SOLUTIONS TO PRACTICE EXAM 1

MATH 3400 — DR. RADU CASCAVAL

#1 $ty' + (1+t)y = 2e^{-2t}$

(a) linear 1st order: (*) $y' + \frac{1+t}{t}y = \frac{2}{t}e^{-2t}$ $/\cdot\mu$

Integrating factor $\mu = e^{\int \frac{1+t}{t}dt} = e^{\int(\frac{1}{t}+1)dt} = e^{\ln t + t} = te^t$

$\Rightarrow$ equation becomes (after multiplying (*) with μ in both sides)

$$\frac{d}{dt}(y\cdot te^t) = \frac{2}{t}e^{-2t}\cdot te^t = 2e^{-t}$$

$$\Rightarrow \quad y\cdot te^t = \int 2e^{-t}dt = -2e^{-t} + c$$

$$\Rightarrow \quad y = \frac{-2e^{-t}+c}{te^t} \quad \text{or} \quad \boxed{y = \frac{c}{t}e^{-t} - \frac{2}{t}e^{-2t}}$$

c = arbitrary

(b) As $t \to 0^+$, $y(t)$ goes to $+\infty$ or $-\infty$

UNLESS the numerator in $\frac{c-2e^{-t}}{te^t}$ also goes to 0

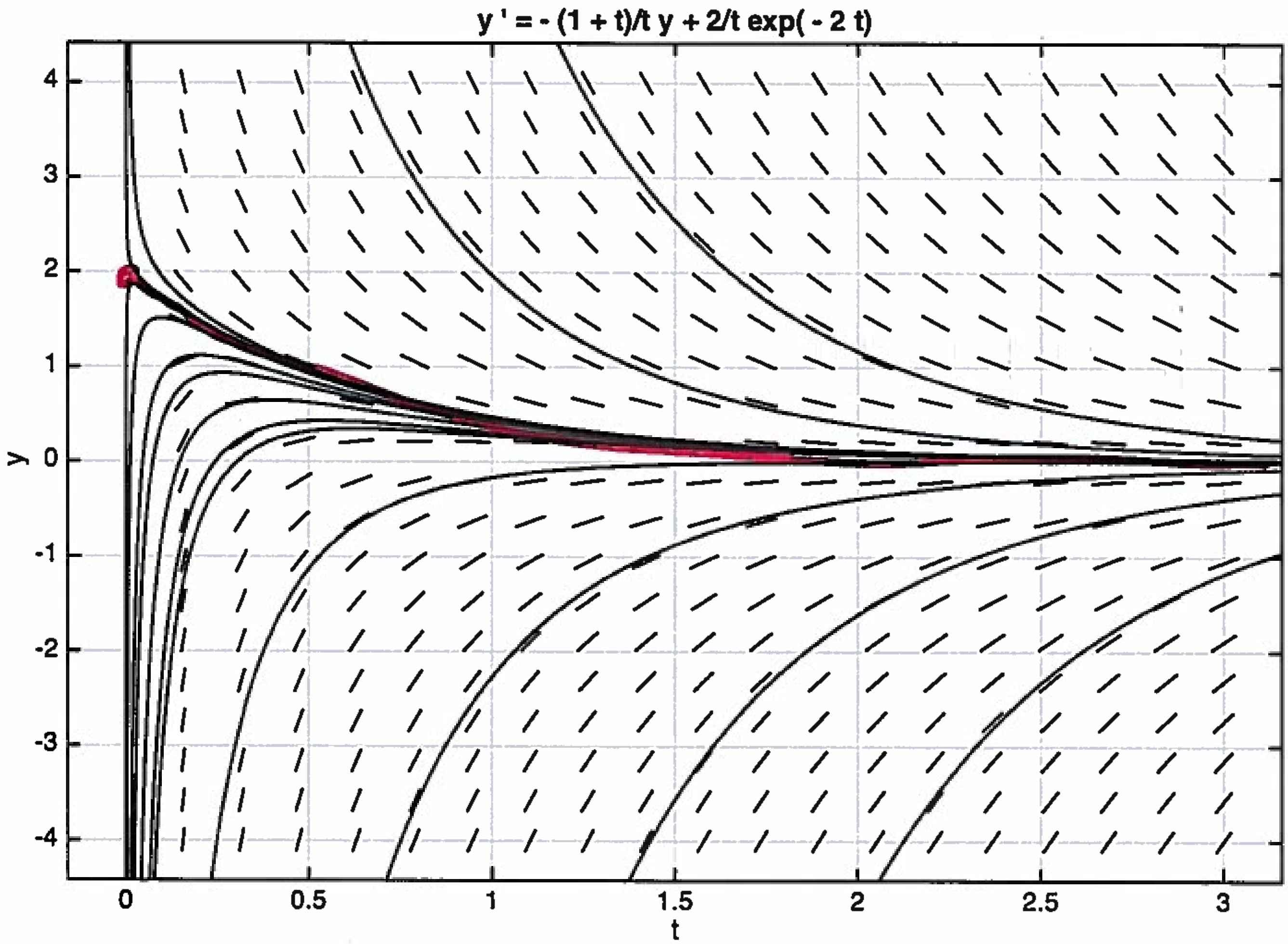
which is precisely the case when $c=2$:

$$\lim_{t\to 0} \frac{2-2e^{-t}}{te^t} = 2\underbrace{\lim_{t\to 0}\frac{1-e^{-t}}{t}} = 2\cdot 1 = 2$$

So the particular solution corresponding to $c=2$

$$y(t) = \frac{2-2e^{-t}}{te^t} \quad \text{or} \quad y(t) = \frac{2}{t}(e^{-t} - e^{-2t})$$

satisfies $y(0+) = 2$.


y ' = - (1 + t)/t y + 2/t exp(- 2 t)
4
3
2
1
0
-1
-2
-3
-4
y
0
0.5
1
1.5
2
2.5
3
t

#2 Consider $y' - y = 3te^{-2t}$

(a) Linear 1st order $\mu = e^{\int -1 dt} = e^{-t}$.

$$\Rightarrow \frac{d}{dt}\left(ye^{-t}\right) = 3te^{-2t} \cdot e^{-t} = 3te^{-3t}$$

$$\Rightarrow ye^{-t} = \int 3te^{-3t}dt.$$ Integrating by parts,

$u = 3t \quad du = 3dt$

$dv = e^{-3t}dt \quad v = \frac{e^{-3t}}{-3}$

we have $$\int \underbrace{3t}_{u}\underbrace{e^{-3t}dt}_{dv} = \underbrace{3t}_{u} \cdot \underbrace{\frac{e^{-3t}}{-3}}_{v} - \int \underbrace{\frac{e^{-3t}}{-3}}_{v}\underbrace{3dt}_{du}$$

$$= -te^{-3t} + \int e^{-3t}dt$$

$$= -te^{-3t} - \frac{e^{-3t}}{3} + c$$

So $ye^{-t} = -te^{-3t} - \frac{e^{-3t}}{3} + c \Rightarrow$

$$\Rightarrow \boxed{y = -te^{-2t} - \frac{e^{-2t}}{3} + ce^{t}}$$ general solution

(b) when $t = 0$: $y = 2$

$$2 = 0 - \frac{1}{3} + c \cdot 1 \Rightarrow c = \frac{1}{3} + 2 = \frac{7}{3}$$

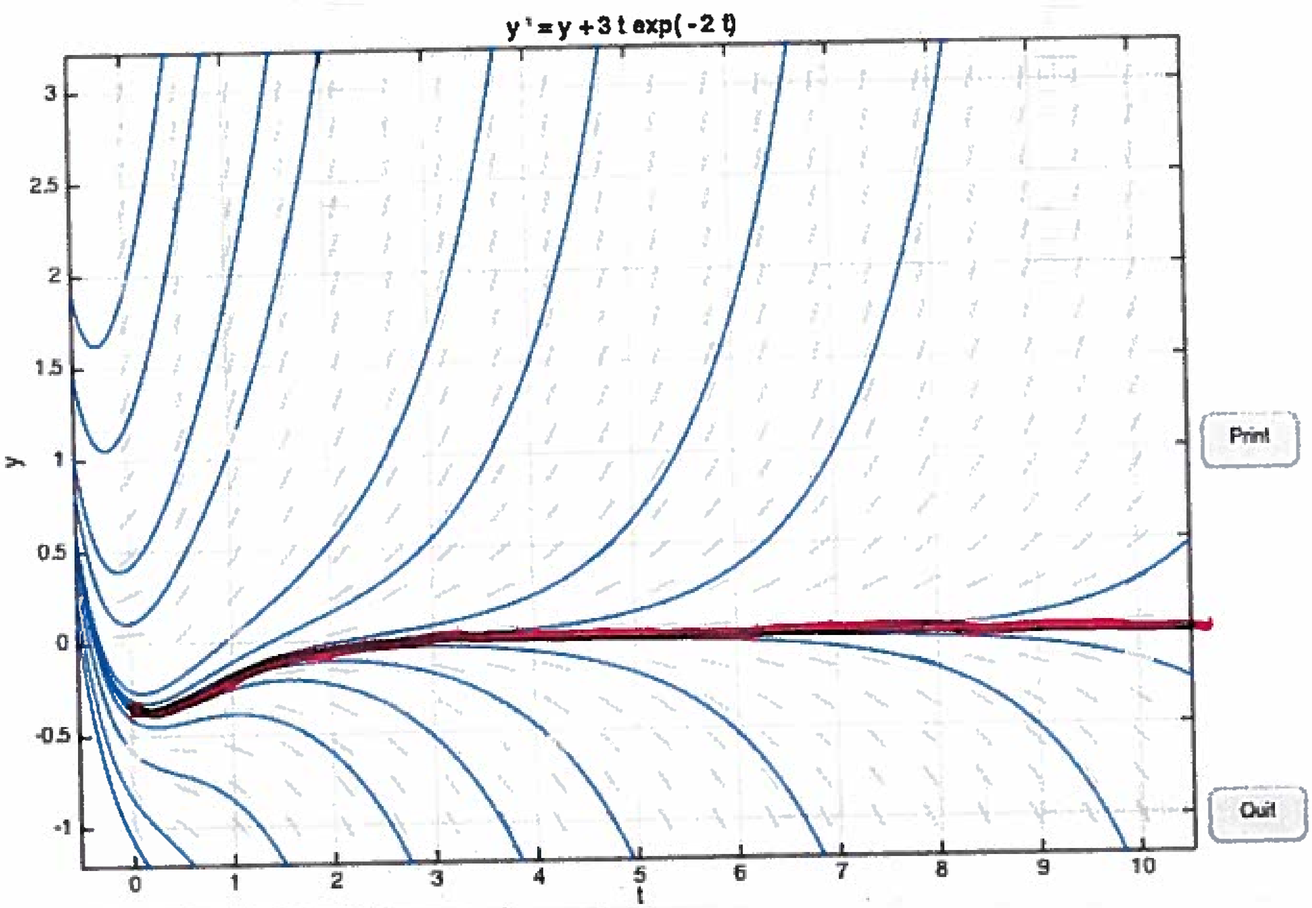
$$\Rightarrow \boxed{y = -te^{-2t} - \frac{e^{-2t}}{3} + \frac{7}{3}e^{t}}$$ Solution of IVP

(c) As $t \to +\infty$, $$\left.\begin{array}{l} te^{-2t} \to 0 \\ e^{-2t} \to 0 \\ e^{t} \to +\infty \end{array}\right\} \Rightarrow y(t) \to +\infty$$

(d) If $y(0)=y_0 \Rightarrow y_0 = 0 - \frac{1}{3} + c\cdot 1 \Rightarrow c = y_0 + \frac{1}{3}$

$$\Rightarrow y(t) = -t e^{-2t} - \frac{e^{-2t}}{3} + \left(y_0 + \frac{1}{3}\right)e^{t}$$

The only way $y(t)$ can remain finite (bounded) as $t \to \infty$ is if $y_0 + \frac{1}{3} = 0 \Rightarrow y_0 = -\frac{1}{3}$



#3 $\frac{dy}{dx} = \frac{x}{4y + x^2 y}$, $y(0) = 2$

Separable DE:

$$y\,dy = \frac{x}{4+x^2}\,dx$$

$$\int y\,dy = \int \frac{x}{4+x^2}\,dx$$

$$\frac{y^2}{2} = \frac{1}{2}\ln(4+x^2) + c$$

$$y^2 = \ln(4+x^2) + c$$

$$\boxed{y = \pm\sqrt{\ln(4+x^2)+c}}$$ general solution

$x=0 \Rightarrow y=2$:

$$2 = \sqrt{\ln(4+0)+c} \Rightarrow 4 = \ln 4 + c \Rightarrow c = 4 - \ln 4$$

$$\Rightarrow \boxed{y = \sqrt{\ln(4+x^2)+4-\ln 4}}$$ solution of the IVP.

#4 $\quad 3y + x^3y^4 + 3x\dfrac{dy}{dx} = 0$

Not separable, not linear, not homogeneous.
It is Bernoulli:

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{x^2}{3}y^4 \qquad \text{Bernoulli with } (n=4)$$

So $v = y^{1-4} = y^{-3}$ is the substitution to use.

$$\Rightarrow \quad y = v^{-1/3} \quad \Rightarrow \quad \frac{dy}{dx} = -\tfrac{1}{3}v^{-4/3}\frac{dv}{dx}$$

$\Rightarrow$ Diff eq becomes:

$$-\tfrac{1}{3}v^{-4/3}\frac{dv}{dx} + \frac{1}{x}v^{-1/3} = -\frac{x^2}{3}v^{-4/3}$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{3}{x}v = x^2 \quad \rightarrow \text{linear 1}^{st} \text{ order in } v$$

Integrating factor $\mu = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$

$$\Rightarrow \quad \frac{d}{dx}\left(v \cdot x^{-3}\right) = x^2 \cdot x^{-3} = \frac{1}{x}$$

$$\Rightarrow \quad v x^{-3} = \int \frac{1}{x}dx = \ln x + c$$

$$\Rightarrow \quad v = x^3(\ln x + c) \quad \text{or} \quad v = x^3\ln x + cx^3$$

Back to y: $\quad y = v^{-1/3} \Rightarrow y(x) = \left[x^3(\ln x + c)\right]^{-1/3}$ or

$$\boxed{y(x) = \frac{1}{x\cdot(\ln x + c)^{1/3}}}$$

#5 $\quad 2x \sin y \cos y \dfrac{dy}{dx} + 4x^2 + \sin^2 y = 0$

We check for exactness:

$$\underbrace{(4x^2+\sin^2 y)}_{M}\,dx + \underbrace{2x \sin y \cos y}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2\sin y \cos y = \frac{\partial N}{\partial x} = 2\sin y \cos y$$

exact!!!

So we seek $F = F(x,y)$ such that $\begin{cases} \dfrac{\partial F}{\partial x} = M \\ \dfrac{\partial F}{\partial y} = N \end{cases}$

$$\frac{\partial F}{\partial x} = 4x^2 + \sin^2 y \Rightarrow F(x,y) = \frac{4}{3}x^3 + x\sin^2 y + c(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = 2x \sin y \cos y + c'(y) = N(x,y) = 2x\sin y \cos y$$

$$\Rightarrow c'(y) = 0 \Rightarrow c(y) = c \text{ (constant!)}$$

$$\Rightarrow F(x,y) = \frac{4}{3}x^3 + x\sin^2 y$$ [we pick $c=0$ for simplicity]

$\Rightarrow$ General solution: $F(x,y) = \text{Const}$

$$\Rightarrow \boxed{\frac{4}{3}x^3 + x\sin^2 y = C}$$ General solution in implicit form (C = arbitrary)

It is possible to solve for y:

$$\sin y = \pm\sqrt{\frac{C - \frac{4}{3}x^3}{x}} \Rightarrow \boxed{y = \pm \arcsin\sqrt{\frac{C - \frac{4}{3}x^3}{x}}}$$

#6 $(2y-5x)\frac{dy}{dx} = x+5y$

Homogeneous DE:

$$\frac{dy}{dx} = \frac{x+5y}{2y-5x} = \frac{1+5\frac{y}{x}}{2\frac{y}{x}-5}$$

Substitution $v = \frac{y}{x} \Rightarrow y = x \cdot v(x) \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow$ Diff Eq becomes $v + x\frac{dv}{dx} = \frac{1+5v}{2v-5}$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+5v}{2v-5} - v = \frac{1+5v-2v^2+5v}{2v-5} = \frac{1+10v-2v^2}{2v-5}$$

Separable DE in v:

$$\frac{2v-5}{1+10v-2v^2}\,dv = \frac{dx}{x} \Rightarrow \int \frac{2v-5}{\underbrace{1+10v-2v^2}_{}}\,dv = \int\frac{dx}{x}$$

Subs $u = 1+10v-2v^2$

$du = 10-4v = 2(5-2v)$

$$\Rightarrow -\frac{1}{2}\int\frac{du}{u} = \ln|x|+c$$

$$\Rightarrow -\frac{1}{2}\ln|1+10v-2v^2| = \ln|x|+c$$

$$\Rightarrow \ln|1+10v-2v^2| = -2\ln|x|+c$$

$$\Rightarrow 1+10v-2v^2 = \frac{c}{x^2}$$

Back to y

$\Rightarrow 1+10\frac{y}{x} - 2\frac{y^2}{x^2} = \frac{c}{x^2}$ or

$$\boxed{2y^2 - 10xy - x^2 = c}$$

One can solve for $y = y(x)$ ...

#7 $\quad xy' - 3y = 3x\sqrt[3]{y^4}$.

Definitely Bernoulli

$$y' - \frac{3}{x}y = 3\,y^{4/3} \qquad n = 4/3$$

Substitution $\quad v = y^{1-4/3} = y^{-1/3} \Rightarrow \quad y = v^{-3}$

$\Rightarrow \quad -3v^{-4}v' - \frac{3}{x}v^{-3} = 3(v^{-3})^{4/3} \qquad y' = -3v^{-4}v'$

$$= 3v^{-4}$$

$\Rightarrow \quad \boxed{v' + \frac{1}{x}v = -1}$ linear in v:

$$xv' + v = -x \quad (\text{integrating factor } \mu = x)$$

$$\frac{d}{dx}(xv) = -x$$

$$xv = -\frac{x^2}{2} + c \Rightarrow \boxed{v = -\frac{x}{2} + \frac{c}{x}}$$

Back to y:

$$y = v^{-3} \Rightarrow \boxed{y(x) = \left(-\frac{x}{2} + \frac{c}{x}\right)^{-3}}$$

#8

$2y^3 y'' = 1$. This is second order DE

But it is reducible

$F(y, y', y'') = c$

(not explicitly dependent on x!)

Subs:

$p = \frac{dy}{dx} \Rightarrow$ Treat it as $p = p(y)$!

$y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p\frac{dp}{dy}$

$$2y^3 \cdot p\frac{dp}{dy} = 1$$

Separable in $p = p(y)$:

$$2p\,dp = \frac{1}{y^3}dy \Rightarrow \int 2p\,dp = \int \frac{1}{y^3}dy$$

$$\Rightarrow p^2 = -\frac{1}{2y^2} + c \Rightarrow p = \pm\sqrt{\frac{2cy^2 - 1}{2y^2}}$$

Back to y:

$$\frac{dy}{dx} = \pm\sqrt{\frac{cy^2 - \frac{1}{2}}{y^2}} \Rightarrow \frac{dy}{dx} = \pm\frac{\sqrt{cy^2 - \frac{1}{2}}}{y}$$

Separable in $y = y(x)$:

$$\frac{y}{\sqrt{cy^2 - \frac{1}{2}}}dy = \pm dx \Rightarrow \int \frac{y}{\sqrt{\underbrace{cy^2 - \frac{1}{2}}_{u}}}dy = \pm x + d$$

arbitrary constant

Subs $u = cy^2 - \frac{1}{2}$

$du = 2cy\,dy$

$$\Rightarrow \frac{1}{2c}\int \frac{du}{\sqrt{u}} = \pm x + d \Rightarrow 2\sqrt{u} = \pm 2cx + d \quad \text{(new "d")}$$

or $\sqrt{u} = \pm cx + d \Rightarrow u = (\pm cx + d)^2$

$$\Rightarrow \quad cy^2 - \tfrac{1}{2} = (\pm cx + d)^2$$

$$\Rightarrow \quad y^2 = \frac{1}{c}\left[\tfrac{1}{2} + (\pm cx + d)^2\right]$$

$$\Rightarrow \quad \boxed{y(x) = \pm\sqrt{\frac{1}{2c} + \frac{(\pm cx + d)^2}{c}}}$$

c, d = arbitrary!

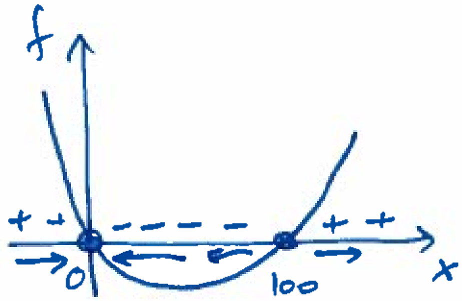
#9 $\quad \dfrac{dx}{dt} = 0.0001x^2 - 0.01x = f(x)$

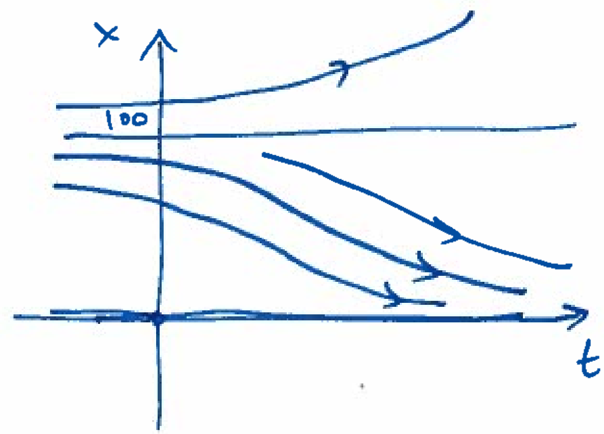
Equilibria: where RHS $= 0$: $f(x) = 0$

$$0.0001x^2 - 0.01x = 0 \Rightarrow 0.0001(x^2 - 100x) = 0$$

$$\Rightarrow x(x - 100) = 0 \Rightarrow \boxed{x = 0} \text{ or } \boxed{x = 100}$$

Stability





$x = 0$ is stable (asymptotically)

$x = 100$ is unstable

(b) $X(0) = 25$

$$\frac{dx}{dt} = 0.0001\, x(x-100)$$

$$\frac{dx}{x(x-100)} = 0.0001 dt$$

$$\int \frac{1}{x(x-100)} dx = \int 0.0001 dt = 0.0001t + c$$

Partial Fractions

$$\frac{1}{x(x-100)} = \frac{1}{100}\left(-\frac{1}{x} + \frac{1}{x-100}\right)$$

$$\Rightarrow \int \frac{1}{x(x-100)} dx = \frac{1}{100}\left(-\ln|x| + \ln|x-100|\right) = \frac{1}{100} \ln\left|\frac{x-100}{x}\right|$$

DE: $\Rightarrow \frac{1}{100} \ln\left|\frac{x-100}{x}\right| = 0.0001t + c \Rightarrow$

$$\Rightarrow \ln\left|\frac{x-100}{x}\right| = 0.01t + c \Rightarrow \frac{x-100}{x} = ce^{0.01t}$$

$$\Rightarrow (x-100) = x c e^{0.01t} \Rightarrow \boxed{x = \frac{100}{1+ce^{0.01t}}}$$

$$t=0 \Rightarrow x = 25 \Rightarrow 25 = \frac{100}{1+c} \Rightarrow 1+c = 4 \Rightarrow c = 3 \Rightarrow \boxed{x = \frac{100}{1+3e^{0.01t}}}$$

As $t \to \infty$, $x(t) \to 0$ (so extinction!)

#10 $\dfrac{dP}{dt} = P(10-P) - h$

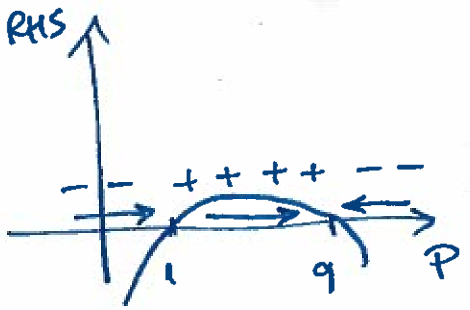
(logistic growth with harvest)

Equilibria: $P = P^*$ where RHS$=0$

$$P(10-P)-h=0 \Rightarrow 10P-P^2-h=0$$

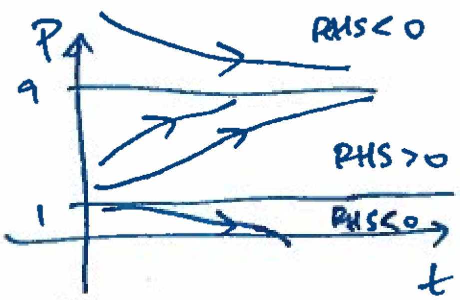
$$\Rightarrow P^2-10P+h=0 \Rightarrow P^*_{1,2} = 5 \pm \sqrt{25-h}$$

So if $h=9$: $P_{1,2} = 5 \pm \sqrt{25-9} = 5 \pm 4 = \begin{cases} P_1 = 1 \\ P_2 = 9 \end{cases}$



So $P_2 = 9$ is a stable (limiting) equilibria

while $P_1 =$ [redacted] is unstable (critical value)



#11 (a)

Ascend

$$ma = F_R - mg = -kv - mg$$

$$m\frac{dv}{dt} = -kv - mg$$

$$\Rightarrow \frac{dv}{dt} = -\frac{k}{m}v - g$$

$$\Rightarrow \frac{dv}{dt} + \frac{k}{m}v = -g \quad / \; e^{\frac{k}{m}t}$$

$$\Rightarrow \frac{d}{dt}\left(ve^{\frac{k}{m}t}\right) = -ge^{k/m\, t}$$

$$\Rightarrow ve^{k/m\, t} = \int -g e^{k/m\, t}\, dt = -g\frac{m}{k}e^{\frac{k}{m}t} + c$$

$$\Rightarrow \boxed{v = -g\frac{m}{k} + ce^{-k/m\, t}}$$

At $t=0$, $v(0) = v_0 \Rightarrow v_0 = -g\frac{m}{k} + ce^{0} \Rightarrow c = v_0 + g\frac{m}{k}$

So $\boxed{v(t) = -g\frac{m}{k} + \left(v_0 + g\frac{m}{k}\right)e^{-k/m\, t}}$

$v_0 = 5$ m/s
$g = 10$ m/s^2
$k = 5$ kg/s
$m = 1$ kg

$$\Rightarrow v(t) = -2 + (5+2)e^{-5t} = -2 + 7e^{-5t}$$

(b) Max height is when $v=0$

$$v(t) = -2+7e^{-5t} = 0 \Rightarrow e^{-5t} = \frac{2}{7} \Rightarrow -5t = \ln\frac{2}{7}$$

$$\Rightarrow t^* = \frac{1}{5}\ln\frac{7}{2} = 0.2506$$

$$\frac{dx}{dt} = v(t) = -2+7e^{-5t}$$

$$\Rightarrow x(t) = \int(-2+7e^{-5t})dt = -2t - \frac{7}{5}e^{-5t} + d$$

$$t=0 \to x=0 \Rightarrow 0 = -\frac{7}{5} + d \Rightarrow d = \frac{7}{5} \Rightarrow$$

$$\Rightarrow \boxed{x(t) = -2t - \frac{7}{5}e^{-5t} + \frac{7}{5}}$$

At $t^* = \frac{1}{5}\ln\frac{7}{2} \Rightarrow x^* = x(t^*) = 0.498$ m

(c)

Descend

$$ma = F_R - mg \qquad F_R = kv$$

$$m\frac{dv}{dt} = kv - mg$$

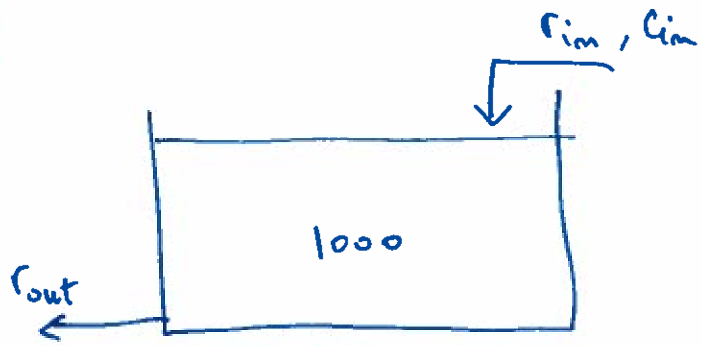
$$\frac{dv}{dt} = \frac{k}{m}v - g$$

$$\Rightarrow \frac{dv}{dt} - \frac{k}{m}v = -g \quad / e^{-\frac{k}{m}t}$$

$$\Rightarrow \frac{d}{dt}\left(v e^{-\frac{k}{m}t}\right) = -g e^{-\frac{k}{m}t}$$

$$v e^{-\frac{k}{m}t} = g\frac{m}{k}e^{-\frac{k}{m}t} + c \Rightarrow \boxed{v(t) = g\frac{m}{k} + ce^{\frac{k}{m}t}}$$

#12



$r_{in} = 5$ gal/s , $c_{in} = 0$

$r_{out} = 5$ gal/s

$V = V_0 = 1{,}000$ gals

$x(t) =$ amount of salt in tank

$$\Rightarrow \quad \frac{dx}{dt} = r_{in} C_{in} - r_{out} C_{out}$$

$$= 5 \cdot 0 - 5 \cdot \frac{x}{V}$$

$$\frac{dx}{dt} = -5 \frac{x}{V_0} = -\frac{5}{1000} x = -0.005 x$$

$$\Rightarrow \quad x(t) = x_0 e^{-0.005 t}$$

$x_0 = 500$ lb

$$\Rightarrow \boxed{x(t) = 500 e^{-0.005 t}} \text{ lb}$$

To get half the initial ~~c~~ concentration :

$$C(t) = \frac{x(t)}{V_0} = \frac{500}{1000} e^{-0.005 t} = \frac{1}{2} C(0) = \frac{1}{2} \frac{x_0}{V_0} = \frac{1}{2} \frac{500}{1000}$$

$$\Rightarrow \quad e^{-0.005 t} = \frac{1}{2} \Rightarrow \quad 0.005 t = \ln 2$$

$$t = \frac{\ln 2}{0.005} = 138.62 \, (s).$$