SOLUTIONS PRACTICE EXAM 2

1. *[3 pts each]* Determine whether each of the following series is convergent (absolutely or conditionally) or divergent. Indicate which test(s) you are using, and show all your work.

(a) $\sum_{n=0}^{\infty} ne^{-n}$

RATIO TEST : $\frac{a_{n+1}}{a_n} = \frac{(n+1)e^{-(n+1)}}{ne^{-n}} = \frac{n+1}{n}\cdot\frac{1}{e} \to \frac{1}{e} < 1$ as $n \to \infty$

$\Rightarrow$ (absolutely) convergent series.

(b) $\sum_{n=1}^{\infty}(\sqrt{n^2+n}-n)$

$a_n = \sqrt{n^2+n} - n = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{2} \neq 0$

so series diverges (by the Test for divergence!)

(c) $\sum_{n=1}^{\infty}(\sqrt{n^2+1}-n)$

$a_n = \sqrt{n^2+1} - n = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$; $b_n = \frac{1}{n}$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$ (see (b)) $\Longrightarrow$ by Limit Comparison test, $\sum a_n$ is divergent (same as $\sum b_n = \sum \frac{1}{n} = +\infty$)

(d) $\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$

By Ratio Test, $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{1\cdot 3 \cdots (2n+1)} \cdot \frac{1\cdot 3 \cdots (2n-1)}{n!} = \frac{n+1}{2n+1} \longrightarrow \frac{1}{2} < 1$

the series in (d) converges (absolutely)

(e) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2+1}$

By Comparison Test $\left|\frac{\sin n}{n^2+1}\right| \le \frac{1}{n^2+1}$ and $\sum \frac{1}{n^2+1} < +\infty \Rightarrow$ Series in (e) is abs. convergent!

(f) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$, $a_n = \frac{\cos n\pi}{n}$, $a_1 = \frac{-1}{1}$, $a_2 = \frac{1}{2}$, $a_3 = \frac{-1}{3}$, ... so in general $a_n = \frac{(-1)^n}{n} \Rightarrow \sum \frac{(-1)^n}{n}$ is conv. conditionally (alt series test and $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n} = +\infty$)

(g) $\sum_{n=1}^{\infty} \log\frac{n+1}{n}$

$a_n = \log\frac{n+1}{n} = \log(n+1) - \log n$ so telescopic series

$\sum_{n=1}^{\infty} \log\frac{n+1}{n} = \log 2 - \log 1 + \log 3 - \log 2 + \ldots \Rightarrow$ partial sum $S_N = \sum_{1}^{N} \log\frac{n+1}{n} = \log N+1$

$\Rightarrow S_N \to +\infty \Rightarrow$ divergent series.

2. *[5 pts]* Determine whether the series below is convergent or not

$$\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \frac{3}{16} + \frac{3}{32} + \ldots$$

If convergent, find its sum.

We easily recognize this to be geometric series, $\sum_{n=0}^{\infty} ar^n$, where $a = \frac{3}{2}$, $r = -\frac{1}{2}$. Since $|r| = |-\frac{1}{2}| < 1$, the series converges (absolutely) and its sum equals $\frac{a}{1-r}$:

$$\sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{1}{2}\right)^n = \frac{\frac{3}{2}}{1-(-\frac{1}{2})} = \frac{\frac{3}{2}}{\frac{3}{2}} = 1.$$

3. *[8 pts]* Prove that $\lim s_n = 3$ where the sequence (s_n) is defined by

$$s_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \cdots + \frac{2n-1}{2^n}, \quad n = 1, 2, 3, \ldots.$$

(Hint: see Problem 14.13(c)).

<u>Note</u>: (s_n) is the sequence of partial sums for the series $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$, which is convergent, by Ratio Test. We also know $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent (geom series)

So $\sum_{n=1}^{\infty} \frac{2n-1}{2^n} = \sum_{n=1}^{\infty} \frac{2(n-1)+1}{2^n} = \sum_{n=1}^{\infty} \frac{2(n-1)}{2^n} + \sum_{n=1}^{\infty} \frac{1}{2^n}$

$= 2\sum_{n=1}^{\infty} \frac{n-1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{2^n}$

$= 2\underbrace{\sum_{n=1}^{\infty}\left(\frac{n}{2^{n-1}} - \frac{n+1}{2^n}\right)}_{\text{telescopic series whose sum} = 1} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^n}}_{=1} = 2\cdot 1 + 1 = 3.$

If $p=1$ then $\sum_{n=1}^{\infty} \frac{\log n}{n}$ is divergent by the integral test, since $\int_1^{\infty} \frac{\log x}{x}dx = \log(\log x)\Big|_1^{\infty} = +\infty$

4. *[6 pts]* Show that $\sum_{n=1}^{\infty} \frac{\log n}{n^p}$ diverges when $0 < p < 1$ and converges when $p > 1$. What happens when $p = 1$?

If $0<p<1$, then using the comparison test with $\sum_{n=3}^{\infty} \frac{1}{n^p} = +\infty$ we get $\sum_{n=3}^{\infty} \frac{\log n}{n^p} \ge \sum_{n=3}^{\infty} \frac{1}{n^p} = +\infty$ so $\sum \frac{\log n}{n^p}$ = divergent.

If $p>1$, let $\varepsilon > 0$ such that $p - \varepsilon > 1$. We know there exists N such that $\log n \le n^{\varepsilon}$ for all $n \ge N$. So $\sum_{n=N}^{\infty} \frac{\log n}{n^p} \le \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^p} = \sum_{n=N}^{\infty} \frac{1}{n^{p-\varepsilon}} < \infty$

so $\sum_{n=1}^{\infty} \frac{\log n}{n^p}$ is convergent! (by Comparison test again).

5. *[8 pts]* Prove that the monotonicity assumption in the Alternating Series Test is essential. That is, show that there exists an alternating series $\sum(-1)^n a_n$, for which its sequence of positive terms (a_n) satisfies $\lim a_n = 0$, but which is a divergent series.

Example: Let $a_{2k-2} = \frac{1}{k}$ and $a_{2k} = \frac{1}{2k}$, $k \geq 1$

$$\sum_{n=1}^{\infty} a_n = \underbrace{1 - \frac{1}{2}} + \underbrace{\frac{1}{2} - \frac{1}{4}} + \underbrace{\frac{1}{3} - \frac{1}{6}} + \underbrace{\frac{1}{4} - \frac{1}{8}} + \underbrace{\frac{1}{5} - \frac{1}{10}} + \dots$$

Clearly $a_n \to 0$ but not monotonically.

The series diverges (to $+\infty$) since $S_{2N} = \sum_{n=1}^{2N-1} a_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \dots + \frac{1}{2N}$

and $S_{2N} > S_{2N-1}$. $\quad = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N}\right) \to +\infty$ as $N \to \infty$

6. *[12 pts]* (a) Show that if $\sum a_n$ converges and $\sum b_n$ converges absolutely, then $\sum a_n b_n$ converges absolutely.

Since $\sum a_n$ converges $\Rightarrow (a_n)$ converges to $0 \Rightarrow (a_n)$ is bounded.

Let $M = \sup\{|a_n|\}$. So $|a_n| \leq M$ for all n.

Then $\sum_n |a_n b_n| \leq \sum_n M|b_n| < +\infty$ $\quad (\sum |b_n| < +\infty)$

$\Rightarrow \sum a_n b_n$ is absolutely convergent.

(b) Construct two convergent series $\sum a_n$ and $\sum b_n$ for which the series $\sum a_n b_n$ diverges.

$\sum_n a_n = \sum_n \frac{(-1)^n}{\sqrt{n}} = \sum b_n$ are convergent (cond) series by the Alt. Series Test but

$$\sum a_n b_n = \sum \frac{(-1)^n}{\sqrt{n}} \cdot \frac{(-1)^n}{\sqrt{n}} = \sum \frac{1}{n} = +\infty \text{ divergent.}$$

(c) Suppose $\sum a_n$ is a convergent series of positive terms. Show that the series $\sum \sqrt{a_n}/n$ is also convergent.

Since $\frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left[\left(\sqrt{a_n}\right)^2 + \left(\frac{1}{n}\right)^2\right]$ $\quad \left(\text{using } ab \leq \frac{1}{2}(a^2+b^2)\right)$

we can use Comparison test to conclude

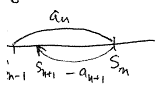
$$\sum \frac{\sqrt{a_n}}{n} \leq \frac{1}{2} \sum \left[\left(\sqrt{a_n}\right)^2 + \frac{1}{n^2}\right]$$

$$= \frac{1}{2}\left(\sum a_n + \sum \frac{1}{n^2}\right) < +\infty.$$

7. *[8 pts]* Suppose an alternating series $\sum(-1)^{n+1} a_n$, $a_n > 0$ and (a_n) monotone, converges to a sum s. Show that the remainder $R_n = s - s_n$ satisfies $|R_n| \leq a_{n+1}$ where s_n is the n^{th} partial sum of the series.

Let $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$, $\quad a_{n+1} \leq a_n$.

If n=odd

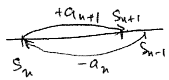


We have that for n=odd: $S_{n+1} \geq S_{n-1}$ since $S_{n+1} - S_{n-1} = a_n - a_{n+1} \geq 0$

for n=even: $S_{n+1} \leq S_{n-1}$ since $S_{n+1} - S_{n-1} = -a_n + a_{n+1} \leq 0$

So $S_2 \leq S_4 \leq \dots \leq S_{2k} \leq \dots$ and $\dots \leq S_{2k+1} \leq S_{2k-1} \leq \dots \leq S_3 \leq S_1$

If n = even



Since (S_n) is convergent to S $\Rightarrow (S_{2k})$ and (S_{2k+1}) are both convergent to S

So $S_{2k} \leq S \leq S_{2k+1}$ for all k.

If n=odd: $S_{n+1} \leq S \leq S_n \Rightarrow S_n - S \leq S_n - S_{n+1} = a_{n+1} \Rightarrow |R_n| \leq a_{n+1}$

If n = even $S_n \leq S \leq S_{n+1} \Rightarrow S - S_n \leq S_{n+1} - S_n = a_{n+1} \Rightarrow |R_n| \leq a_{n+1}$

8. *[8 pts]* Suppose a series $\sum_{n=1}^{\infty} a_n$ is convergent. Show that for every $\varepsilon > 0$, there exists N such that the remainder

$$\left|\sum_{n=N+1}^{\infty} a_n\right| \leq \varepsilon$$

Let $S_n = \sum_{k=1}^{n} a_k$ the sequence of partial sums. Since $\sum a_n$ is convergent (to S) it means $S_n \to S$ as $n \to \infty$. So for $\varepsilon > 0$, there exists N such that $|S_n - S| < \varepsilon$ for $n \geq N$. In particular $|S_N - S| < \varepsilon$ or $\left|\sum_{n=N+1}^{\infty} a_n\right| < \varepsilon$.

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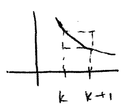
9. *[6 pts]* Define

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n.$$

Prove that a_n is a decreasing sequence and that $0 < a_n < 1$, hence the limit exists and is a number between 0 and 1. (Hint: See Problem 16.9)

Since $\ln x = \int_1^x \frac{1}{x}\,dx$ we have



$$a_{n+1} - a_n = \left(1 + \frac{1}{2} + \dots + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n\right)$$

$\Rightarrow (a_n)$ decreasing. $\quad = \frac{1}{n+1} - \int_1^{n+1} \frac{1}{x}\,dx + \int_1^{n} \frac{1}{x}\,dx = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x}\,dx < 0.$

Also $\ln n = \int_1^n \frac{1}{x}\,dx = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x}\,dx < \sum_{k=1}^{n-1} \frac{1}{k} = 1 + \frac{1}{2} + \dots + \frac{1}{n-1} < 1 + \frac{1}{2} + \dots + \frac{1}{n}$

so $\boxed{a_n > 0}$. Since a_n is decreasing and bounded below $\Rightarrow$ it is convergent to a number in $[0,1]$.

Similarly $\ln n > \sum_{k=1}^{n-1} \frac{1}{k+1} = \frac{1}{2} + \dots + \frac{1}{n} \Rightarrow \boxed{a_n < 1}$.

10. *[18 pts]* Determine the radius of convergence and the interval of convergence for each of the power series

$$(a) \sum_{n=1}^{\infty} n^2 x^n$$

$c_n = n^2$

We have $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$ so $R = \frac{1}{\lim\left|\frac{c_{n+1}}{c_n}\right|} = 1$

To check the end points

$x = 1 \Rightarrow \sum_{n=1}^{\infty} n^2 = +\infty$ (test for divergence)

$x = -1 \Rightarrow \sum (-1)^n n^2 =$ divergent (same test)

$\Rightarrow$ interval of convergence is $\boxed{(-1,1)}$.

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^{2n} = \sum_{n=1}^{\infty} c_n (x^2)^n$$

power series for x^2.

$c_n = \frac{(-1)^n}{n}$

Radius of conv (same as above) is $R = \frac{1}{\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|} = 1$

So series is convergent if $x^2 \in (-1,1)$.

For $x^2 = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the alt series $\Rightarrow$ convergent

$\Rightarrow$ ~~Series~~ Series is convergent when $x^2 \in (-1,1] \Leftrightarrow x^2 \in [0,1] \Leftrightarrow \boxed{x \in [-1,1]}$

$$(c) \sum_{n=1}^{\infty} \frac{(4x-3)^n}{n^2 2^n}$$

We can rewrite $\sum_{n=1}^{\infty} \frac{4^n (x-\frac{3}{4})^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(x - \frac{3}{4}\right)^n$ power series centered at $\frac{3}{4}$

$c_n = \frac{2^n}{n^2} \Rightarrow \left|\frac{c_{n+1}}{c_n}\right| = \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} \to 2 \Rightarrow R = \frac{1}{\lim\left|\frac{c_{n+1}}{c_n}\right|} = \frac{1}{2}$

So series is convergent for all x such that

$$\left|x - \frac{3}{4}\right| < \frac{1}{2} \text{ or } \frac{1}{4} < x < \frac{5}{4}$$

To check end points

$x = \frac{1}{4} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} =$ convergent (abs)

$x = \frac{5}{4} \Rightarrow \sum \frac{1}{n^2} =$ convergent

$\Rightarrow$ interval of convergence is $\boxed{\left[\frac{1}{4}, \frac{5}{4}\right]}$