SOLUTIONS

Student Name: ____________________

Exam 1 Math 3410

Intro to Analysis

Dr. Radu C. Cascaval, Spring 2016

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3 pts each

1. Determine whether the following statements are TRUE or FALSE. Justify your answers.

(a) The sum of two irrational numbers is irrational.

FALSE $\quad a=\sqrt{2}$, $b=1-\sqrt{2}$ are both irrationals but $a+b=1\in\mathbb{Q}$.

(b) If a and b are real numbers satisfying $a < b+1/n$, for all $n\in\mathbb{N}$, then $a\le b$.

TRUE $\quad$ Assume the contrary: that $a>b \Rightarrow \exists\,\varepsilon>0$ such that $b+\varepsilon<a$ $\Rightarrow b+\varepsilon<a<b+1/n$ for all $n \Rightarrow \varepsilon<1/n$ for all $n \Rightarrow n<\frac{1}{\varepsilon}$ for all $n\in\mathbb{N}$ which is a contradiction (enough to pick $n=[\frac{1}{\varepsilon}]+1>\frac{1}{\varepsilon}$)! Therefore $a\le b$!

(c) Every monotone sequence is bounded.

FALSE $\quad \{s_n=n\}_{n\ge1}$ is monotonically increasing but not bounded (above)

(d) Every bounded sequence of real numbers contains a convergent subsequence.

TRUE $\quad$ This is the statement of the Bolzano-Weierstrass theorem.

(e) For any sequence of real numbers, $\liminf s_n$ and $\limsup s_n$ always exist and, furthermore, $\liminf s_n\le\limsup s_n$.

TRUE $\quad \liminf s_n=\lim_{N\to\infty}\left(\inf\{s_n \mid n>N\}\right)$ exists in $\mathbb{R}\cup\{\pm\infty\}$

Similarly $\limsup s_n=\lim_{N\to\infty}\left(\sup\{s_n \mid n>N\}\right)$ — " —

and $\liminf s_n\le\limsup s_n$

(f) If $\liminf s_n=+\infty$ then $\lim s_n=+\infty$.

TRUE $\quad$ If $\liminf s_n=+\infty$, since $\liminf s_n\le\limsup s_n$

$\Rightarrow \limsup s_n=\liminf s_n=+\infty \Rightarrow \lim s_n=+\infty$!

[Alternately, one can argue using the def using M:

$\liminf s_n=+\infty \Rightarrow \forall\, M>0,\ \exists\, N_M\in\mathbb{N}: \ s_n>M,\ \forall\, n\ge N_M$

$\Rightarrow \lim s_n=+\infty$!

16pts 2. (a) Given $b > 0$, show by mathematical induction that $\boxed{(1+b)^n > 1+nb,}$ for all $n > 1$.

"r_n"

$n=2$: $(1+b)^2 = 1+2b+b^2 > 1+2b \Rightarrow P_2$ true

Assume P_n = true, we prove P_{n+1} = true,

$$(1+b)^{n+1} = (1+b)^n(1+b) \underset{P_n = \text{true}}{>} (1+nb)(1+b) = 1+(n+1)b+nb^2 > 1+(n+1)b$$

$\Rightarrow P_{n+1}$ is true!

By mathematical induction, $(1+b)^n > 1+nb$ for all $n>1$.

(b) Using (a), prove that $\lim_{n\to\infty} r^n = +\infty$ for $r > 1$.

Enough to take $b = r-1 > 0 \Rightarrow r^n = (1+b)^n > 1+nb$

and since $\lim_{n\to\infty}(1+nb) = +\infty$, it follows $\lim_{n\to\infty} r^n = +\infty$.

15pts 3. Compute the following limit

"S_n"

$$\lim_{n\to\infty} \boxed{\frac{n^2-3n}{n^2+(-1)^n(\tan^{-1}n+n)}} =$$

Divide both num & denom. by n^2.

$$S_n = \frac{1-\frac{3}{n}}{1+(-1)^n\frac{\tan^{-1}n+n}{n^2}} = \frac{1-\frac{3}{n}}{1+(-1)^n\frac{\tan^{-1}n}{n^2}+(-1)^n\frac{1}{n}}$$

Now $\lim_{n\to\infty}(-1)^n\frac{1}{n} = 0$ (by squeezing property $\left|(-1)^n\frac{1}{n}\right| \le \frac{1}{n} \to 0$ as $n\to\infty$)

and similarly

$$\lim_{n\to\infty}(-1)^n\frac{\tan^{-1}n}{n^2} = 0 \quad \left(\left|(-1)^n\frac{\tan^{-1}n}{n^2}\right| \le \frac{\pi/2}{n^2} \to 0 \text{ as } n\to\infty\right)$$

Using properties of limits,

$$\lim_{n\to\infty} S_n = \frac{\lim_{n\to\infty}\left(1-\frac{3}{n}\right)}{\lim_{n\to\infty}\left(1+(-1)^n\frac{\tan^{-1}n}{n^2}+(-1)^n\frac{1}{n}\right)} = \frac{1}{1} = \boxed{1}.$$

16 pts

4. Let (s_n) be defined by $s_1 = 2$ and

$$s_{n+1} = 2 - \frac{1}{s_n}, \quad \text{for all } n \in \mathbb{N}.$$

(a) Compute the first 4 terms of the sequence.

$S_1 = 2, \quad S_2 = 2 - \frac{1}{S_1} = 2 - \frac{1}{2} = \frac{3}{2}$

$S_3 = 2 - \frac{1}{S_2} = 2 - \frac{1}{3/2} = 2 - \frac{2}{3} = \frac{4}{3}$

$S_4 = 2 - \frac{1}{S_3} = 2 - \frac{1}{4/3} = 2 - \frac{3}{4} = \frac{5}{4}$

Note that, even it is not necessarily for the subsequent solution, one can prove $S_n = \frac{n+1}{n} \; \forall n$.

(b) Show that the sequence (s_n) is bounded below by 1.

We show $S_n > 1$ for all n by mathematical induction

Indeed, for $n=1$, $S_1 = 2 > 1$

Assuming $S_n > 1 \Rightarrow S_{n+1} = 2 - \frac{1}{S_n} > 2 - \frac{1}{1} = 1$

So, by induction, $\boxed{S_n > 1 \text{ for all } n \geq 1}$

(c) Prove that (s_n) is convergent *and* find $\lim_n s_n$.

Enough to show (S_n) is monotone decreasing. (being also bounded by 1 from below implies that it is convergent)

Indeed, to show $S_{n+1} \leq S_n$ is equivalent to show

$2 - \frac{1}{S_n} \leq S_n \Leftrightarrow \frac{2S_n - 1}{S_n} \leq S_n \underset{S_n > 0}{\Leftrightarrow} S_n^2 - 2S_n + 1 \geq 0$

$\Rightarrow (S_n - 1)^2 \geq 0$. This is true always (for all $n \geq 1$).

So (S_n) decreasing & bounded below by $1 \Rightarrow S_n$ converges

Let $S = \lim_{n \to \infty} S_n$. Then taking the limit in $S_{n+1} = 2 - \frac{1}{S_n} \Rightarrow$

$S = 2 - \frac{1}{S} \Rightarrow S = \frac{2S-1}{S} \Rightarrow S^2 - 2S + 1 = 0 \Rightarrow (S-1)^2 = 0 \Rightarrow$

$\Rightarrow S = 1 \Rightarrow \boxed{\lim S_n = 1}$

17 pts 5. Given S and T, two nonempty bounded sets of real numbers, consider their union $S \cup T$ and intersection $S \cap T$.

(a) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Since S, T are bounded, so is $S \cup T$ so $\sup S, \sup T, \sup(S\cup T) < +\infty$

We prove "=" by "≤" and "≥".

Proof of "≤":

$\left.\begin{array}{l}\forall s \in S,\ s \le \sup S \\ \forall t \in T,\ t \le \sup T\end{array}\right] \Rightarrow \forall r \in S\cup T,\ r \le \max\{\sup S, \sup T\} \Rightarrow \max\{\sup S, \sup T\}$ is an upper bound for $S \cup T \Rightarrow$

$\Rightarrow \sup(S\cup T) \le \max\{\sup S, \sup T\}$

Proof of "≥" For $\left.\begin{array}{l}\varepsilon > 0, \exists\, s_0 \in S \text{ with } s_0 > \sup S - \varepsilon \\ \exists\, t_0 \in T \text{ with } t_0 > \sup T - \varepsilon\end{array}\right] \Rightarrow \exists\, r_0 \in S \cup T$ with

$r_0 > \max\{\sup S, \sup T\} - \varepsilon \Rightarrow \sup(S\cup T) > \max\{\sup S, \sup T\} - \varepsilon$. (either s_0 or t_0) Since this is true for all $\varepsilon > 0$

$\rightarrow \sup(S \cup T) \ge \max\{\sup S, \sup T\}$

(b) Derive (and briefly justify) the similar formula for $\inf(S \cup T) =$ ________

For (b) : $\inf(S \cup T) = \min\{\inf S, \inf T\}$.

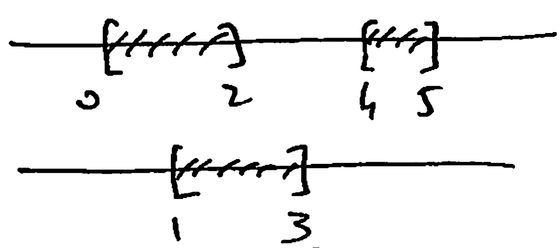
The proof mirrors the proof of (a), or one can simply apply (a) to $-S$ and $-T$.

(b) Is there a similar formula for $\sup(S \cap T)$ in general?

No, there is no connection between $\sup(S\cap T)$ and $\sup S$ & $\sup T$ (in general).

Ex: $S = [0,2] \cup [4,5]$

$T = [1,3]$



$\Rightarrow S \cap T = [1,2] \Rightarrow \sup(S\cap T) = 2$ while $\sup S = 5$
$\sup T = 3$.

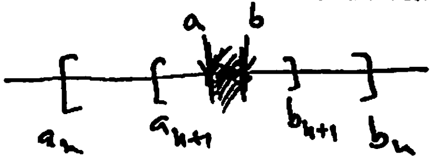
18pts 6. (a) State the Completeness Axiom for $\mathbb{R}$..

Any nonempty set $S \subset \mathbb{R}$ which is bounded from above admits a least upper bound in $\mathbb{R}$ (that is $\sup S \in \mathbb{R}$

(b) Using the Completeness Axiom, prove the Nested Interval Property: For a sequence of intervals $I_n = [a_n, b_n]$ satisfying $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for all n, that is

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$$

there exists at least one real number in the intersection of all these intervals, that is



$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

The sequence $\{a_n\}_{n \ge 1}$ is monotonically increasing and bounded (by any b_n's)

So it is convergent to its $\sup\{a_n \mid n \in \mathbb{N}\}$:

$\lim_{n\to\infty} a_n = a = \sup\{a_n \mid n \in \mathbb{N}\}$. Similarly, $\lim_{n\to\infty} b_n = b = \inf\{b_n \mid n \in \mathbb{N}\}$

Then $\bigcap_{n=1}^{\infty} I_n = \bigcap_{n \ge 1}^{\infty} [a_n, b_n] = [a, b]$ with $a \le b$. So it is $\neq \emptyset$.

(c) Using (b), prove that every real number in $[0, 1]$ can be obtained as the limit of a sequence of rational numbers of the form $\frac{m}{2^k}$, $k \in \mathbb{N}$, $m = 0, 1, \ldots 2^k - 1$.

[Hint: Start with the interval $I_0 = [0, 1]$, then pick I_1 to be either $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$, whichever contains the real number in question. Repeat the process.]

Start with any $x \in [0,1]$. We consider $I_0 = [0,1]$

Then let $I_1 = [0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$, depending on whether $x < \frac{1}{2}$ or $x \ge \frac{1}{2}$

$I_1 = \begin{cases} [0, \frac{1}{2}], & \text{if } x < \frac{1}{2} \\ [\frac{1}{2}, 1] & \text{if } x \ge \frac{1}{2} \end{cases}$. Thus $x \in I_1$.

We repeat this process (of bisecting the interval) and obtain $\forall n \in \mathbb{N}$, $I_n = [a_n, b_n]$, with $x \in I_n \Rightarrow a_n \le x \le b_n$

Enough to show $\{x\} = \bigcap_{n \ge 1} I_n$. This follows from (b) and the fact that $|b_n - a_n| = \frac{1}{2^n} \to 0 \Rightarrow x = \lim a_n$.