

in the two unknowns k and M . Nonlinear systems like this ordinarily are solved numerically using an appropriate computer system. But with the right algebraic trick (Problem 36 in this section) the equations in (10) can be solved manually for $k = 0.000167716$, $M = 188.121$. Substitution of these values in Eq. (7) yields the logistic model

$$P(t) = \frac{998.546}{5.308 + (182.813)e^{-(0.031551)t}} \tag{11}$$

The table in Fig. 2.1.4 compares the actual 1800–1990 U.S. census population figures with those predicted by the exponential growth model in (9) and the logistic model in (11). Both agree well with the 19th-century figures. But the exponential model diverges appreciably from the census data in the early decades of the 20th century, whereas the logistic model remains accurate until 1940. By the end of the 20th century the exponential model vastly overestimates the actual U.S. population—predicting over a billion in the year 2000—whereas the logistic model somewhat underestimates it.

Year	Actual U.S. Pop.	Exponential Model	Exponential Error	Logistic Model	Logistic Error
1800	5.308	5.308	0.000	5.308	0.000
1810	7.240	6.929	0.311	7.202	0.038
1820	9.638	9.044	0.594	9.735	−0.097
1830	12.861	11.805	1.056	13.095	−0.234
1840	17.064	15.409	1.655	17.501	−0.437
1850	23.192	20.113	3.079	23.192	0.000
1860	31.443	26.253	5.190	30.405	1.038
1870	38.558	34.268	4.290	39.326	−0.768
1880	50.189	44.730	5.459	50.034	0.155
1890	62.980	58.387	4.593	62.435	0.545
1900	76.212	76.212	0.000	76.213	−0.001
1910	92.228	99.479	−7.251	90.834	1.394
1920	106.022	129.849	−23.827	105.612	0.410
1930	123.203	169.492	−46.289	119.834	3.369
1940	132.165	221.237	−89.072	132.886	−0.721
1950	151.326	288.780	−137.454	144.354	6.972
1960	179.323	376.943	−197.620	154.052	25.271
1970	203.302	492.023	−288.721	161.990	41.312
1980	226.542	642.236	−415.694	168.316	58.226
1990	248.710	838.308	−589.598	173.252	76.458
2000	281.422	1094.240	−812.818	177.038	104.384

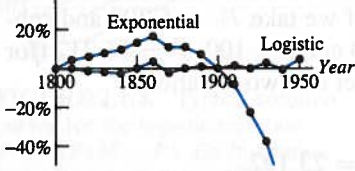


FIGURE 2.1.5. Percentage errors in the exponential and logistic population models for 1800–1950.

FIGURE 2.1.4. Comparison of exponential growth and logistic models with U.S. census populations (in millions).

The two models are compared in Fig. 2.1.5, where plots of their respective errors—as a *percentage* of the actual population—are shown for the 1800–1950 period. We see that the logistic model tracks the actual population reasonably well throughout this 150-year period. However, the exponential error is considerably larger during the 19th century and literally goes off the chart during the first half of the 20th century.