Solutions to Practice Exam 2

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#1 $y''+y'-6y=0$. Characteristic equation

$$r^2+r-6=0$$

$$(r-2)(r+3)=0$$

$$r_1=2,\ r_2=-3 \quad \text{distinct real roots}$$

(a) General solution

$$\boxed{y(x)=c_1e^{2x}+c_2e^{-3x}}$$

(b) $\begin{cases} y(0)=0 \\ y'(0)=3 \end{cases}$ $\qquad y(x)=c_1e^{2x}+c_2e^{-3x}$, $\quad y'(x)=2c_1e^{2x}-3c_2e^{-3x}$

$\Rightarrow \begin{cases} c_1+c_2=0 \\ 2c_1-3c_2=3 \end{cases} \Rightarrow c_2=-c_1 \Rightarrow 5c_1=3 \Rightarrow c_1=\frac{3}{5} \Rightarrow c_2=-\frac{3}{5}$

$\Rightarrow$ solution of IVP: $\boxed{y(x)=\frac{3}{5}e^{2x}-\frac{3}{5}e^{-3x}}$

#2 $y''+y'-2y=0$. Characteristic equation

$$r^2+r-2=0$$

$$(r-1)(r+2)=0$$

$$r_1=1,\ r_2=-2 \quad \text{distinct real roots}$$

(a) Two linearly independent solutions

$$\boxed{y_1(x)=e^{x},\quad y_2(x)=e^{-2x}}$$

$\to v(x)=e^x \Rightarrow \boxed{y_2(x)=xe^x}$

General solution:

$$y(x)=c_1y_1(x)+c_2y_2(x) \Rightarrow \boxed{y(x)=c_1\cancel{x}+c_2xe^x}$$

#5 $y''-2y'+y=2e^x$

(a)
$1/\ y_p(x)=x^2e^x$
$-2/\ y_p'(x)=2xe^x+x^2e^x$
$1/\ y_p''(x)=2e^x+4xe^x+x^2e^x$

$$y_p''-2y_p'+y_p=(2+4x+x^2)e^x-2(2x+x^2)e^x+x^2e^x$$
$$=(2+4x+x^2-4x-2x^2+x^2)e^x=2e^x \checkmark$$

(b) Homogeneous solution: $y''-2y'+y=0$

$r^2-2r+1=0 \to (r-1)^2=0$ repeated roots $r_1=r_2=1$

$\to y_h(x)=c_1e^x+c_2xe^x$

General solution of non-homogeneous eq:

$$\to \boxed{y(x)=y_h(x)+y_p(x)=c_1e^x+c_2xe^x+x^2e^x}.$$

$$\Rightarrow \int \frac{dz}{z} = \int\left(1-\frac{1}{x}\right)dx \Rightarrow \ln|z| = x - \ln|x| + c \quad , x>0$$

Take $c=0$ $\rightarrow$ $z = e^{x-\ln x} \Rightarrow z = \frac{e^x}{x}$

$$\Rightarrow v' = \frac{e^x}{x} \Rightarrow v(x) = \int \frac{e^x}{x}\,dx .$$

So $\boxed{y_2(x) = e^{-x}\int \frac{e^x}{x}\,dx}$ (integral does not have an explicit representation).

General solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \Rightarrow \boxed{y(x) = c_1 e^{-x} + c_2 e^{-x}\int \frac{e^x}{x}\,dx}$$

#4 $\quad x^2 y'' - x(x+2)y' + (x+2)y = 0$

$y_1(x) = x$. Seek $y_2(x) = x\,v(x)$

$(x+2)/\ y_2 = xv$

$-x(x+2)/\ y_2' = v + xv'$

$x^2/\ y_2'' = 2v' + xv''$

$$0 = x^2 y_2'' - x(x+2)y_2' + (x+2)y_2 = x^3 v'' + 2x^2 v' - x^2(x+2)v' - \cancel{x(x+2)v} + \cancel{x(x+2)v}$$

$$\Rightarrow 0 = x^3 v'' - x^3 v' \Rightarrow v'' - v' = 0 \quad (1^{st}\text{ order in } v' = z)$$

$$z' - z = 0 \Rightarrow z(x) = e^x \Rightarrow v'(x) = e^x$$

$\Rightarrow$ General solution $\boxed{y(x) = c_1 e^x + c_2 e^{-2x}}$

$$y'(x) = c_1 e^x - 2c_2 e^{-2x}$$

(b) $y(0) = 1$, $y'(0) = -2$ $\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ c_1 - 2c_2 = -2 \end{cases} \Rightarrow c_1 = \dfrac{\begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = 0$

$$c_2 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = 1$$

$\Rightarrow$ Solution of IVP: $\boxed{y(x) = e^{-2x}}$

#3 $\quad xy'' + (x+1)y' + y = 0 \quad , x>0$

$y_1(x) = e^{-x}$. Reduction of order: Seek $y_2(x) = v(x)e^{-x}$

$1/\ y_2 = ve^{-x}$

$(x+1)/\ y_2' = v'e^{-x} - ve^{-x}$

$x/\ y_2'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$

$$0 = xy_2'' + (x+1)y_2' + y_2 = x(v'' - 2v' + v)e^{-x} + (x+1)(v' - v)e^{-x} + ve^{-x}$$

Divide by e^{-x}:

$$xv'' - 2xv' + \underline{vx} + (x+1)v' - \underline{(x+1)v} + \underline{v} = 0$$

$$xv'' - (x-1)v' = 0 \quad (1^{st}\text{ order in } v' = z)$$

$$xz' - (x-1)z = 0 \Rightarrow \frac{dz}{dx} = \frac{x-1}{x}z \Rightarrow \frac{1}{z}dz = \left(1 - \frac{1}{x}\right)dx$$

#6 $y'' + y = 2\cos x$

Method of undetermined coefficients:

Homogeneous sol: $y''+y=0 \Rightarrow r^2+1=0 \Rightarrow r=\pm i$

$\Rightarrow y_h(x) = c_1\cos x + c_2 \sin x$

$\Rightarrow$ Seek $y_p(x) = x(A\cos x + B\sin x)$

$y_p'(x) = A\cos x + B\sin x + x(-A\sin x + B\cos x)$

$y_p''(x) = 2(-A\sin x + B\cos x) + x(-A\cos x - B\sin x)$

So $2\cos x = y_p'' + y_p = 2(-A\sin x + B\cos x) \Rightarrow A=0,\ B=1$

$\Rightarrow y_p(x) = x\sin x$

$\Rightarrow$ General solution $\boxed{y(x) = c_1\cos x + c_2\sin x + x\sin x}$.

$y'(x) = -c_1\sin x + c_2\cos x + \sin x + x\cos x$

(b) $y(0)=-1$, $y'(0)=0$

$\Rightarrow \begin{cases} c_1 = -1 \\ c_2 = 0 \end{cases} \Rightarrow$ Solution to IVP $\boxed{y(x) = -\cos x + x\sin x}$.

#7 $y'' + 4y = \sin^2 x$

Homogeneous solution: $y_h(x) = c_1\cos 2x + c_2\sin 2x$

$r^2+4=0 \Rightarrow r=\pm 2i$

Variation of parameters:

$y(x) = v_1(x)\cos 2x + v_2(x)\sin 2x$

$y'(x) = v_1'\cos 2x + v_1(-2\sin 2x) + v_2'\sin 2x + v_2(2\cos 2x)$

Set $\boxed{v_1'\cos 2x + v_2'\sin 2x = 0}$

$\Rightarrow y'(x) = -2v_1\sin 2x + 2v_2\cos 2x$

$\Rightarrow y''(x) = -2v_1'\sin 2x - 4v_1\cos 2x + 2v_2'\cos 2x - 4v_2\sin 2x$

$\sin^2 x = y'' + 4y = -2v_1'\sin 2x + 2v_2'\cos 2x$

$\Rightarrow \begin{cases} v_1'\cos 2x + v_2'\sin 2x = 0 \\ -2v_1'\sin 2x + 2v_2'\cos 2x = \sin^2 x \end{cases}$

$\Rightarrow v_1' = \dfrac{\begin{vmatrix} 0 & \sin 2x \\ \sin^2 x & 2\cos 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}} = \dfrac{-\sin^2 x\sin 2x}{2} \Rightarrow v_1(x) = -\dfrac{1}{2}\int \sin^2 x \sin 2x\,dx$

$v_2' = \dfrac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sin^2 x \end{vmatrix}}{|\text{same}|} = \dfrac{\sin^2 x\cos 2x}{2} \Rightarrow v_2(x) = \dfrac{1}{2}\int \sin^2 x\cos 2x\,dx$

We can compute these integrals using the trig identity

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

Instead, we observe that

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

makes the eq:

$$y''+4y = \frac{1}{2}(1-\cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x$$

So we can use method of undetermined coeff:

$y_{p_1}(x)$: $y''+4y = \frac{1}{2} \Rightarrow y_{p_1}(x) = \text{const}\,(A)$

$$4A = \frac{1}{2} \Rightarrow A = \frac{1}{8}$$

$y_{p_2}(x)$: $y''+4y = -\frac{1}{2}\cos 2x$ [resonant case]

$\to y_{p_2}(x) = x(A\cos 2x + B\sin 2x)$

$y_{p_2}'(x) = A\cos 2x + B\sin 2x + x(-2A\sin 2x + 2B\cos 2x)$

$y_{p_2}''(x) = 2(-2A\sin 2x + 2B\cos 2x) + x(-4A\cos 2x - 4B\sin 2x)$

$-\frac{1}{2}\cos 2x = -4A\sin 2x + 4B\cos 2x \to A = 0,\ B = -\frac{1}{8}$

So $y_{p_2}(x) = -\frac{1}{8}x\sin 2x$

$$\to \boxed{y(x) = y_h(x) + y_{p_1}(x) + y_{p_2}(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{8} - \frac{1}{8}x\sin 2x}$$

#8 $y'' - 4y = xe^x$

Again we could use method of undet coeff or variation of parameters:

$$y_h(x) = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_p(x) = v_1(x) e^{2x} + v_2(x) e^{-2x}$$

Set $v_1'(x) e^{2x} + v_2'(x) e^{-2x} = 0$

then $v_1'(x) 2e^{2x} + v_2'(x)(-2e^{-2x}) = xe^x$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{-2x} \\ xe^x & -2e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}} = \frac{-xe^{-x}}{-4} = \frac{1}{4} x e^{-x}$$

$$v_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & xe^x \end{vmatrix}}{|\text{same}|} = \frac{xe^{3x}}{-4} = -\frac{1}{4} x e^{3x}$$

$$\Rightarrow v_1(x) = \frac{1}{4}\int xe^{-x}\,dx \overset{\text{int. by parts}}{=} \dots = -\frac{1}{4}(x+1)e^{-x}$$

$$v_2(x) = -\frac{1}{4}\int xe^{3x}\,dx \overset{\text{int by parts}}{=} \dots = -\frac{1}{4}\left(\frac{x}{3} - \frac{1}{9}\right)e^{3x}$$

Then $y_p(x) = v_1(x)e^{2x} + v_2(x)e^{-2x}$

$$= -\frac{1}{4}(x+1)e^x - \frac{1}{4}\left(\frac{x}{3} - \frac{1}{9}\right)e^x = -\frac{1}{3}xe^x + -\frac{2}{9}e^x$$

$$\Rightarrow \boxed{y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} x e^x - \frac{2}{9} e^x}$$

#9 $4x'' + 16x = 12\cos\omega t$, $x(0) = x'(0) = 0$

or $x'' + 4x = 3\cos\omega t$

Resonance occurs if RHS $(3\cos\omega t)$ is a solution of the homogeneous eq $x'' + 4x = 0$

or $x(t) = c_1\cos 2t + c_2 \sin 2t$

So $\omega_0 = 2$ is the natural frequency $\left(\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{4}} = 2\right)$

If $\omega = 2$, then $x'' + 4x = 3\cos 2t$.

Solution (using method of undetermined coeff)

$$x_p(x) = t\,(A\cos 2t + B\sin 2t)$$

$\vdots$

$\Rightarrow A = 0$
$B = \frac{3}{4}$

$$\Rightarrow x_p(t) = \frac{3}{4} t \sin 2t$$

$$\Rightarrow x(t) = \frac{3}{4} t \sin 2t + c_1 \cos 2t + c_2 \sin 2t$$

$c_1 = ?$ $x'(t) = \frac{3}{4}\sin 2t + \frac{3}{4} t\cos 2t - 2c_1 \sin 2t + 2c_2 \cos 2t$
$c_2 = ?$

$$\Rightarrow \begin{cases} 0 = x(0) = c_1 \\ 0 = x'(0) = -2c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases} \Rightarrow \boxed{x(t) = \frac{3}{4} t \sin 2t}$$

(b) If $\omega \neq 2 \Rightarrow x'' + 4x = 3\cos\omega t$

Particular sol $x_p(t) = A\cos\omega t + B\sin\omega t$

$x_p'(t) = -\omega A\sin\omega t + \omega B\cos\omega t$

$x_p''(t) = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$

$\Rightarrow 3\cos\omega t = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t + 4A\cos\omega t + 4B\sin\omega t$

$= (4-\omega^2)A\cos\omega t + (4-\omega^2)B\sin\omega t$

$\Rightarrow \begin{array}{l} 3 = (4-\omega^2)A \\ 0 = (4-\omega^2)B \end{array} \Rightarrow A = \frac{3}{4-\omega^2}$

$B = 0$

$\Rightarrow x_p(t) = \frac{3}{4-\omega^2}\cos\omega t$

$\Rightarrow x(t) = \frac{3}{4-\omega^2}\cos\omega t + c_1\cos 2t + c_2\sin 2t$

$c_1 = ?$
$c_2 = ?$ $\quad x'(t) = -\frac{3\omega}{4-\omega^2}\cos\omega t - 2c_1\sin 2t + 2c_2\cos 2t$

$\Rightarrow \begin{cases} 0 = x(0) = \frac{3}{4-\omega^2} + c_1 \\ 0 = x'(0) = 2c_2 \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{3}{4-\omega^2} \\ c_2 = 0 \end{cases}$

$\Rightarrow x(t) = \frac{3}{4-\omega^2}\cos\omega t - \frac{3}{4-\omega^2}\cos 2t$

$= \frac{3}{4-\omega^2}(\cos\omega t - \cos 2t)$

(c) From (b)

$x(t) = \frac{3}{4-\omega^2}\, 2\sin\frac{2-\omega}{2}t \sin\frac{2+\omega}{2}t$

$= \frac{6}{(2+\omega)(2-\omega)}\sin\frac{2-\omega}{2}t \sin\frac{2+\omega}{2}t$

(d) Since $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$

We have $\frac{\sin\frac{2-\omega}{2}t}{\frac{2-\omega}{2}t} \to 1$ as $\omega \to 2$

So $x_\omega(t) = \frac{6}{2+\omega}\underbrace{\left[\frac{2}{(2-\omega)t}\sin\frac{2-\omega}{2}t\right]}_{\to 1}\frac{t}{2}\sin\frac{2+\omega}{2}t$

$\xrightarrow[\omega \to 2]{} \frac{6}{2+2}\frac{t}{2}\sin\frac{2+2}{2}t = \frac{3}{4}t\sin 2t$

Which is exactly the solution found in (a).