Practice for Exam 2 - Math 3410

Intro to Analysis - Spring 2016

1. Determine whether each of the following series is convergent (absolutely or conditionally) or divergent. Indicate which test(s) you are using, and show all your work.

   (a) $\displaystyle\sum_{n=0}^{\infty} ne^{-n}$ (b) $\displaystyle\sum_{n=1}^{\infty}(\sqrt{n^2+n}-n)$

   (c) $\displaystyle\sum_{n=1}^{\infty}(\sqrt{n^2+1}-n)$ (d) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$

   (e) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2+1}$ (f) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$ (g) $\displaystyle\sum_{n=1}^{\infty}\log\frac{n+1}{n}$

2. Determine whether the series below is convergent or not
$$\frac{3}{2}-\frac{3}{4}+\frac{3}{8}-\frac{3}{16}+\frac{3}{32}+\ldots$$
   If convergent, find its sum.

3. Prove that $\lim s_n = 3$ where the sequence (s_n) is defined by
$$s_n = \frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\cdots+\frac{2n-1}{2^n}, \quad n=1,2,3,\ldots.$$
   (Hint: see Problem 14.13(c)).

4. Show that $\displaystyle\sum_{n=1}^{\infty}\frac{\log n}{n^p}$ diverges when $0<p<1$ and converges when $p>1$. What happens when $p=1$?

5. Prove that the monotonicity assumption in the Alternating Series Test is essential. That is, show that there exists an alternating series $\sum(-1)^n a_n$, for which its sequence of positive terms (a_n) satisfies $\lim a_n = 0$, but which is a divergent series.

6. (a) Show that if $\sum a_n$ converges and $\sum b_n$ converges absolutely, then $\sum a_n b_n$ converges absolutely.
   (b) Construct two convergent series $\sum a_n$ and $\sum b_n$ for which the series $\sum a_n b_n$ diverges.
   (c) Suppose $\sum a_n$ is a convergent series of positive terms. Show that the series $\sum \sqrt{a_n}/n$ is also convergent.

7. Suppose an alternating series $\sum(-1)^{n+1}a_n$, $a_n>0$ and (a_n) monotone, converges to a sum s. Show that the remainder $R_n = s - s_n$ satisfies $|R_n| \le a_{n+1}$ where s_n is the n^{th} partial sum of the series.

8. Suppose a series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent. Show that for every $\varepsilon>0$, there exists N such that the remainder
$$\Big|\sum_{n=N+1}^{\infty} a_n\Big| \le \varepsilon$$

9. Define
$$a_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\,.$$
   Prove that a_n is a decreasing sequence and that $0<a_n<1$, hence the limit exists and is a number between 0 and 1. (Hint: See Problem 16.9)

10. *[18 pts]* Determine the radius of convergence and the interval of convergence for each of the power series

    (a) $\displaystyle\sum_{n=1}^{\infty} n^2x^n$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^{2n}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{(4x-3)^n}{n^2 2^n}$