (with x explicitly missing), then the substitution $p = \frac{dy}{dx}$ reduces it to a 1st order DE in $p = p(y)$. Here $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$.

Applications:

Newton's Law of Heating/Cooling: The temperature $u = u(t)$ of an object brought into a large room, with room temperature $T = T(t)$ is given by:

$$\frac{du}{dt} = k(T - u)$$

Population Logistic Growth: A population $P(t)$ satisfying the logistic equation $\frac{dP}{dt} = aP(M - P)$ with initial population $P(0) = P_0$ is given by

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mat}}$$

Mixing Problems: The amount of substance in a water solution in a well mixed tank is given by

$$\frac{dx}{dt} = r_{in}c_{in} - r_{out}c_{out}$$

where r_{in} is the volumetric rate of the flow coming into the tank etc.