Review for Exam 1
Math 3410 - Intro to Analysis

Spring 2015 - Dr. Radu Cascaval

TRUE/FALSE questions. Give a brief explanation for each answer.

- (a) Every ordered field has the Archimedean property.
- (b) The ordered eld axioms imply $|a - b| \leq |a| + |b|$ for all a, b.
- (c) If $\lim a_n = -\infty$ then $\limsup_n a_n = -\infty$.
- (d) For any sequence of real numbers, the lim inf and the lim sup always exist and furthermore the $\liminf$ is always $\leq \limsup$.
- (e) The equation $3x^3 + 2x^2 + 3x + 2 = 0$ has a rational solution.
- (f) $\sqrt[3]{216}$ is an irrational number.
- (g) The limit of a convergent sequence of negative numbers is negative.
- (h) The limit of a convergent sequence of rational numbers is rational.
- (i) Every interval contains at least three rational numbers.
- (j) Every bounded sequence of real numbers is convergent.
- (k) Every convergent sequence of real numbers is bounded.
- (l) Every monotone sequence of real numbers is convergent.
- (m) If $\{a_n\}$ is a monotone sequence of real numbers, then $\lim a_n$ exists and belongs to the interval $(-\infty, +\infty)$.

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1. Show that every real number is the limit of a sequence of rational numbers.
2. Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences, with $\lim a_n = a$ and $\lim b_n = b$. Assume that $a_n \leq b_n$ for all by finitely many terms. Show that $a \leq b$.
3. Let $\{s_n\}$ and $\{t_n\}$ be two bounded sequences. Show that

$$\limsup_n (s_n + t_n) \leq \limsup_n s_n + \limsup_n t_n$$

Give an example of two bounded sequences $\{s_n\}$ and $\{t_n\}$ such that

$$\limsup_n (s_n t_n) \neq \limsup_n s_n \limsup_n t_n$$

4. It can be shown that $\sqrt[3]{1+\sqrt{5}}$ is an algebraic number, i.e., it is a solution of some polynomial equation with integer coefficients. Find such an equation.

5. (a) Let $b > 0$. Use mathematical induction to prove that $(1+b)^n \geq 1 + nb$ for all $n \in \mathbb{N}$.

   (b) Show that $\lim a^n = \infty$ when $a > 1$. (Hint: use part(a)).

   (c) Show that for any real number $\varepsilon > 0$, there is a $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$.

6. (a) Find $\inf S$ and $\sup S$ for the following sets. Otherwise write DNE.

$$(i)\ S = \{x \in \mathbb{R} : x^2 > 1\} \cap \{x \in \mathbb{R} : x \leq 0\} \qquad (ii)\ S = \{(-n)^n : n \in \mathbb{N}\}.$$

   (b) Let $S = \{r \in \mathbb{Q} : r < a\}$ where a is a real number. Show that $\sup S = a$.

   (c) Give an example of a set of irrational numbers whose infimum is a rational number.

7. Calculate the limit of each of the following sequences, if it exists. Otherwise write DNE. **Justify** your answer but formal proof is NOT necessary.

$$(i)\ s_n = n\sin(\frac{1}{n}) \qquad (ii)\ s_n = \frac{2^n}{n!} \qquad (iii)\ s_n = (-1)^n \frac{n}{n+1} \qquad (iv)\ s_n = (2^n + 3^n)^{1/n}$$

8. (a) Use the definition of a Cauchy sequence to show that $s_n = n^{-1}\sin n$ is a Cauchy sequence.

   (b) Let (s_n) be a convergent sequence and $\lim s_n = a$. Prove that $\lim \sigma_n = a$ where the sequence (σ_n) is defined by $\sigma_n = (s_1 + s_2 + \ldots + s_n)/n$.

   (c) Let (σ_n) be defined as in part (c). Find $\lim \sigma_n$ when $s_n = 1 + (-1)^n$, and prove your claim.

9. (a) Show that the sequence $s_n = (1 + n^{-1})^n$ is monotone increasing and bounded. (Hint: expand s_n using binomial theorem).

   (b) By Theorem 10.2, the sequence (s_n) in part (a) converges to a limit. Calculate lim s_n.

   (c) A sequence is defined as $s_1 = 0.d$, $s_2 = 0.dd$, $s_3 = 0.ddd, \ldots$ where $d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Show that the sequence (s_n) is monotone increasing and bounded. Then find $\sup\,(s_n)$.

10. (a) Determine $\liminf s_n$ and $\limsup s_n$ for the sequence $s_n = (1 + \cos n\pi)^n$.

    (b) For a sequence (a_n), prove that $\lim a_n = \infty$ if and only if $\liminf a_n = \limsup a_n = \infty$.

    (c) Consider the sequence of partial sums defined by $s_n = 1 + 2^{-1} + 3^{-1} + \ldots + n^{-1}$. Show that (s_n) is *not* a Cauchy sequence.

    (d) By Theorem 10.11, the sequence (s_n) in part (c) does not converge. Prove that $\lim s_n = \infty$.