STUDENT NAME: ______________________

STUDENT ID ______________________

EXAM II – MATH 3400, SPRING 2016

READ EACH PROBLEM CAREFULLY! To get full credit, you must show all work! **The exam has 5 problems on 5 pages. The 6th page contains the review sheet of methods for solving DEs which can be used during the exam. Make sure you turn in ALL pages. No calculators that can graph or integrate is allowed!**

Exercise 1

Consider the differential equation

$$y'' - 4y' + 4y = 0$$

(a) Find two linearly independent solutions and write the general solution.

(b) Find the solution satisfying the initial conditions

$$y(0) = 1, y'(0) = 0$$

Exercise 2
Use the method of reduction of order to find a second solution $y_2(t)$ for

$$(x-1)y'' - xy' + y = 0$$

given that $y_1(x) = x$ is one solution. Write down the general solution of the differential equation.

Exercise 3 Given the non-homogeneous equation

$$y'' + 3y' + 2y = e^{-x}$$

(a) Find the general solution $y(x)$.

(b) Solve the initial value problem with initial conditions

$$y(0) = 2, y'(0) = 3$$

Exercise 4

(a) Using the Variation of Parameters method, find a particular solution of the differential equation

$$y'' + y = 1 + \sin x$$

(b) How is this different from the particular solution found using the method of undetermined coefficients? Explain why in this case both methods are applicable, and describe a situation (right hand side?) when the only applicable method is the first one.

Exercise 5

A spring-mass system ($m = 2$, $k = 4$) with damping ($c = 4$) is initially stretched ($x(0) = 1$) and has zero velocity ($x'(0) = 0$). Recall that, given an external forcing $f(t)$, the motion is governed by the differential equation

$$mx'' + cx' + kx = f(t)$$

(a) Is it an overdamped, underdamped or critically damped system? Determine its motion $x = x(t)$ in absence of any external forcing ($f(t) = 0$).

(b) Determine its motion $x = x(t)$ in presence of the external forcing $f(t) = 2\sin(t)$ (same initial conditions).

(c) Is it possible to create resonance in this system? Why or why not? Explain.

METHODS FOR SOLVING 2nd order DIFFERENTIAL EQUATIONS

Math 3400 - Spring 2016
Instructor: Radu C. Cascaval

2nd order linear with constant coefficients, homogeneous of the form $ay''+by'+cy=0$. Characteristic equation $ar^2+br+c=0$ has the roots r_1, r_2. The general solution is of the form

- $y(x)=c_1e^{r_1x}+c_2e^{r_2x}$, in the case of **two distinct *real* roots**, $r_1 \neq r_2$;
- $y(x)=c_1e^{rx}+c_2xe^{rx}$, in the case of **one repeated root** $r_1=r_2=r$;
- $y(x)=c_1e^{\alpha x}\cos\beta x+c_2e^{\alpha x}\sin\beta x$, in the case of **two complex (conjugate) roots** $r_1=\alpha+i\beta, r_2=\alpha-i\beta$.

Reduction of order: If the equation $y''+p(x)y'+q(x)y=0$ has one known solution $y_1(x)$, then a second, linearly independent, solution can be found in the form $y_2(x)=v(x)y_1(x)$. One needs to solve only a 1st order equation in $z(x)=v'(x)$.

Method of undetermined coefficients (or judicious guessing) The nonhomogeneous 2nd order equation $ay''+by'+cy=f(x)$ with $f(x)$ of a special form (for example x^n, e^{rx}, $\sin kx$, $\cos kx$ or any combination of them, including sums and products) has a particular solution given by this table

$f(x)$	y_p
$P_m=b_0+b_1x+b_2x^2+\cdots+b_mx^m$	$x^s(A_0+A_1x+A_2x^2+\cdots+A_mx^m)$
$a\cos kx+b\sin kx$	$x^s(A\cos kx+B\sin kx)$
$e^{rx}(a\cos kx+b\sin kx)$	$x^se^{rx}(A\cos kx+B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0+A_1x+A_2x^2+\cdots+A_mx^m)e^{rx}$
$P_m(x)(a\cos kx+b\sin kx)$	$x^s[(A_0+A_1x+\cdots+A_mx^m)\cos kx + (B_0+B_1x+\cdots+B_mx^m)\sin kx]$

FIGURE 3.5.1. Substitutions in the method of undetermined coefficients.

Variation of parameters: If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions for the 2nd order homogeneous equation $y''+p(x)y'+q(x)y=0$, then a **particular solution of the nonhomogeneous equation** $y''+p(x)y'+q(x)y=f(x)$ is found in the form

$$y(x)=v_1(x)y_1(x)+v_2(x)y_2(x),$$

where $v_1'(x)=-\frac{y_2(x)f(x)}{W(y_1,y_2)(x)}$ and $v_2'(x)=\frac{y_1(x)f(x)}{W(y_1,y_2)(x)}$