April 15, 2015

SOLUTIONS Student Name: ____________________

Exam 2 - Math 3410

Intro to Analysis

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The exam has 5 problems on 5 pages. Students need to make sure ALL pages are returned. To receive credit, students need to show all work. No calculators allowed!

1. Determine whether the following statements are TRUE or FALSE. Justify your answers.

(a) If $\limsup |a_n| = 0$ then $\lim a_n = 0$.

$\boxed{\text{TRUE}}$ Solution 1: $\limsup |a_n| = 0 \Rightarrow \forall \varepsilon > 0$, $|a_n| < \varepsilon$ for all n except finitely many $\Rightarrow \forall \varepsilon > 0, \exists N_\varepsilon : |a_n| < \varepsilon, \forall n > N_\varepsilon \Rightarrow \lim_{n\to\infty} a_n = 0$ (by def.)

Solution 2: Since $0 \le |a_n| \Rightarrow 0 \le \liminf |a_n| \le \limsup |a_n| = 0 \Rightarrow \lim a_n = 0$.

(b) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for all $p \ge 1$.

$\boxed{\text{FALSE}}$ The series $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) diverges ($p = 1$). Statement would be true if $p > 1$.

(c) $\sum_{n=1}^{\infty} \frac{1}{10^n} = \frac{1}{9}$.

$$\sum_{n=1}^{\infty} \frac{1}{10^n} = \sum_{n=1}^{\infty} \frac{1}{10}\left(\frac{1}{10}\right)^{n-1} = \sum_{n=0}^{\infty} \frac{1}{10}\left(\frac{1}{10}\right)^{n} = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{1}{9}.$$

$\boxed{\text{TRUE}}$ This is a geometric series:

Another way: $\sum_{n=1}^{\infty} \frac{1}{10^n} = \left(\sum_{n=0}^{\infty} \frac{1}{10^n}\right) - 1 = \frac{1}{1-\frac{1}{10}} - 1 = \frac{10}{9} - 1 = \frac{1}{9}$.

(d) If the series $\sum a_n$ is convergent, then so is $\sum a_n^2$.

$\boxed{\text{FALSE}}$ $\sum_{n \ge 1} \frac{(-1)^n}{\sqrt{n}}$ converges (by Alternating Series Test),

but $\sum_{n \ge 1} \left[\frac{(-1)^n}{\sqrt{n}}\right]^2 = \sum_{n \ge 1} \frac{1}{n} = +\infty$ (diverges, as harmonic series).

(e) If the series $\sum |a_n|$ is convergent, then so is $\sum a_n^2$

$\boxed{\text{TRUE}}$ $\sum |a_n| < +\infty \Rightarrow \lim a_n = 0 \Rightarrow \exists N : |a_n| < 1, \forall n \ge N$

So $a_n^2 = |a_n|^2 < |a_n|, \forall n \ge N$. By Comparison Test, $\sum_{n \ge N} a_n^2 < +\infty$ as well!

(f) If $\liminf_n a_n = 0$ then there exist a subsequence (a_{n_k}) of (a_n) with $\sum_k a_{n_k}$ convergent.

$\boxed{\text{TRUE}}$ $\Downarrow$ $\forall k \in \mathbb{N}, \exists a_{n_k} : |a_{n_k}| < \frac{1}{k^2}$. Moreover, one can pick $n_1 < n_2 < \ldots < n_k < \ldots$ having this property $\Rightarrow \sum_k |a_{n_k}| < \sum \frac{1}{k^2} < +\infty \Rightarrow \sum a_{n_k}$ is (abs) convergent!

(g) The series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$ is absolutely convergent.

$\boxed{\text{TRUE}}$ This is (in compact form) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ and we know that

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$ (p-test series)!

2. For each of the following series, determine whether it is absolutely convergent, conditionally convergent or divergent. Please indicate which test you are using and show all your work.

Cond Conv

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. This is convergent, by Alternating series test ($b_n = \frac{1}{n} \searrow 0$). It is not absolutely convergent, since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n\geq 1}^{\infty} \frac{1}{n} = +\infty$ (harmonic series)

Hence, it is conditionally convergent series!

div

(b) $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2+n} - n)$. By rationalizing the numerator

$$\sqrt{n^2+n} - n = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \longrightarrow \frac{1}{2}$$

Hence $a_n = (-1)^n(\sqrt{n^2+n} - n)$ does not converge to 0 (it is in fact divergent!) so $\sum_{n=1}^{\infty} a_n$ is divergent as well!

Cond Conv

(c) $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2+1} - n)$. Again, rationalizing the numerator

$$\sqrt{n^2+1} - n = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} \searrow 0 \text{ as } n\to\infty,$$

so by alternating series test, $\sum_{n=1}^{\infty} (-1)^n(\sqrt{n^2+1} - n)$ is convergent. It is not absolutely convergent though, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}+n} = +\infty$ (limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$)

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2+1}+n}} = \sqrt{1+\frac{1}{n^2}}+1 \to 2$$

abs Conv

(d) $\sum_{n\geq 1} \frac{1}{1.01^n - 1000}$

Since $\sum_{n\geq n_0} \frac{1}{1.01^n} < +\infty$ (geometric series with $r = \frac{1}{1.01} < 1$)

By limit comparison test, we conclude $\sum_{n\geq n_0} \frac{1}{1.01^n - 1000}$ is also (abs) convergent! $\lim_{n\to\infty} \frac{\frac{1}{(1.01)^n}}{\frac{1}{(1.01)^n - 1000}} = 1$

[first few negative terms don't matter].

abs Conv

(e) $\sum_{n\geq 1} \frac{1}{n^2 + \sqrt{n^2+1}}$

By Comparison Test: $0 \leq \frac{1}{n^2+\sqrt{n^2+1}} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$

... continued on next page ...

div

(f) $\sum_{n=1}^{\infty} \log \frac{n+1}{n} = \sum_{n=1}^{\infty} [\log(n+1) - \log n]$. This is a telescopic series

n^{th} partial sum $S_n = \sum_{k=1}^{n} \log(k+1) - \log k = \cancel{\log 2} - \log 1 + \cancel{\log 3} - \cancel{\log 2}$

$+ \dots + \log(n+1) - \cancel{\log n} = \log(n+1) - \underbrace{\log 1}_{0} = \log(n+1) \xrightarrow[n\to\infty]{} +\infty$

So our original series is divergent!

Cond Conv

(g) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$. First,

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \log n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \log n}$ integral test $\int_2^{\infty} \frac{1}{x \log x} dx = \log \log x \Big|_2^{\infty} = +\infty$

So by integral test, the series is <u>not</u> abs convergent.

It is (conditionally) convergent though, by Alternating Series test

since $b_n = \frac{1}{n \log n} \searrow 0$.

abs conv

(h) $\sum_{n=1}^{\infty} \left(\frac{5n}{6n+1} \right)^n$

Using Root test $\lim_{n\to\infty} \sqrt[n]{\left(\frac{5n}{6n+1}\right)^n} = \lim_{n\to\infty} \frac{5n}{6n+1} = \frac{5}{6} < 1$

$\Rightarrow$ series is (abs). convergent!

3. Calculate

$$\lim_{n\to\infty} \left(\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \dots + (-1)^n \frac{3}{2^n} \right)$$

We notice ~~these~~ are the partial sums of a geometric series with $a = \frac{3}{2}$, $r = -\frac{1}{2}$

So the limit will equal the sum of the series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

We can verify this by using the identity $1 + r + \dots + r^{n-1} = \frac{1-r^n}{1-r}$

Indeed: $\frac{3}{2} - \frac{3}{4} + \dots + (-1)^n \frac{3}{2^n} = \frac{3}{2}\left(1 - \frac{1}{2} + \frac{1}{4} - \dots + \left(-\frac{1}{2}\right)^{n-1}\right)$

$= \frac{3}{2} \frac{1 - \left(-\frac{1}{2}\right)^n}{1 - \left(-\frac{1}{2}\right)} = \frac{3}{2} \frac{1 - \left(-\frac{1}{2}\right)^n}{\frac{3}{2}} = 1 - \left(-\frac{1}{2}\right)^n$.

As $n \to \infty$, $\lim_{n\to\infty} \left(\frac{3}{2} - \frac{3}{4} + \dots + (-1)^n \frac{3}{2^n} \right) = \lim_{n\to\infty} \left[1 - \left(-\frac{1}{2}\right)^n \right] = \boxed{1}$

4. (a) State the Cauchy criterium (property) for a series $\sum_n a_n$ and explain why it is equivalent with the convergence of the series.

$\sum_{n\geq 1} a_n$ satisfies the Cauchy criterium if $\forall \varepsilon>0, \exists N=N(\varepsilon)$ such that $\left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon \ \forall m>n>N$

This is the same with saying that the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$ is a Cauchy sequence, which is then equivalent with (S_n) convergent sequence in $\mathbb{R}$!!

[This is fundamental property in $\mathbb{R}$, derived from the Completeness Axiom]

(b) Using (a), prove that if $\sum_n |a_n|$ is convergent, then $\sum_n a_n$ is convergent. [Hint: Use Triangle inequality]

If $\sum_n |a_n|$ convergent $\Rightarrow \sum |a_n|$ satisfies the Cauchy criterium, that is, $\forall \varepsilon>0, \exists N_\varepsilon$: $\sum_{k=n+1}^{m} |a_k| < \varepsilon \ \forall m>n>N_\varepsilon$.

But by triangle inequality:

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k| < \varepsilon, \ \forall m>n>N_\varepsilon,$$

which simply means $\sum a_n$ also has Cauchy property, hence by (a) it is convergent

(c) Using (a), prove that if a series $\sum_n a_n$ is convergent, then $\lim_n a_n = 0$.

If $\sum_n a_n$ convergent, then $\forall \varepsilon>0, \exists N_\varepsilon$ such that

$$\left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon, \ \forall m>n>N_\varepsilon. \text{ (Cauchy property).}$$

If we take $m=n+1 \Rightarrow |a_m| < \varepsilon, \ \forall m>N_\varepsilon$.

Hence $\forall \varepsilon>0, \exists N_\varepsilon : |a_m| < \varepsilon, \forall m>N_\varepsilon \Rightarrow$ this simply means $\lim_{n\to\infty} a_n = 0$ (by def. of the limit!)

5. For each of the following series, determine the radius of convergence and the interval of convergence:

(a) $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1}$

By Ratio test : $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}}\right| = x^2 \frac{2n+1}{2n+3} \longrightarrow x^2$ as $n \to \infty$

So if $|x^2| < 1 \Rightarrow$ abs convergent
if $|x^2| > 1 \Rightarrow$ divergent
$\Rightarrow$ Radius of convergence $R = 1$ & interval of convergence include all $x \in (-1, 1)$.

For $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{1}{2n+1} = +\infty$

For $x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{2n+1}$ = cond convergent (by Alt. Series Test)

So, in conclusion, interval of convergence is $[-1, 1)$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n$

By Ratio test again

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1} x^{n+1}}{(n+1)!}}{\frac{(-1)^n x^n}{n!}}\right| = x \frac{n!}{(n+1)!} = x \cdot \frac{1}{n+1} \xrightarrow[n \to \infty]{} 0 < 1$

So series is (abs) convergent for all $x \in \mathbb{R} \Rightarrow R = +\infty$ and interval of convergence is $(-\infty, \infty)$.

(c) $\sum_{n=1}^{\infty} n(x-3)^n$

By Ratio Test again:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x-3)^{n+1}}{n(x-3)^n}\right| = \frac{n+1}{n}|x-3| \to (x-3)$ as $n \to \infty$

So if $|x-3| < 1 \Rightarrow$ abs conv
$|x-3| > 1 \Rightarrow$ divergence
$\Rightarrow$ Interval of convergence includes $(2, 4)$ so Radius of convergence $\boxed{R = 1}$

At $x = 2 \Rightarrow \sum_{n=1}^{\infty} n(2-3)^n = \sum_{n=1}^{\infty} (-1)^n n$ = divergent $((-1)^n n \not\to 0)$

At $x = 4 \Rightarrow \sum_{n=1}^{\infty} n(4-3)^n = \sum_{n=1}^{\infty} n$ = divergent.

So, in conclusion, Interval of Convergence is $(2, 4)$.