Solutions to Practice for Final Exam
Math 3400 - Intro to Differential Equations

Spring 2016 - Dr. Radu Cascaval

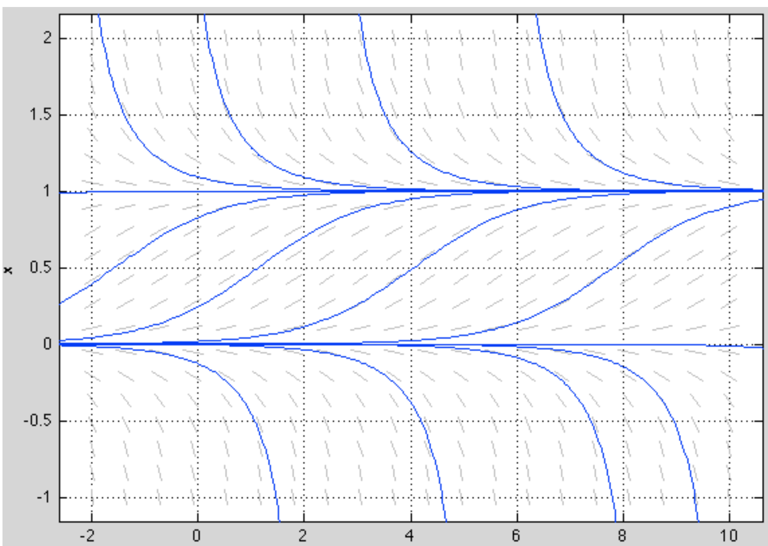
The first 15 (multiple-choice) problems are 5 points each, the remaining 5 (essay) problems are 10 points each (Show all work to get full credit on the essay problems).

1. (Circle all that apply!) The differential equation $x\dfrac{dy}{dx} = x - y$ is
   (a) separable, (b) linear, (c) homogeneous, (d) exact

   Answer: (b), (c), (d)

2. The general solution of the equation $x' + 5x = 3$ is
   (a) $x(t) = \frac{3}{5} + e^{-5t}$ (b) $x(t) = 3 + C\sin 5t$ (c) $x(t) = \frac{3}{5} + Ce^{-5t}$ (d) $x(t) = C\cos 3t$
   Answer: (c)

3. The following represents the slope field and several solution curves of the equation

   (a) $x' = t(1-x)$ (b) $x' = x(x-1)$ (c) $x' = t^2 - t$ (d) $x' = x(1-x)$
   Answer: (d)

4. A solution of the initial value problem $x' + 8x = 1 + e^{-6t}$, $x(0) = 0$, is
   (a) $x(t) = \frac{1}{8} + \frac{1}{2}e^{6t} - \frac{5}{8}e^{8t}$
   (b) $x(t) = \frac{1}{8} + \frac{1}{2}e^{-6t} - \frac{5}{8}e^{-8t}$
   (c) $x(t) = 4 - e^{2t} + 3e^{8t}$
   (d) $x(t) = 4 - e^{-2t} + 3e^{-8t}$
   Answer: (b)

5. Two linearly independent solutions of the equation $y'' + y' - 6y = 0$ are
   (a) e^{-3x} and e^{2x}, (b) e^{-2x} and e^{3x}, (c) e^{-x} and e^{6x}, (d) e^{-6x} and e^{x}
   Answer: (a)

6. A particular solution for the differential equation $y'' + 2y' + y = 3 - 2\sin x$ is
   (a) $A + B\sin x$

(b) $A + Bx^2 + C\cos x + D\sin x$
(c) $A + Bx\cos x + Cx\sin x$
(d) $A + B\cos x + C\sin x$
Answer: (d)

7. The solution of the initial value problem $x^2y'' - xy' - 3y = 0, y(1) = 1, y'(1) = -2$ is
(a) $\frac{5}{4}x^{-1} - \frac{1}{4}x^3$, (b) $\frac{1}{4}x + \frac{3}{4}x^{-3}$, (c) $\frac{5}{4}e^{-x} - \frac{1}{4}e^{3x}$, (d) $\frac{1}{4}e^x + \frac{3}{4}e^{-3x}$
Answer: (a)

8. The Laplace Transform of $f(t) = te^t\sin(2t)$ is
(a) $\frac{4(s+1)}{(s-1)^2(s^2+4)}$, (b) $\frac{4(s-1)}{((s-1)^2+4)^2}$, (c) $\frac{2}{(s-1)^2+4}$, (d) $\frac{2(s-1)}{(s-1)^2+4}$
Answer: (b)

9. The inverse Laplace Transform of the function $F(s) = \frac{4s+2}{s^2+2s+5}$ is
(a) $e^{-t}\cos 2t + 2e^{-t}\sin 2t$, (b) $e^{-t}\cos 2t - 4e^{-t}\sin 2t$, (c) $4e^{-t}\cos 2t + e^{-t}\sin 2t$,
(d) $4e^{-t}\cos 2t - e^{-t}\sin 2t$
Answer: (d)

10. The Laplace Transform of the function $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 2-t, & t \geq 1 \end{cases}$ is
(a) $\frac{1-2e^{-s}}{s^2}$, (b) $\frac{2}{s}$, (c) $\frac{1}{s^2} - \frac{2e^{-s}}{s}$, (d) $\frac{1}{s^2} + \left(\frac{2}{s} - \frac{1}{s^2}\right)e^{-s}$
Answer: (a)

11. For the initial value problem $x'' + 4x' + 13x = t, \quad x(0) = -1, x'(0) = 1$, the Laplace transform $X(s)$ of the solution $x(t)$ is
(a) $\frac{1}{s^2+4s}\left(-s - 3 - \frac{1}{s} + \frac{1}{s^2}\right)$
(b) $\frac{1}{s^2+4s}\left(-s - \frac{1}{s} + \frac{1}{s^2}\right)$
(c) $-\frac{s+3}{s^2+4s+13} + \frac{1}{s^2(s^2+4s+13)}$
(d) $-\frac{s}{s^2+4s+13} + \frac{1}{s^2(s^2+4s+13)}$
Answer: (c)

12. The partial fraction decomposition of $\frac{s+4}{(s-1)^2(s^2+4)}$ is
(a) $\frac{A}{(s-1)^2} + \frac{Bs+C}{s^2+4}$
(b) $\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+4}$
(c) $\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s^2+4}$
(d) $\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs}{s^2+4}$
Answer: (b)

13. Which of the following equations can be rearranged into separable equations?
(a) $(x + y)y' = x - y$
(b) $y' - e^y = e^{x+y}$
(c) $y' = \ln(xy)$
(d) None of the above
Answer: (b)

14. A body with mass $m = \frac{1}{2}\,kg$ is attached to the end of a spring that is stretched $2\,m$ by a force of $100\,N$. It is set in motion with initial position $x_0 = 1$ m and initial velocity $v_0 = -5\,m/s$. The position function of the body is given by
(a) $x(t) = \cos 10t - \frac{1}{2} sin 10t$
(b) $x(t) = \cos 5t - \sin 5t$
(c) $x(t) = e^{-t}(\cos 9t - 3\sin 9t)$
(d) $x(t) = e^{-10t}(\frac{1}{10}\cos t - \sin t)$
Answer: (a)

15. The appropriate form of a particular solution y_p of the equation $y'' + 2y' + y = e^{-t}$ is
(a) Ae^{-t} (b) Ate^{-t} (c) At^2e^{-t} (d) $(A + Bt)e^{-t}$
Answer: (c)

The following problems will be graded based on the complete work!! You must show all work to get full credit.

Note Title 5/4/2012

Problem 1

(i) $y' = -2xy \Rightarrow \frac{dy}{y} = -2x\,dx \Rightarrow \ln|y| = -x^2 + C$

$\Rightarrow y = e^{-x^2+c}$ or $\boxed{y(x) = c e^{-x^2}}$

(ii) $(\underbrace{x^3 + \frac{y}{x}}_{M})\,dx + (\underbrace{y^2 + \ln x}_{N})\,dy = 0$

is exact since $M_y = \frac{1}{x} = N_x$ on the region $x > 0$ in the (x,y) plane

To determine F such that $F_x = M$, $F_y = N$

we solve $F_x(x,y) = x^3 + \frac{y}{x} \Rightarrow F(x,y) = \frac{x^4}{4} + y\ln x + g(y)$

$F_y(x,y) = y^2 + \ln x$

$\Downarrow$

$F_y(x,y) = \ln x + g'(y)$

$\Rightarrow y^2 + \ln x = \ln x + g'(y) \Rightarrow g'(y) = y^2$

$\Rightarrow g(y) = \frac{y^3}{3}$

So $F(x,y) = \frac{x^4}{4} + y\ln x + \frac{y^3}{3}$ and the general sol is $\boxed{F(x,y) = c}$

(#2) $xy'' + (x-2)y' + y = 0$ has $y_1(x) = x^3 e^{-x}$ as a sol.

Method of reduction of order: Seek $v = v(x)$ such that

$$y_2(x) = v(x)\, x^3 e^{-x} \text{ solves the diff eq}$$

$$y_2'(x) = v'(x)\, x^3 e^{-x} + v(x)\left(3x^2 e^{-x} - x^3 e^{-x}\right)$$

$$= v'(x)\, x^3 e^{-x} + v(x)\left(3x^2 - x^3\right) e^{-x}$$

$$y_2''(x) = v''(x)\, x^3 e^{-x} + 2v'(x)\left(3x^2 - x^3\right) e^{-x} + v(x)\left(\left(6x - 3x^2\right) - \left(3x^2 - x^3\right)\right) e^{-x}$$

$$\Rightarrow 0 = L[y_2] = x y_2'' + (x-2) y_2' + y_2$$

$$= v''(x) \cdot \left(x \cdot x^3 e^{-x}\right) + v'(x)\left[2x\left(3x^2 - x^3\right) e^{-x} + (x-2) \cdot x^3 e^{-x}\right]$$

$$+ v(x)\underbrace{\left[\ \cdots\cdots\ \right]}_{=0}$$

$$0 = v'' x^4 e^{-x} + v'\underbrace{\left(6x^3 - 2x^4 + x^4 - 2x^3\right)}_{4x^3 - x^4} e^{-x}$$

$$\Rightarrow \quad v'' x^4 e^{-x} + v' x^3 (4 - x) e^{-x} = 0$$

$$\Rightarrow \quad x v'' + (4 - x) v' = 0$$

$$\Rightarrow \quad v'' = \frac{x-4}{x} v'$$

Set $v' = u$: $\quad u' = \frac{x-4}{x} u \Rightarrow u(x) = e^{\int \frac{x-4}{x} dx}$

$$\Rightarrow u(x) = e^{\int 1 - \frac{4}{x}\, dx} = e^{x - 4\ln x}$$

$$\Rightarrow u(x) = \frac{e^x}{x^4}$$

$$\Rightarrow v'(x) = \int \frac{e^x}{x^4}\, dx$$

$$\Rightarrow \boxed{y_2(x) = x^3 e^{-x} \int \frac{e^x}{x^4}\, dx}$$

#3 $\frac{dT}{dt} = -(T - Q_0) \Rightarrow \frac{dT}{dt} = -T + Q_0$ or $\frac{dT}{dt} + T = Q_0$

If $Q_0(t) = 20 + 10t$

$\Rightarrow$ $T(t)$ solves $\begin{cases} \frac{dT}{dt} + T = 20 + 10t \\ T(0) = 40 \end{cases}$

Using method of integrating factor. ($\rho(t) = e^t$)

$$\frac{d}{dt}T + T = 20 + 10t \quad / e^t$$

$$\frac{d}{dt}(Te^t) = (20 + 10t)e^t$$

$$Te^t = \int 20e^t + 10te^t\, dt$$

$$= 20e^t + 10 \int te^t\, dt$$

int by parts

$\int te^t dt = te^t - e^t$

$$= 20e^t + 10(te^t - e^t) + C$$

$$\Rightarrow T(t) = 20 + 10t - 10 + Ce^{-t} = 10 + 10t + Ce^{-t}$$

$t=0$: $40 = 10 + C \Rightarrow C = 30$

$$\Rightarrow \boxed{T(t) = 10 + 10t + 30e^{-t}}$$

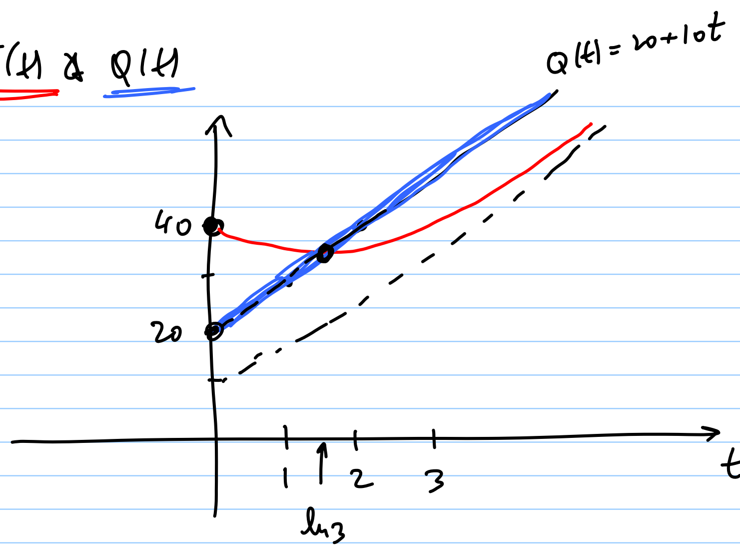
(b) $Q(t) = T(t) \Rightarrow 20 + \cancel{10t} = 10 + \cancel{10t} + 30e^{-t} \Rightarrow$

$$\Rightarrow 10 = 30e^{-t}$$

$$\Rightarrow \frac{1}{3} = e^{-t}$$

$$\Rightarrow \boxed{t = \ln 3}$$

Graphs of $T(t)$ & $Q(t)$



#4, #5 See previous exams