Solutions

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EXAM II – MATH 3400, SPRING 2016

READ EACH PROBLEM CAREFULLY! To get full credit, you must show all work! **The exam has 5 problems on 5 pages. The 6th page contains the review sheet of methods for solving DEs which can be used during the exam. Make sure you turn in ALL pages. No calculators that can graph or integrate is allowed!**

Exercise 1
Consider the differential equation

$$y'' - 4y' + 4y = 0$$

(a) Find two linearly independent solutions and write the general solution.

Characteristic eq: $r^2 - 4r + 4 = 0 \iff (r-2)^2 = 0 \Rightarrow r_1 = r_2 = 2$ (repeated roots)

$y_1(x) = e^{2x}$
$y_2(x) = xe^{2x}$ are two linearly independent solutions.

General solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$
$= c_1 e^{2x} + c_2 x e^{2x}$

$$\boxed{y(x) = (c_1 + c_2 x) e^{2x}}$$

(b) Find the solution satisfying the initial conditions

$$y(0) = 1, y'(0) = 0$$

$y'(x) = c_2 e^{2x} + (c_1 + c_2 x) 2e^{2x}$
$= (c_2 + 2c_1 + 2c_2 x) e^{2x}$

At $x = 0$:

$$\begin{cases} 1 = y(0) = (c_1 + c_2 \cdot 0) e^{2\cdot 0} = c_1 \\ 0 = y'(0) = (c_2 + 2c_1 + 2c_2 \cdot 0) e^{2 \cdot 0} = c_2 + 2c_1 \end{cases} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = -2c_1 = -2 \end{array}$$

$\Rightarrow$ Solution is $\boxed{y(x) = (1 - 2x) e^{2x}}$

Exercise 2

Use the method of reduction of order to find a second solution $y_2(t)$ for

$$(x-1)y'' - xy' + y = 0$$

given that $y_1(x) = x$ is one solution. Write down the general solution of the differential equation.

We seek $y_2(x) = x \cdot v(x)$, $v(x) = ?$

$1 \;/\; y_2(x) = x \cdot v$

$(-x) \;/\; y_2'(x) = 1 \cdot v + x \cdot v' = v + xv'$

$(x-1) \;/\; y_2''(x) = v' + 1 \cdot v' + xv'' = 2v' + xv''$

$$0 = (x-1)y_2'' - xy_2' + y_2 = (x-1)(2v' + xv'') - x(v + xv') + xv$$

$$= x(x-1)v'' + [2(x-1) - x^2]v' + \underbrace{[(-x)+x]}_{=0}v$$

[confirmation that $y_1(x) = x$ is a solution!

$$0 = x(x-1)v'' - (x^2 - 2x + 2)v'$$

This is a first order eq in $v' = z$

$x(x-1)z' = (x^2-2x+2)z$. Separable 1st order!

$$\frac{dz}{z} = \frac{x^2 - 2x + 2}{x(x-1)}\,dx$$ [Partial Fraction decomp]

$$\Rightarrow \int \frac{dz}{z} = \int \left(1 - \frac{2}{x} + \frac{1}{x-1}\right) dx \Rightarrow$$

$$\Rightarrow \ln|z| = x - 2\ln|x| + \ln|x-1|$$

$$\Rightarrow z = e^{x - \ln x^2 + \ln|x-1|}$$

$$\Rightarrow z = \frac{e^x(x-1)}{x^2} \Rightarrow v' = \frac{e^x(x-1)}{x^2} \Rightarrow v(x) = \int \frac{e^x(x-1)}{x^2}\,dx$$

So $y_2(x) = x \int \frac{e^x(x-1)}{x^2}\,dx$ and Gen sol $\boxed{y(x) = c_1 x + c_2 x \int \frac{e^x(x-1)}{x^2}\,dx}$

Exercise 3 Given the non-homogeneous equation

$$y'' + 3y' + 2y = e^{-x}$$

(a) Find the general solution $y(x)$.

Homogeneous solution: $y'' + 3y' + 2y = 0 \to r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$

$r_1 = -1,\ r_2 = -2$

$\Rightarrow y_h(x) = c_1 e^{-x} + c_2 e^{-2x}$

Particular solution (using method of undetermined coeff.)

$y_p(x) = Axe^{-x}$

(b) Solve the initial value problem with initial conditions

$$y(0) = 2, y'(0) = 3$$

$y_p'(x) = Ae^{-x} - Axe^{-x} = A(1-x)e^{-x}$

$y_p''(x) = A(-1)e^{-x} + A(1-x)(-e^{-x}) = A(x-2)e^{-x}$

$e^{-x} = y_p'' + 3y_p' + 2y_p = A(x-2)e^{-x} + 3A(1-x)e^{-x} + 2Axe^{-x}$

$= Ae^{-x} \Rightarrow A = 1 \Rightarrow y_p(x) = xe^{-x}$

$\Rightarrow$ General solution $\boxed{y(x) = c_1 e^{-x} + c_2 e^{-2x} + xe^{-x}}$

(b) $y'(x) = -c_1 e^{-x} - 2c_2 e^{-2x} + e^{-x} - xe^{-x}$

$\begin{cases} 2 = y(0) = c_1 + c_2 \\ 3 = y'(0) = -c_1 - 2c_2 + 1 \end{cases} \Rightarrow \begin{cases} 2 = c_1 + c_2 \\ 2 = -c_1 - 2c_2 \end{cases} \Rightarrow \begin{matrix} c_2 = -4 \\ c_1 = 6 \end{matrix}$

$\Rightarrow \boxed{y(x) = 6e^{-x} - 4e^{-2x} + xe^{-x}}$

Exercise 4

(a) Using the Variation of Parameters method, find a particular solution of the differential equation

$$y'' + y = \underbrace{1 + \sin x}_{f(x)}$$

Homogeneous $y''+y=0$ has $y_1 = \cos x$, $y_2 = \sin x$

Seek

$$y(x) = v_1 y_1 + v_2 y_2 \quad , v_1 = ? , v_2 = ?$$

Set

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = 1 + \sin x$$

$$\Rightarrow \begin{cases} v_1' \cos x + v_2' \sin x = 0 \\ v_1'(-\sin x) + v_2' \cos x = 1 + \sin x \end{cases} \qquad W = 1$$

$$\Rightarrow v_1' = \frac{1}{W}\begin{vmatrix} 0 & \sin x \\ 1+\sin x & \cos x \end{vmatrix} = -\sin x(1+\sin x) \Rightarrow v_1(x) = -\int \sin x + \sin^2 x \, dx$$

$$v_2' = \frac{1}{W}\begin{vmatrix} \cos x & 0 \\ -\sin x & 1+\sin x \end{vmatrix} = \cos x(1+\sin x) \Rightarrow v_2(x) = \int \cos x + \sin x \cos x \, dx$$

(b) How is this different from the particular solution found using the method of undetermined coefficients? Explain why in this case both methods are applicable, and describe a situation (right hand side?) when the only applicable method is the first one.

(a) Cont...

$$v_1(x) = \cos x - \int \frac{1}{2}(1-\cos 2x)\, dx = \cos x - \frac{x}{2} + \frac{\sin 2x}{4} + c_1$$

$$v_2(x) = \sin x + \int \frac{1}{2} \sin 2x \, dx = \sin x - \frac{1}{4}\cos 2x + c_2$$

So

$$y(x) = \left(\cos x - \frac{x}{2} + \frac{\sin 2x}{4} + c_1\right)\cos x + \left(\sin x - \frac{1}{4}\cos 2x + c_2\right)\sin x$$

$$= 1 - \frac{x}{2}\cos x + \frac{1}{4}\underbrace{(\sin 2x \cos x - \cos 2x \sin x)}_{\sin(2x - x) = \sin x} + c_1 \cos x + c_2 \sin x$$

$$= 1 - \frac{x}{2}\cos x + c_1 \cos x + c_2 \sin x \; , \quad c_1, c_2 = \text{arbitrary!}$$

(b) This leads to the same solution as the method of undetermined coefficients [where one solves $y''+y=1$ and $y''+y=\sin x$ separately]. In this problem both methods are applicable due to the simple form of the RHS, but only the variation of parameters would be possible given say $y''+y=\tan x$

Exercise 5

A spring-mass system ($m = 2$, $k = 4$) with damping ($c = 4$) is initially stretched ($x(0) = 1$) and has zero velocity ($x'(0) = 0$). Recall that, given an external forcing $f(t)$, the motion is governed by the differential equation

$$mx'' + cx' + kx = f(t)$$

(a) Is it an overdamped, underdamped or critically damped system? Determine its motion $x = x(t)$ in absence of any external forcing ($f(t) = 0$).

$$2x''+4x'+4x=0 \Rightarrow x''+2x'+2x=0 \Rightarrow r^2+2r+2=0$$

$$\Rightarrow (r+1)^2+1=0 \Rightarrow r=-1\pm i \Rightarrow \text{underdamped!}$$

$$x_1(t)=e^{-t}\cos t,\ x_2(t)=e^{-t}\sin t \Rightarrow x(t)=c_1e^{-t}\cos t+c_2e^{-t}\sin t$$

$$\Rightarrow \boxed{x(t)=e^{-t}(c_1\cos t+c_2\sin t)}. \Rightarrow x'(t)=-e^{-t}(c_1\cos t+c_2\sin t)+$$
$$+e^{-t}(-c_1\sin t+c_2\cos t)$$

$$\begin{cases} x(0)=1 \\ x'(0)=0 \end{cases} \Rightarrow \begin{cases} c_1=1 \\ -c_1+c_2=0 \end{cases} \Rightarrow c_1=c_2=1$$

$$\Rightarrow \boxed{x(t)=e^{-t}(\cos t+\sin t)}$$

(b) Determine its motion $x = x(t)$ in presence of the external forcing $f(t) = 2\sin(t)$ (same initial conditions).

$$2x''+4x'+4x=2\sin t \Rightarrow x''+2x'+2x=\sin t$$

Method of undet. coef.: Seek

$$x_p(t)=A\cos t+B\sin t$$
$$x_p'(t)=-A\sin t+B\cos t$$
$$x_p''(t)=-A\cos t-B\sin t$$

$$\Rightarrow \sin t = x_p''+2x_p'+2x_p = -A\cos t - B\sin t+2(-A\sin t+B\cos t)+2(A\cos t+B\sin t)$$

$$=(A+2B)\cos t+(B-2A)\sin t \Rightarrow \begin{cases} -A+2B=0 \\ B-2A=1 \end{cases} \Rightarrow \begin{matrix} A=2B \\ -3B=1 \end{matrix}$$

$$\Rightarrow B=-\frac{1}{3},\ A=-\frac{2}{3} \Rightarrow x_p(t)=-\frac{2}{3}\cos t-\frac{1}{3}\sin t$$

(c) Is it possible to create resonance in this system? Why or why not? Explain.

Resonance occurs when RHS is a solution of the homogeneous equation. In our case, due to presence of damping, one cannot experience pure resonance

[the RHS could be $e^{-t}\cos t$ or $e^{-t}\sin t$ but then solution would be $te^{-t}(A\cos t+B\sin t)$ which is not increasing!]