METHODS FOR SOLVING 2nd order DIFFERENTIAL EQUATIONS

Math 3400 - Spring 2016
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2nd order linear with constant coefficients, homogeneous of the form $ay''+by'+cy=0$. Characteristic equation $ar^2+br+c=0$ has the roots r_1, r_2. The general solution is of the form

- $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$, in the case of **two distinct *real* roots**, $r_1 \neq r_2$;
- $y(x) = c_1e^{rx} + c_2xe^{rx}$, in the case of **one repeated root** $r_1 = r_2 = r$;
- $y(x) = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x$, in the case of **two complex (conjugate) roots** $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$.

Reduction of order: If the equation $y'' + p(x)y' + q(x)y = 0$ has one known solution $y_1(x)$, then a second, linearly independent, solution can be found in the form $y_2(x) = v(x)y_1(x)$. One needs to solve only a 1st order equation in $z(x) = v'(x)$.

Method of undetermined coefficients (or judicious guessing) The nonhomogeneous 2nd order equation $ay'' + by' + cy = f(x)$ with $f(x)$ of a special form (for example x^n, e^{rx}, $\sin kx$, $\cos kx$ or any combination of them, including sums and products) has a particular solution given by this table

$f(x)$	y_p
$P_m = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^se^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1x + \cdots + A_mx^m)\cos kx + (B_0 + B_1x + \cdots + B_mx^m)\sin kx]$

FIGURE 3.5.1. Substitutions in the method of undetermined coefficients.

Variation of parameters: If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions for the 2nd order homogeneous equation $y'' + p(x)y' + q(x)y = 0$, then a **particular solution of the nonhomogeneous equation** $y'' + p(x)y' + q(x)y = f(x)$ is found in the form

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x),$$

where $v_1'(x) = -\frac{y_2(x)f(x)}{W(y_1,y_2)(x)}$ and $v_2'(x) = \frac{y_1(x)f(x)}{W(y_1,y_2)(x)}$

Euler Equations: The equation $ax^2y''+bxy'+cy=0$ admits solutions of the form $y(x)=x^r$, where r solves a quadratic equation.

- If roots are real distinct, the solutions are $y_1(x) = x^{r_1}, y_2(x) = x^{r_2}$.
- If the roots are repeated, the solutions are $y_1(x) = x^r$ and $y_2(x) = x^r\ln x$.
- If the roots are complex, the solutions are $y_1(x) = x^\alpha\cos(\beta\ln x)$ and $y_2(x) = x^\alpha\sin(\beta\ln x)$