Practice for Final Exam - Math 3410

Intro to Analysis

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In addition to the previous exams, sample exams, homework problems assigned throughout the semester and examples given in the textbook and in the lectures, here is a list of textbook problems you can use as a review of the topics covered on the final.

Sect 1. Exercises 1.3, 1.5, 1.8
Sect 2. Exercises 2.4
Sect 3. Exercises 3.5, 3.8
Sect 4. Exercises 4.1, 4.2, 4.7, 4.8, 4.10, 4.15, 4.16
Sect 5. Exercises 5.3
Sect 7. Exercises 7.2, 7.3, 7.4, 7.5
Sect 8. Exercises 8.5, 8.6, 8.8, 8.9, 8.10
Sect 9. Exercises 9.4, 9.5, 9.6, 9.9, 9.12, 9.13, 9.14
Sect 10. Exercises 10.4, 10.7, 10.9, 10.11
Sect 11. Exercises 11.2, 11.4, 11.5, 11.9, 11.10
Sect 12. Exercises 12.2, 12.4, 12.8, 12.9, 12.10
Sect 14. Exercises 14.1, 14.4, 14.5, 14.8,14.10, 14.12
Sect 15. Exercises 15.3, 15.4, 15.5, 15.6
Sect 16. Exercises 16.3, 16.4, 16.9
Sect 23. Exercises 23.1, 23.2, 23.5, 23.6
Sect 26. Exercises 26.3, 26.5, 26.7
Sect 31. Exercises 31.1, 31.2, 31.5

1. Determine whether the following statements are TRUE or FALSE. **Justify your answers**.

   (a) (s_n) is convergent (to $s \in \mathbb{R}$) if and only if $\limsup s_n = \liminf s_n = s$.

   (b) If (s_n) is unbounded then $\lim \frac{1}{s_n} = 0$.

   (c) If s_n converges to s then $|s_n|$ converges to $|s|$.

   (d) If A is bounded below, and $\alpha = \inf A$, then there exists a non-increasing sequence $(s_n) \subset A$ such that s_n converges to α.

   (e) The series $\sum \dfrac{s_n}{n}$ is convergent if and only if $\lim s_n = 0$.

   (f) For any $x \in \mathbb{R}$, $\lim \dfrac{x^n}{n!} = 0$.

2. (a) Prove by induction that $(1+a)^n > 1 + na$ for all $n \in \mathbb{N}$, $a > 0$.

   (b) Use part (a) to show that $\lim b^n = +\infty$ for $b > 1$.

   (c) Use part (b) to show that $\lim r^n = 0$ for $|r| < 1$.

3. (a) Prove that for every $\epsilon > 0$, there exists a natural number N such that $0 < 1/N < \epsilon$.

   (b) Use the ϵ-definition of the limit to prove that if (s_n) and (t_n) are two sequences of real numbers such that s_n converges to s and $|s_n - t_n| < 1/n$ for all n, then t_n also converges to s.

4. Let (s_n) be a sequence of real numbers.
   (i) If $s_n < b$ for *all but finitely many* n, prove that

$$\limsup s_n \leq b.$$

   (ii) If $s_n < a$ for *infinitely many* n, prove that

$$\liminf s_n \leq a.$$

   (iii) If (s_n) is nondecreasing, prove that

$$\lim s_n = \sup\{s_n, n \in \mathbb{N}\}$$

5. Given the sequence defined recursively by

$$s_{n+1} = 2 + \frac{1}{s_n}, \quad s_1 = 1$$

   (a) Show $s_{n+1} > s_n$ if and only if $s_n < s_{n-1}$ for all $n > 1$.

   (b) Using (a), prove that the subsequences (s_{2k}) and (s_{2k+1}) are monotone and convergent to the same limit.

   (c) Show that s_n converges and find its limit.

6. Determine whether the following series converges (absolutely, conditionally) or diverges.

   (a) $\displaystyle\sum_n \sin\frac{1}{n}$

   (b) $\displaystyle\sum_n (-1)^n \frac{\sqrt{n+1} - \sqrt{n}}{n}$

   (c) $\displaystyle\sum_n (-1)^n \frac{n}{n^2+1}$

   (d) $\displaystyle\sum_n n^p r^n, \quad p, r \in \mathbb{R}, |r| < 1$

7. (a) Find the Taylor series of the function $f(x) = \sin^2 x$, centered at $x = 0$. [You may use the trig identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$] and prove that this Taylor series of f converges to $f(x)$ for all real numbers x.

   (b) Show that the Taylor series for $g(x) = \sqrt{1+x}$ centered at $x = 0$ is

$$1 + \frac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot \ldots (2n-3)}{2^n n!} x^n$$

   (c) Find the radius of convergence and the interval of convergence of this series in (b) .

   (d) Show that $g(x)$ equals the sum of its Taylor series in (b) , wherever convergent.