Thursday, March 3, 2016 STUDENT NAME: ____________________

EXAM I – MATH 3400, SPRING 2016

READ EACH PROBLEM CAREFULLY! To get full credit, you must show all work! **The exam has 6 problems on 6 pages. The 7th page contains the review sheet of methods for solving DEs which can be used during the exam. Make sure you turn in ALL pages. No graphing calculator is allowed!**

Exercise 1 *[16 points]*
(a) Solve the initial value problem

$$y' + 2y = 2, \quad y(0) = 2$$

Linear 1st order: Method of integrating factor:

$$\mu = e^{\int 2dt} = e^{2t}$$

$$\frac{d}{dt}(y e^{2t}) = 2e^{2t} \Rightarrow y e^{2t} = \int 2e^{2t} dt = \frac{2}{2} e^{2t} + c$$

$$\Rightarrow \underbrace{\boxed{y = 1 + ce^{-2t}}}_{\text{General Solution}} \quad c = \text{arbitrary constant}$$

For $\left.\begin{matrix} x=0 \\ y=2 \end{matrix}\right] \Rightarrow 2 = 1 + ce^{0} = 1 + c \Rightarrow c = 1 \Rightarrow$

Solution of IVP is $\boxed{y(t) = 1 + e^{-2t}}$

(b) Find $\lim_{t\to\infty} y(t)$.

As $t \to \infty$, $e^{-2t} \to 0$, $\lim_{t \to \infty} y(t) = 1 + 0 = 1$.

Exercise 2 *[16 points]*

Solve the differential equations

$$(a) \quad y' - 3y = e^{3x}$$

Integrating factor : $\mu = e^{\int -3\,dx} = e^{-3x}$.

$(y\,e^{-3x})' = e^{3x} \cdot e^{-3x} = 1 \rightarrow$

$\rightarrow \; y e^{-3x} = x + c \Rightarrow y = (x+c)\,e^{3x}$ or

$\boxed{y(x) = x e^{3x} + c e^{3x}}$ c = arbitrary

$$(b) \quad xy' = y^2 + 1$$

$x \dfrac{dy}{dx} = y^2 + 1$

$\displaystyle\int \frac{dy}{y^2+1} = \int \frac{dx}{x}$

$\arctan y = \ln|x| + c$

$\boxed{y = \tan(\ln|x| + c)}$

Exercise 3 *[17 points]*
Determine whether the following equations are exact or not. Then **solve only those** that are exact.

$$(a) \quad \underbrace{(2y - x)}_{M}dx + \underbrace{(y + 2x)}_{N}dy = 0$$

$\frac{\partial M}{\partial y} = 2$, $\frac{\partial N}{\partial x} = 2 \Rightarrow$ exact! Need F : $\begin{cases} \frac{\partial F}{\partial x} = M = 2y - x \\ \frac{\partial F}{\partial y} = N = y + 2x \end{cases}$

$\Rightarrow F(x,y) = 2xy - \frac{x^2}{2} + c(y)$

$\rightarrow \frac{\partial F}{\partial y} = 2x + c'(y) \Rightarrow 2x + c'(y) = y + 2x \Rightarrow c'(y) = y \rightarrow c(y) = \frac{y^2}{2}$

So $F(x,y) = 2xy - \frac{x^2}{2} + \frac{y^2}{2}$. General solution $F(x,y) = c$ or

$$\boxed{2xy - \frac{x^2}{2} + \frac{y^2}{2} = \text{const}}$$

$$(b) \quad \underbrace{(y^2 + 1)}_{M}dx + \underbrace{(x^2 + 1)}_{N}dy = 0$$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2x$. Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$

$\rightarrow$ equation is not exact.

Exercise 4 *[7 points]*

(a) Find the general solution for the differential equation below

$$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$$

Homogeneous $\quad \frac{dy}{dx} = \frac{x^2}{2xy} + \frac{3y^2}{2xy} = \frac{1}{2}\frac{x}{y} + \frac{3}{2}\frac{y}{x}$

$\boxed{v = \frac{y}{x} \Rightarrow y = xv, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}} \Rightarrow \quad v + x\frac{dv}{dx} = \frac{1}{2}\frac{1}{v} + \frac{3}{2}v \Rightarrow x\frac{dv}{dx} = \frac{1}{2}\frac{1}{v} + \frac{1}{2}v$

$= \frac{1}{2}\frac{1+v^2}{v}$

Separable eq in $v=v(x)$ $\Rightarrow \frac{2v}{1+v^2}dv = \frac{dx}{x} \Rightarrow \int \frac{2v}{1+v^2}dv = \int \frac{dx}{x} \Rightarrow \ln(1+v^2) = \ln|x| + c$

$\Rightarrow 1+v^2 = cx \Rightarrow v^2 = cx-1 \Rightarrow v = \pm\sqrt{cx-1}$

$\frac{y}{x} = \pm\sqrt{cx-1} \Rightarrow \boxed{y = \pm x\sqrt{cx-1}}$ general solution

(b) Find the solution satisfying the initial condition $y(1) = -2$.

$x=1, \; y=-2 \Rightarrow \quad -2 = -1\sqrt{c\cdot 1-1} = -\sqrt{c-1}$

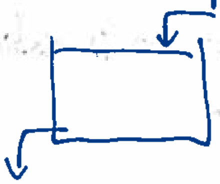
$\Rightarrow 2 = \sqrt{c-1} \Rightarrow c-1 = 4 \Rightarrow c = 5$

$\Rightarrow$ solution is $\boxed{y = -x\sqrt{5x-1}}$

Exercise 5 *[18 points]* A tank initially contains 300 gal of pure water. A salt water mixture with a concentration of 5 pounds per gallon enters the tank at the rate of 20 gallon per minute. The well mixed solution leaves the tank at the same rate. Let $x(t)$ be the quantity of salt (pounds) in the tank at time t. [Hint: The differential equation $\frac{dx}{dt} = r_{in}c_{in} - r_{out}c_{out}$ describes the rate of change of the amount of salt in the tank]

(a) Set up the initial value problem for $x(t)$.

$c_{in} = 5\,lb/gal$, $r_{in} = 20\,gal/min$



$r_{out} = 20\,gal/min$

Since $r_{in} = r_{out}$, volume stays consta

$$\frac{dV}{dt} = r_{in} - r_{out} = 20 - 20 = 0$$

$$\rightarrow V(t) \equiv V_0 = 300$$

$$\frac{dx}{dt} = r_{in}c_{in} - r_{out}c_{out}$$

$$= 20 \cdot 5 - 20 \cdot \frac{x}{V}$$

$$\Rightarrow \frac{dx}{dt} = 100 - 20\frac{x}{300} = 100 - \frac{2}{30}x \Rightarrow \boxed{\frac{dx}{dt} = 100 - \frac{1}{15}x}$$

(b) Find the particular solution of the above IVP.

$$\frac{dx}{dt} + \frac{1}{15}x = 100 \Rightarrow \frac{d}{dt}\left(x e^{1/15 t}\right) = 100e^{1/15 t}$$

$$\Rightarrow x e^{\frac{1}{15}t} = \frac{100 e^{1/15 t}}{1/15} + c = 1500e^{1/15 t} + c$$

$$\rightarrow x = 1500 + ce^{-\frac{1}{15}t}$$

$$x(0) = 0 \Rightarrow 0 = 1500 + c \cdot 1 \Rightarrow c = -1500$$

$$\Rightarrow \boxed{x = 1500\left(1 - e^{-\frac{1}{15}t}\right)}$$

(c) How much salt is in the tank after 30 minutes?

After 30 min, $t = 30$

$$\Rightarrow x = 1500\left(1 - e^{-\frac{30}{15}}\right) = 1500\left(1 - e^{-2}\right) = 1{,}297$$

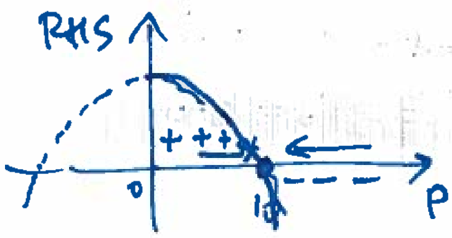
Exercise 6 *[16 points]* A population of a certain species satisfies the following differential equation:

$$\frac{dP}{dt} = 20 - \frac{1}{5}P^2$$

(a) Find the equilibria, and determine their stability.

Equilibria: RHS $= 0 \Rightarrow 20 - \frac{1}{5}P^2 = 0 \Rightarrow P^2 = 100 \rightarrow P = 10$

($P = -10$ is extraneous)



$P = 10$ is a stable equilibrium

(b) Assuming the initial population is $P(0) = 5$, solve the IVP.

$$\frac{dP}{dt} = \frac{100-P^2}{5} \Rightarrow \frac{dP}{100-P^2} = \frac{1}{5}\,dt \Rightarrow \int \frac{dP}{100-P^2} = \frac{t}{5} + c$$

$$\Rightarrow \int \frac{1}{20}\left(\frac{1}{10-P} + \frac{1}{10+P}\right) dP = \frac{t}{5} + c \Rightarrow \frac{1}{20}\ln\left|\frac{10+P}{10-P}\right| = \frac{t}{5} + c \Rightarrow \ln\left|\frac{10+P}{10-P}\right| = 4t + c$$

$$\Rightarrow \frac{10+P}{10-P} = ce^{4t} \Rightarrow (10+P) = (10-P)ce^{4t} \Rightarrow P = \frac{10ce^{4t} - 10}{1 + ce^{4t}}$$

$$P(t) = 10\,\frac{ce^{4t} - 1}{1 + ce^{4t}} \text{ or}$$

$$\cancel{P(t) \text{ or } 10\,\frac{1 - ce^{-4t}}{ce^{-4t} + 1}}$$

At $P(0) = 5$

$$5 = 10\,\frac{c-1}{1+c} \Rightarrow \frac{1}{2} = \frac{c-1}{c+1}$$

$$\Rightarrow 2c - 2 = c + 1$$

$$\boxed{c = 3}$$

$$\Rightarrow \boxed{P(t) = 10\,\frac{3e^{4t} - 1}{1 + 3e^{4t}}}$$

(c) Is there a limiting population? If so, what is $\lim_{t\to+\infty} P(t)$?

As $t \to \infty$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} 10\,\frac{3 - e^{-4t}}{e^{-4t} + 3} = 10\,\frac{3 - 0}{0 + 3} = 10$$

(not by accident the limiting population = equilibrium)