SOLUTIONS TO PRACTICE FINAL

MATH 3410 - SPRING 2015

#1 (a) $\boxed{\text{TRUE}}$ $\lim s_n = s \in \mathbb{R}$ means $\forall \varepsilon > 0, \exists N = N(\varepsilon)$ with $s-\varepsilon < s_n < s+\varepsilon$ $\forall n > N$

$\liminf s_n = \limsup s_n = s \in \mathbb{R}$ means $\forall \varepsilon > 0$, $\exists N_1$ & N_2

$s - \varepsilon < s_n$ $\forall n > N_1$ and $s_n < s+\varepsilon$ $\forall n > N_2$

so $s - \varepsilon < s_n < s + \varepsilon$ $\forall n > \max\{N_1, N_2\}$.

(b) $\boxed{\text{FALSE}}$ Counterexample : $s_n = \begin{cases} 1, & n = \text{odd} \\ n, & n = \text{even} \end{cases}$

is an unbounded sequence for which $\lim \frac{1}{s_n} \neq 0$

[in fact $\frac{1}{s_n} = \begin{cases} 1, & n = \text{odd} \\ \frac{1}{n}, & n = \text{even} \end{cases}$ is divergent!]

(c) $\boxed{\text{TRUE}}$: By the Triangle Inequality :

$$\big||s_n| - |s|\big| \leq |s_n - s|$$

Since $\lim_{n\to\infty} s_n = s \Rightarrow \forall \varepsilon > 0, \exists N_\varepsilon$: $|s_n - s| < \varepsilon$ $\forall n > N_\varepsilon$

$\Rightarrow \big||s_n| - |s|\big| \leq |s_n - s| < \varepsilon$, $\forall n > N_\varepsilon \Rightarrow \lim_{n\to\infty} |s_n| = |s|$.

(d) $\boxed{\text{TRUE}}$: A = bounded below and $\alpha = \inf A \in \mathbb{R}$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists$ $s \in A$ with $s < \alpha + \varepsilon$.

Iteratively, we pick $\varepsilon = \frac{1}{n}$, $n = 1, 2, \ldots$, and choose $s_n \in A$ ~~construct~~ with $\alpha \leq s_n < \alpha + \frac{1}{n}$ and $s_n \leq s_{n-1}$. In this way, $(s_n) \subset A$ is a non-increasing sequence which converges to α.

(e) $\boxed{\text{FALSE}}$ $\sum \frac{s_n}{n}$ is convergent is NOT equivalent to $\lim_{n\to\infty} s_n =$

In fact, neither statement implies the other.

Ex: $s_n = (-1)^n$, $\sum \frac{(-1)^n}{n}$ is convergent but $(-1)^n$ does not converge to 0

or $s_n = \begin{cases} 1, & n = \text{perfect square } k^2 \\ 0, & \text{otherwise} \end{cases}$ $\sum_{n\geq 1} \frac{s_n}{n} = \sum_k \frac{1}{k^2} < +\infty$

while s_n does not converge to zero.

Also, one can construct a sequence convergent to zero while $\sum \frac{s_n}{n}$ divergent.

Ex: $s_n = \frac{1}{\log n} \to 0$ as $n \to \infty$ but $\sum \frac{\frac{1}{\log n}}{n} = \sum \frac{1}{n \log n} = +\infty$

diverges (by integral test).

(f) $\boxed{\text{TRUE}}$ Fix $x \in \mathbb{R}$. Consider the infinite series

$\sum_{n=1}^{\infty} \frac{x^n}{n!}$. By Ratio Test, $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$

so $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges $\Rightarrow$ $\cancel{\lim_{n\to\infty}}$ $\frac{x^n}{n!}$ converges to 0!

#2 (a) We prove by induction $P_n: (1+a)^n \geq 1 + na$, $n \in \mathbb{N}, a \geq$

Indeed we first show $P_1: (1+a)^1 = 1 + a = 1 + 1.a \Rightarrow P_1$ true.

Induction step. Assume P_n has been already proved to be true, then

$$P_{n+1}: (1+a)^{n+1} = (1+a)^n . (1+a) \geq (1+na)(1+a) =$$

$$= 1 + (n+1)a + na^2 \geq 1 + (n+1)a \Rightarrow$$

$P_{n+1} = $ true!

(b) For $b>1$, let $b=1+a$, $a>0$

then $b^n = (1+a)^n \geq 1+na = 1+n(b-1) \to \infty$

so $\lim_{n\to\infty} b^n = +\infty$ (for $b>1$)

(c) Indeed, if $|r|<1 \Rightarrow \lim_{n\to\infty} |r|^n = \cancel{\lim_{n\to\infty} |1+a|^n}$

$= \dfrac{1}{\lim_{n\to\infty} \left(\frac{1}{|r|}\right)^n} = 0$ as $n\to\infty$.

#3 (a) Using Archimedean property $\left[\begin{array}{l}\forall a,b>0,\ \exists\, n\in\mathbb{N} \text{ such} \\ \text{that } na>b\end{array}\right]$

with $a=\varepsilon, b=1 \Rightarrow \exists N:\ N\varepsilon>1$ or $\frac{1}{N}<\varepsilon$.

(b) If $(s_n), (t_n)$ are sequences with $s_n \to s$

and $|s_n - t_n| < 1/n$, $\forall n$, then $t_n \to s$.

Indeed $\forall \varepsilon>0$, $\exists N_1$ such that $|s_n - s| < \varepsilon/2$ $\forall n>N_1$,

and $\exists N_2$ such that $\frac{1}{N_2} < \varepsilon/2 \Rightarrow |s_n - t_n| < \frac{1}{n} < \frac{1}{N_2} < \varepsilon/2$

$\forall n > N_2$

$\Rightarrow \forall n > N = \max\{N_1, N_2\}$,

$$|s - t_n| = |(s-s_n)+(s_n-t_n)| \leq |s-s_n| + |s_n - t_n| <$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (\forall n>N)$$

$\Rightarrow t_n \to s$ as $n\to\infty$!

#4 (i) Assume $s_n < b$ for all but finitely many n's.

then $\exists N_0 \in \mathbb{N} : s_n < b \ \forall n > N_0$.

$\Rightarrow \sup\{s_n \mid n \geq N_0\} \leq b$. But $\sup\{s_n \mid n > N\}$ is a non-increasing sequence in N, which also converges to lim sup

so $\limsup s_n = \lim_{N\to\infty}\left(\sup\{s_n \mid n \geq N\}\right) \leq b$.

(ii) If $s_n < a$ for infinitely many n's

$\Rightarrow \inf\{s_n \mid n > N\} < a$, $\forall N \in \mathbb{N}$

$\Rightarrow \lim_{N\to\infty}\left(\inf\{s_n \mid n > N\}\right) \leq a \Rightarrow \liminf s_n \leq a$.

(iii) If (s_n) is nondecreasing (so $s_n \leq s_{n+1}$, $\forall n$)

then $\lim s_n = s = \sup\{s_n \mid n \in \mathbb{N}\}$ (given $s =$ finite)

Indeed, if $\lim s_n = s \in \mathbb{R}$ then $s = \sup\{s_n \mid n \in \mathbb{N}\}$

because $s_n \leq s_{n+1} \leq s$, $\forall n$.

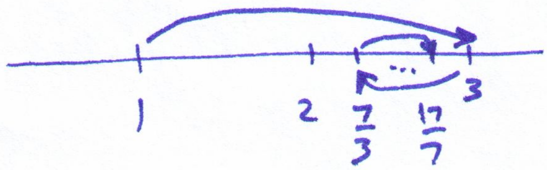
(otherwise, if $s_n > s$ for some s, then $\lim s_n > s$ contradiction!)

If $\lim s_n = +\infty$, then $\sup\{s_n \mid n \in \mathbb{N}\} = +\infty$

(otherwise s_n were bounded and $\lim s_n \in \mathbb{R}$).

(#5) $s_{n+1} = 2 + \frac{1}{s_n}$, $s_1 = 1$

$s_2 = 2 + \frac{1}{1} = 3 \Rightarrow s_3 = 2 + \frac{1}{3} = \frac{7}{3} \Rightarrow s_4 = 2 + \frac{3}{7} = \frac{17}{7}, \ldots$ etc



(a) $s_{n+1} > s_n$ if and only if $s_n < s_{n-1}$

Indeed, $s_{n+1} - s_n = 2 + \frac{1}{s_n} - \left(2 + \frac{1}{s_{n-1}}\right) = \frac{1}{s_n} - \frac{1}{s_{n-1}}$

$$= \frac{s_{n-1} - s_n}{s_n s_{n-1}}$$

So $s_{n+1} - s_n > 0 \iff s_{n-1} - s_n > 0$ (as long as we establish $s_n > 0$)

[The fact $s_n > 0$ follows by induction. In fact we can establish $s_n > 2$, $\forall n > 1$ by induction

$s_2 = 3 > 2$; Assume $s_n > 2 \Rightarrow s_{n+1} = 2 + \frac{1}{s_n} > 2$.]

(b) Since $s_2 = 3 > 1 = s_1 \Rightarrow s_3 < s_2$.

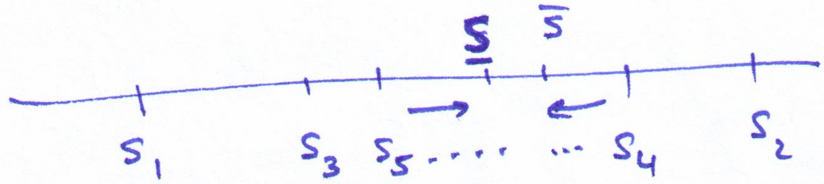
$\Rightarrow s_{n+1} > s_n$ for $n =$ odd and $s_{n+1} < s_n$ for $n =$ even

Moreover, $s_{n+1} - s_n = \frac{s_{n-1} - s_n}{s_n s_{n-1}} \Rightarrow |s_{n+1} - s_n| \le \frac{|s_{n-1} - s_n|}{4}$

For $n =$ odd:

$\Rightarrow s_{n+2} - s_n = (s_{n+2} - s_{n+1}) + (s_{n+1} - s_n) > (s_{n+1} - s_n) - \frac{(s_{n+1} - s_n)}{4} > 0$

$\Rightarrow s_{n+2} > s_n \Rightarrow (s_{2k+1})$ is increasing and similarly (s_{2k}) decreasing



Denote $\lim_{k \to \infty} s_{2k+1} = \underline{s}$, and $\lim s_{2k} = \bar{s}$

From $|s_{n+1} - s_n| \le \dfrac{|s_{n-1} - s_n|}{4} \Rightarrow |\underline{s} - \bar{s}| \le \dfrac{|\bar{s} - \underline{s}|}{4}$

$\Rightarrow \underline{s} = \bar{s}$ so $\lim_{n \to \infty} s_n$ exists and equals $\bar{s} = \underline{s}$.

(c) If $\lim_{n \to \infty} s_n = s \Rightarrow$ taking the limit in the recursion relation $s_{n+1} = 2 + \dfrac{1}{s_n} \Rightarrow$

$$s = 2 + \frac{1}{s} \Rightarrow s^2 - 2s - 1 = 0 \Rightarrow$$

$$\Rightarrow s = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2}.$$

Since $s_n > 2$, $\forall n \Rightarrow s = \lim_n s_n \geq 2 \Rightarrow \boxed{s = 1 + \sqrt{2}}$.

Note: This sequence's behavior resembles the behavior of partial sums of an alternating series. Indeed, one can make a similar argument based on the infinite series $\sum_{n=1}^{\infty} (s_{n+1} - s_n)$.

#6 (a) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$

By limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$ we have

$$\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$ (e.g via l'Hôpital rule)

so $\sum_{n=1}^{\infty} \sin \frac{1}{n} = +\infty$ (divergent!)

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1} - \sqrt{n}}{n}$ is an alternating series

with $b_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{(n+1) - n}{n(\sqrt{n+1} + \sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \to 0$

in a decreasing fashion, so by Alt Series Test, the series convergent. In fact, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$ is also convergent (compare it with $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} < +\infty$)

so the original series converges absolutely!

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ is conditionally convergent!

[again, it is alternating series with $b_n = \frac{n}{n^2+1} \searrow 0$

and, in this case $\sum_{n=1}^{\infty} b_n = \sum \frac{n}{n^2+1} = +\infty$, either

by limit comparison with $\sum \frac{n}{n^2} = \sum \frac{1}{n} = +\infty$ or by

the integral test: $\int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2} \ln(x^2+1)\Big|_1^{\infty} = +\infty$.

(d) $\sum_{n=1}^{\infty} \underbrace{n^p r^n}_{a_n}$, $p, r \in \mathbb{R}$, $|r|<1$

(this can be thought as a power series in r, for each fixed p)

By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^p r^{n+1}}{n^p \cdot r^n}\right| = |r| \cdot \left(\frac{n+1}{n}\right)^p = |r|\left(1+\frac{1}{n}\right)^p$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |r| < 1 \Rightarrow \text{absolutely convergent!!!!}$$

#7 $f(x) = \sin^2 x$, $x=0$. $\qquad f(0)=0$

$$= \frac{1}{2}(1-\cos 2x)$$

Now $f'(x) = \frac{1}{2}\cdot 2\sin 2x = \sin 2x$ $\qquad f'(0)=0$

$f''(x) = 2\cos 2x$ $\qquad \Rightarrow \qquad f''(0)=2$

$f'''(x) = -2^2 \sin 2x$ $\qquad f'''(0)=0$

$f^{(iv)}(x) = -2^3\cos 2x$ $\qquad f^{(iv)}(0) = -2^3$

$f^{(v)}(x) = +2^4 \sin 2x$ $\qquad f^{(v)}(0) = 0$

So, iteratively, we get $f^{(n)}(0) = \begin{cases} 0, & n=\text{odd} \\ (-1)^{n/2} 2^{n-1}, & n=\text{even}, n\geq 2 \end{cases}$

$\Rightarrow$ Taylor's series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=1}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} = \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1} x^{2k}}{(2k)!}.$$

[This series can be also obtained from the Taylor series for $g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \Rightarrow 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \dots$

$$\Rightarrow f(x) = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \dots\right).$$]

The convergence of this Taylor series to $f(x)$ is established as follows:

Method 1: The series has radius of convergence $R = +\infty$ since

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{2^{2k+1}x^{2(k+1)}}{(2k+2)!}\cdot\frac{(2k)!}{2^{2k-1}x^{2k-1}}\right|$$

$$\lim_{k\to\infty}\frac{|(2x)^2|}{(2k+1)(2k+2)} = 0 < 1, \forall x \in \mathbb{R}.$$

Denoting $h(x)$ = the sum of this series and using the differentiating term by term property of power series,

$$h(x) = \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \dots\right)$$

$$\Rightarrow h''(x) = \frac{1}{2}\cdot 4\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots\right)$$

$$= 2 - 4h(x)$$

$$\Rightarrow h''(x) + 4h(x) = 2.$$

$$h(0) = 0, \quad h'(0) = 0$$

Solving this initial value problem leads to

$$\boxed{h(x) = \frac{1}{2} - \frac{1}{2}\cos 2x}$$

Method 2: The "more standard" method to establish the convergence of Taylor series to the original function recalls the Taylor remainder formula

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(y)}{(n+1)!}(x-x_0)^{n+1}$$

for some $y \in [x, x_0]$

In our case, $x_0 = 0$

$$|R_n(x)| = \left|\frac{f^{(n+1)}(y)}{(n+1)!}\right| |x|^{n+1} \leq$$

$$\leq \frac{2^n}{(n+1)!} x^{n+1} \xrightarrow[\text{as } n\to\infty]{} 0 \quad , \forall x$$

So $\underset{\text{for } x\to\infty}{T_n(x) \to f(x)}$!!

(b) $g(x) = \sqrt{1+x}$, $x=0$ $\qquad g(x) = (1+x)^{1/2} \Rightarrow g(0) = 1$

$g'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad \Rightarrow \qquad g'(0) = \frac{1}{2}$

$g''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2} \qquad g''(0) = \frac{1}{2}\left(-\frac{1}{2}\right)$

$g'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-5/2} \qquad g'''(0) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots$

$\vdots$

$\Downarrow$

$$g^{(n)}(0) = \frac{1}{2}\underbrace{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}_{n-1 \text{ factors}}$$

$$\Rightarrow g^{(n)}(0) = \frac{1}{2}(-1)^{n-1}\frac{1\cdot 3\cdots(2n-3)}{2^{n-1}} = (-1)^{n-1}\frac{1\cdot 3\cdots(2n-3)}{2^n}$$

So its Taylor series is

$$\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}x^n = 1+\frac{1}{2}x+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdots(2n-3)}{2^n n!}x^n$$

(c) Radius of convergence is

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{1\cdot 3\cdots(2n-3)}{2^n\cdot n!}\cdot\frac{2^{n+1}(n+1)!}{1\cdot 3\cdots(2n-3)(2n-1)}$$

$$= \lim_{n\to\infty}\frac{2(n+1)}{2n-1} = 1$$

So interval of convergence includes $(-1,1)$.

(For $x=\pm 1$ one can show that the series converges, hence interval of convergence is indeed $[-1,1]$.)

(d) To show $g(x)$ equals its Taylor series on $(-1,1)$ it is enough to consider again the remainder

$$|R_n(x)| = |g(x)-T_n(x)| = \left|\frac{g^{(n+1)}(y)}{(n+1)!}x^{n+1}\right| \leq$$

$$\leq \frac{1\cdot 3\cdots(2n-1)}{2^{n+1}(n+1)!}|x|^{n+1} \leq \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n\cdot(2n+2)} \longrightarrow 0$$

as $n\to\infty$