Math 3410 - Intro to Analysis - MathOnline
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March 1, 2016 Student Name: ______________________________

Exam 1 - Math 3410 - Intro to Analysis

Dr. Radu C. Cascaval, Spring 2016

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1. Determine whether the following statements are TRUE or FALSE. Justify your answers.

   (a) The sum of two irrational numbers is irrational.

   (b) If a and b are real numbers satisfying $a < b + 1/n$, for all $n \in \mathbb{N}$, then $a \leq b$.

   (c) Every monotone sequence is bounded.

   (d) Every bounded sequence of real numbers contains a convergent subsequence.

(e) For any sequence of real numbers, $\liminf s_n$ and $\limsup s_n$ always exist and, furthermore, $\liminf s_n \leq \limsup s_n$.

(f) If $\liminf s_n = +\infty$ then $\lim s_n = +\infty$.

2. (a) Given $b > 0$, show by mathematical induction that $(1 + b)^n > 1 + nb$, for all $n > 1$.

(b) Using (a), prove that $\lim_{n\to\infty} r^n = +\infty$ for $r > 1$.

3. Compute the following limit

$$\lim_{n\to\infty} \frac{n^2 - 3n}{n^2 + (-1)^n(\tan^{-1} n + n)} =$$

4. Let (s_n) be defined by $s_1 = 2$ and

$$s_{n+1} = 2 - \frac{1}{s_n}, \quad \text{for all } n \in \mathbb{N}.$$

(a) Compute the first 4 terms of the sequence.

(b) Show that the sequence (s_n) is bounded below by 1.

(c) Prove that (s_n) is convergent *and* find $\lim_n s_n$.

5. Given S and T, two nonempty bounded sets of real numbers, consider their union $S \cup T$ and intersection $S \cap T$.
   (a) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

   (b) Derive (and briefly justify) the similar formula for $\inf(S \cup T) =$ ______________________________

   (b) Is there a similar formula for $\sup(S \cap T)$ in general?

6. (a) State the Completeness Axiom for $\mathbb{R}$..

(b) Using the Completeness Axiom, prove the Nested Interval Property: For a sequence of intervals $I_n = [a_n, b_n]$ satisfying $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all n, that is

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$$

there exists at least one real number in the intersection of all these intervals, that is

$$\cup_{n=1}^{\infty} I_n \neq \emptyset$$

(c) Using (b), prove that every real number in $[0,1]$ can be obtained as the limit of a sequence of rational numbers of the form $\frac{m}{2^k}$, $k \in \mathbb{N}$, $m = 0, 1, \ldots 2^k - 1$.

[Hint: Start with the interval $I_0 = [0,1]$, then pick I_1 to be either $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$, whichever contains the real number in question. Repeat the process.]