SOLUTIONS TO PRACTICE EXAM 1 – MATH 3410

TRUE/FALSE

(a) FALSE (There are ordered fields which do not satisfy the Archimedean property, e.g. field of all rational functions)

Ex: field of all rational functions $\frac{f(x)}{g(x)}$

$\frac{f(x)}{g(x)} > 0$ if leading coeff of f & $g > 0$.

(b) TRUE $|a-b| \leq |a| + |-b| = |a| + |b|$

(c) TRUE $\lim_n a_n = -\infty$ implies that for any $M > 0$, there exists $N \in \mathbb{N}$ such that $a_n < -M$ for $n > N$

Then $\sup\{a_n ; n > N\} \leq -M$

$\Rightarrow \limsup_n a_n = \lim \left[\sup\{a_n , n > N\}\right]$

$\leq -M$

for all $M \Rightarrow \limsup_n a_n = -\infty$

(d) TRUE

$\limsup_n a_n \overset{\text{def}}{=} \lim_N v_N$, $v_N = \sup\{a_n ; n > N\}$

v_N is a nonincreasing sequence (in $\mathbb{R} \cup \{\pm\infty\}$)

So $\lim v_N$ exists (either in $\mathbb{R}$ or $\pm\infty$).

Similarly for $\liminf_n$.

Clearly $\liminf_n a_n \leq \limsup_n a_n$

(e) TRUE

$f(x) = 3x^3 + 2x^2 + 3x + 2 = 0$

If x is a rational solution $\Rightarrow x = \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$

Checking all the values we obtain $f(-\frac{2}{3}) = 0$ so $x = -\frac{2}{3}$ is a rational solution!

(f) FALSE $\sqrt[3]{216}$ is NOT irrational!

Since $216 = 2 \times 108 = 2 \times 2 \times 54 = 2 \times 2 \times 2 \times 27$

$= 2^3 \times 3^3 = 6^3$

So $\sqrt[3]{216} = 6$!

(g) FALSE $a_n = -\frac{1}{n} < 0$ for each $n \in \mathbb{N}$

but $\lim_n a_n = 0$

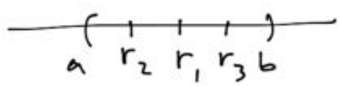
(h) FALSE $\quad s_{n+1} = \dfrac{1}{s_n + 1}, \quad s_1 = 1$

All $s_n \in \mathbb{Q}$ (by induction) but

$\lim_n s_n = \dfrac{\sqrt{5}-1}{2} \notin \mathbb{Q}$

(i) TRUE $\quad a, b \in \mathbb{R},\ a < b \Rightarrow \exists\, r_1 \in (a,b)$

$\Rightarrow \exists\, r_2 \in (a, r_1)$ and $r_3 \in (r_1, b)$



(j) FALSE $\quad \{(-1)^n\}$ bounded by divergent

(k) TRUE $\quad \{s_n\}$ convergent to $s \in \mathbb{R}$

$\varepsilon = 1, \exists N$ such that $|s_n - s| < 1$ for $n > N$

$\Rightarrow s_n \in (s-1, s+1) \Rightarrow$

$\Rightarrow s_n \in [m, M]$ where $M = \max\{s+1, s_1, \dots, s_N\}$

$m = \min\{s-1, s_1, \dots, s_N\}$

$\Rightarrow s_n$ is bounded!

(l) FALSE $\quad s_n = n$ is monotonic but divergent

(m) FALSE (same as above)

#1. Let $a \in \mathbb{R}$

For each $n \in \mathbb{N}$, we can use the density of $\mathbb{Q}$ in $\mathbb{R}$ to find $s_n \in \mathbb{Q}$ such that

$$a < s_n < a + \frac{1}{n}$$

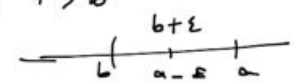
By the squeezing property, $\lim_{n\to\infty} s_n = a$

#2 $\quad \lim_n a_n = a, \ \lim_n b_n = b, \quad a_n \le b_n$ for $n \ge N_0$

Need to show $a \le b$

Assume the contrary, that $a > b$

Then let $\varepsilon = \dfrac{a-b}{2} > 0$



There exists N_1 such that

$|a_n - a| < \varepsilon$ whenever $n > N_1$

and $N_2 \in \mathbb{N}$ such that

$|b_n - b| < \varepsilon$ whenever $n > N_2$

Take $n > \max\{N_1, N_2, N_0\}$

$$b_n < b + \varepsilon = a - \varepsilon < a_n$$

which leads to a contradiction! ($a_n \le b_n$ for all $n \ge N_0$)

#3 $\quad$ Let $s = \limsup_n s_n < +\infty$

$t = \limsup t_n < +\infty$

(since both $\{s_n\}$, $\{t_n\}$ are bounded!)

We need to show $\limsup_n (s_n + t_n) \leq s+t$.

Let $\varepsilon > 0$. Then there exists N_1 such that

$s_n < s + \varepsilon/2$ for all $n > N_1$,

and, similarly, there exists N_2 such that

$t_n < t + \varepsilon/2$ for all $n > N_2$

$\Rightarrow s_n + t_n < s + t + \varepsilon$ for $n > \max\{N_1, N_2\}$

$\Rightarrow \limsup_n (s_n + t_n) \leq s + t + \varepsilon$

This holding true for any $\varepsilon > 0 \Rightarrow \limsup (s_n + t_n) \leq s+t$

QED

$\limsup s_n t_n \neq \limsup s_n \limsup t_n$

$s_n = (-1)^n$, $t_n = (-1)^{n+1}$

Then $s_n t_n = -1$ so $\limsup s_n t_n = -1$

but $\limsup_n s_n = \limsup t_n = 1$.

#4 $x = \sqrt[3]{1+\sqrt{5}} \Rightarrow x^3 = 1+\sqrt{5}$

$\Rightarrow x^3 - 1 = \sqrt{5} \Rightarrow (x^3-1)^2 = 5$

$\Rightarrow x^6 - 2x^3 + 1 = 5 \Rightarrow x^6 - 2x^3 - 4 = 0$

#5 (a) $b > 0$

Basis for induction: $n = 1$ $(1+b)^1 \geq 1+b$ (true!)

Induction step

Assume $(1+b)^n \geq 1+nb$. Prove $(1+b)^{n+1} \geq 1+(n+1)b$

$(1+b)^{n+1} = (1+b)(1+b)^n \geq (1+b)(1+nb) = 1+(n+1)b + nb^2$

$\geq 1+(n+1)b$

QED

(b) When $a > 1$, $a = 1+b$ for $b > 0$

so $a^n = (1+b)^n \geq 1+nb$. Since $\lim_n 1+nb = +\infty$

$\Rightarrow \lim_n a^n = +\infty$!

(c) We know $\lim_{n\to\infty} 2^n = +\infty$

Let $\varepsilon > 0$ and $M = \frac{1}{\varepsilon}$. There exists N such that

$2^n > M = \frac{1}{\varepsilon}$ for $n > N$

$\Rightarrow 2^{-n} < \varepsilon$ whenever $n > N$. QED

#6 (a) (i) $S = \{x \in \mathbb{R} \mid x^2 > 1\} \cap \{x \in \mathbb{R} \mid x \leq 0\}$

$= (-\infty, -1)$

so $\inf S = \text{DNE}$, $\sup S = -1$

(ii) $S = \{(-n)^n \mid n \in \mathbb{N}\} = \{-1, 2^2, -3^3, \dots\}$

$\Rightarrow \inf S = \text{DNE}$, $\sup S = \text{DNE}$

(b) $S = \{ r \in \mathbb{Q} \mid r < a \}$, a = real number

First a is an upper bound, so $\sup S$ exists and $\sup S \le a$

Assume (by contradiction) that $\sup S \neq a$

then $\sup S < a$

But by density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that

$$\left. \begin{array}{l} r \in \mathbb{Q} \text{ such that} \\ \sup S < r < a \end{array} \right] \Rightarrow r \in S \Rightarrow$$

$\Rightarrow r \le \sup S$. This contradiction proves $\sup S = a$ QED

(c) $S = \{ \frac{\sqrt{2}}{n} \mid n \in \mathbb{N} \}$ has $\inf S = 0$

#7 (i) $s_n = n \sin\left(\frac{1}{n}\right) = \dfrac{\sin \frac{1}{n}}{\frac{1}{n}} \longrightarrow 1$.

so $\lim_n s_n = 1$

(ii) $s_n = \dfrac{2^n}{n!}$, $s_n \le \dfrac{s_{n+1}}{2}$ for $n > 10$ (by induction)

so $\lim_n s_n = 0$

(iii) $s_n = (-1)^n \dfrac{n}{n+1}$ is divergent ($\lim s_n$ DNE)

since $\lim s_{2n} = 1$, $\lim s_{2n+1} = -1$

(iv) $s_n = (2^n + 3^n)^{1/n}$

$$\ln s_n = \frac{\ln(2^n + 3^n)}{n}$$

$$\lim_{x \to \infty} \frac{\ln(2^x + 3^x)}{x} \overset{\text{l'Hopital}}{=} \lim_{x \to \infty} \frac{1}{2^x + 3^x}\left(2^x \ln 2 + 3^x \ln 3\right)$$

$$= \frac{3^x \ln 3}{3^x} = \ln 3.$$

So $\lim_{n \to \infty} s_n = 3$.

(Alternate proof: $s_n = (2^n + 3^n)^{\frac{1}{n}} = \left[3^n\left(\left(\frac{2}{3}\right)^n + 1\right)\right]^{1/n}$

$\Rightarrow s_n = 3\left[\left(\frac{2}{3}\right)^n + 1\right]^{1/n} \longrightarrow 3 \cdot 1^0 = 3$.)

#8 $s_n = \dfrac{\sin n}{n}$.

$$|s_n - s_m| = \left| \frac{\sin n}{n} - \frac{\sin m}{m} \right|$$

$$\le \left|\frac{\sin n}{n}\right| + \left|\frac{\sin m}{m}\right|$$

$$\le \frac{1}{n} + \frac{1}{m}.$$

So if $\varepsilon > 0$, take $N \ge \frac{2}{\varepsilon} \Rightarrow$

$$|s_n - s_m| \le \frac{1}{n} + \frac{1}{m} < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

whenever $n, m > N$

$\Rightarrow \{s_n\}$ is Cauchy

(b) Let s_n convergent, $\lim s_n = a$

To show $\lim \sigma_n = a$, $\sigma_n = \frac{s_1 + \dots + s_n}{n}$

we can show

$$\liminf s_n \le \liminf \sigma_n \text{ and}$$

$$\limsup \sigma_n \le \limsup s_n$$

(see hint on ex 12.12)

Indeed, for $n > N$

$$\frac{s_1+\dots+s_n}{n} = \frac{s_1+\dots+s_N}{n} + \frac{s_{N+1}+\dots+s_n}{n}$$

Let $\varepsilon > 0$ and N such that $s_n < \limsup_n s_n + \varepsilon$ for $n > N$

$$\Rightarrow \sigma_n \le \frac{s_1+\dots+s_N}{n} + \frac{(n-N)}{n}\left(\limsup_n s_n + \varepsilon\right)$$

$$\Rightarrow \limsup \sigma_n \le \limsup \frac{s_1+\dots+s_N}{n} + \limsup_n \frac{n-N}{n}(\cdot\text{—}\cdot)$$

$$\Rightarrow \limsup \sigma_n \le 0 + 1\cdot\left(\limsup s_n + \varepsilon\right)$$

$$\le \limsup s_n + \varepsilon$$

Since ε = arbitrary $\Rightarrow \limsup \sigma_n \le \limsup s_n$!

#(c) $s_n = 1 + (-1)^n \Rightarrow (s_n) = (0, 2, 0, 2, \dots)$

Now $\sigma_1 = \frac{s_1}{1} = 0$

$$\sigma_2 = \frac{s_1+s_2}{2} = \frac{0+2}{2} = 1$$

$$\sigma_3 = \frac{s_1+s_2+s_3}{3} = \frac{0+2+0}{3} = \frac{2}{3}$$

$$\Rightarrow \left.\begin{aligned} \sigma_{2k} &= \frac{2k}{2k} = 1 \\ \sigma_{2k+1} &= \frac{2k}{2k+1} \end{aligned}\right] \Rightarrow \sigma_n \to 1 \text{ as } n\to\infty$$

[even though s_n is divergent!]

#9 $s_n = \left(1+\frac{1}{n}\right)^n$. Need to show it is monotonically increasing

$$s_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}\left(\frac{1}{n}\right)^k$$

$$= 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \dots + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!\,n^2} + \frac{n(n-1)(n-2)}{3!\,n^3} + \dots$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots$$

$$\dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

Since $1-\frac{1}{n} < 1-\frac{1}{n+1}$, $1-\frac{2}{n} < 1-\frac{2}{n+1}$ etc

it follows

$$s_n < s_{n+1}$$

So $\{s_n\}$ is monotonically increasing!

To show $\{s_n\}$ bounded, enough to use $1-\frac{1}{n}<1$, $1-\frac{2}{n}<1$, etc.

$$s_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!}$$

$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{n-1}} < 3$$

So $\{s_n\}$ is bounded above by 3.

We conclude that $\{s_n\}$ is convergent to some limit $s \in [2,3]$. This limit is precisely the number $e \approx 2.7182\dots$

(c) $s_1 = 0.d = d\cdot 10^{-1}$

$s_2 = 0.dd = d\cdot 10^{-1} + d\cdot 10^{-2}$

$\vdots$

$s_n = d\,10^{-1} + d\,10^{-2} + \dots + d\cdot 10^{-n}$

Clearly $s_{n+1} - s_n = d\cdot 10^{-(n+1)} > 0$

so $s_n < s_{n+1} \Rightarrow \{s_n\}$ is monotone

also $s_n < 0.(d+1)$ $[=1 \text{ if } d=9] \Rightarrow s_n$ bounded

To find $\lim_n d_n$,

$$10^{-1} + \dots + 10^{-n} = \frac{1-10^{-n-1}}{1-10^{-1}} - 1$$

$$\Rightarrow \lim_{n\to\infty} d_n = d\left(\frac{1}{1-10^{-1}} - 1\right) = \frac{d}{9}.$$

#10 (a) $s_n = (1+\cos n\pi)^n$

$$= \begin{cases} 2^n, & n = \text{even} \\ 0, & n = \text{odd} \end{cases}$$

so $\liminf s_n = 0$, $\limsup s_n = +\infty$.

(b) in the textbook

(c) $s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

$m > n$

$s_m - s_n = \frac{1}{n+1} + \dots + \frac{1}{m}$

if $n = 2^k - 1$, $m = 2^{k+1}$

$$s_m - s_n > \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \dots + \frac{1}{2^{k+1}} \geq \frac{1}{2}$$

So (s_n) is not Cauchy! $(\varepsilon = \frac{1}{2})$

(d) Clearly $s_{2^k} > \frac{k}{2} \to \infty$

So $\lim_{k\to\infty} s_{2^k} = +\infty \Rightarrow \lim s_n = \infty$.

s_n monotone