Wed, Mar 4, 2015

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Solutions

Exam 1 - Math 3410

Intro to Analysis

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1. Determine whether the following statements are TRUE or FALSE. Justify your answers.

(a) $\sqrt{2}+\sqrt{3}$ is irrational.

TRUE: Assume the contrary, that is $\sqrt{2}+\sqrt{3}=r\in\mathbb{Q}$

Then $\sqrt{2}=r-\sqrt{3} \Rightarrow 2=(r-\sqrt{3})^2=r^2-2r\sqrt{3}+3 \Rightarrow \sqrt{3}=\frac{r^2+1}{2r}\in\mathbb{Q}$

But we know $\sqrt{3}\notin\mathbb{Q}$. The contradiction shows $\sqrt{2}+\sqrt{3}\notin\mathbb{Q}$.

(b) Every open interval (a,b) contains infinitely many *irrationals*.

TRUE By density of $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$, any interval in $\mathbb{R}$ contains at least one irrational. We get infinitely many irrationals but dividing (a,b) into half, then repeat the process.

(c) If $\liminf s_n=-\infty$ then $\lim s_n=-\infty$.

FALSE Ex $s_n=(-2)^n$. $\liminf s_n=-\infty$ since the subsequence $s_{2k+1}=-2^{2k+1}\to-\infty$ but the $\lim s_n$ does not exist.

$(\lim_{k\to\infty} s_{2k}=+\infty)$

(d) If (s_n) and (t_n) converge to s and t respectively, and $s_n<t_n$ for all n, then $s<t$.

FALSE Ex: $s_n=1-\frac{1}{n}$, $t_n=1+\frac{1}{n}$, $s_n<t_n$ for all n

but $\lim_{n\to\infty} s_n=1=\lim_{n\to\infty} t_n$ so $s=t$.

~~(e) Every open interval (a,b) contains infinitely many irrationals.~~

(f) If S is a bounded set of real numbers, then $\sup S\in S$.

FALSE $S=(0,1)$ (open interval), then $\sup S=1\notin S$.

(g) If $\lim s_n=+\infty$ then s_n is monotone.

FALSE Ex: $(s_n)_{n\geq1}=(1,-1,2,0,3,1,4,2,\dots)$ has $\lim_{n\to\infty} s_n=+\infty$ but is not monotone.

$(s_{2k+1}=k+1$ and $s_{2k}=k-2)$.

Statement P_n

2. Show by mathematical induction that $n! > 2^n$, for all $n > 3$.

$n! = 1\cdot 2\cdot 3 \cdot \ldots \cdot n$ For $n=4$: $4! = 24 > 16 = 2^4$

$2^n = 2\cdot 2\cdot 2\cdot\ldots\cdot 2$ So P_4 is true

Induction step: Assume P_n, prove P_{n+1}: $(n>3)$

$$(n+1)! = n!(n+1) > 2^n\cdot(n+1) > 2^n\cdot 2 = 2^{n+1}$$

(by induction hypothesis) (since $n>1$)

So P_{n+1} is also true!

3. Find the following limit: $\lim_n(\sqrt{n^2+2n}-\sqrt{n^2-3n}) =$ [rationalize the numerator]

$$= \lim_{n\to\infty} \frac{n^2+2n-(n^2-3n)}{\sqrt{n^2+2n}+\sqrt{n^2-3n}} = \lim_{n\to\infty}\frac{5n}{\sqrt{n^2+2n}+\sqrt{n^2-3n}} =$$

$$= \lim_{n\to\infty}\frac{5}{\sqrt{1+\frac{2}{n}}+\sqrt{1-\frac{3}{n}}} = \frac{5}{\sqrt{1+0}+\sqrt{1+0}} = \frac{5}{2}$$

(using properties of limits)

4. Let (s_n) be defined by $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + s_n}$, for $n \in \mathbb{N}$.

(a) Show that each term s_n is an irrational number.

We show this by induction. We know already (from class) that $s_1 = \sqrt{2}$ is irrational. We now assume s_n is irrational and show s_{n+1} is also irrational.

We assume the contrary, that s_n is irrational but s_{n+1} is rational, that is $s_{n+1} \in \mathbb{Q}$. Then

$$s_{n+1}^2 = 2 + s_n \Rightarrow s_n = s_{n+1}^2 - 2 \in \mathbb{Q},$$

This contradicts the induction hypothesis (that $s_n \notin \mathbb{Q}$) so this completes the induction argument!

(b) Prove that (s_n) is monotone and bounded.

We can see that $s_2 = \sqrt{2+s_1} = \sqrt{2+\sqrt{2}} > \sqrt{2} = s_1$, so we will try to prove that s_n is monotonically increasing

$$s_n < s_{n+1}.$$

To find an upper bound, we notice that if such a sequence is bounded, then it is convergent and the limit has to satisfy $s = \sqrt{2+s} \Rightarrow s^2 - s - 2 = 0$

$\Rightarrow (s+1)(s-2) = 0$

So $s = 2$

(this will be the answer in (c): $\lim_n s_n = 2$)

(c) Find $\lim_n s_n$.

So we prove by induction

$$s_n < s_{n+1} < 2$$

Indeed, clearly $s_1 < s_2 < 2$

Assuming $s_{n-1} < s_n < 2$, then we see

$$s_{n+1} = \sqrt{2+s_n} < \sqrt{2+2} = \sqrt{4} = 2$$

and

$$s_{n+1} > s_n \Leftrightarrow \sqrt{2+s_n} > s_n \Leftrightarrow s_n^2 - s_n - 2 < 0 \quad \text{(true since } 0 < s_n < 2\text{)}$$

5. Given two sets of real numbers S and T, we define the new set

$$S+T=\{s+t \mid s\in S, t\in T\}.$$

(a) Show that, if S and T are bounded

$$\inf(S+T)=\inf(S)+\inf(T)$$

and the same for the sup.

We show $\inf(S+T) \le \inf S + \inf T$ (*)
$\inf(S+T) \ge \inf S + \inf T$. (**)

[all inf's exist as real #s, since S, T and $S+T$ bounded]

Since $s \ge \inf S$ $\forall s \in S$ and $t \ge \inf T$, $\forall t \in T$ $\Rightarrow$
$\Rightarrow$ $s+t \ge \inf S + \inf T$, $\forall s+t \in S+T$ $\Rightarrow$ $\inf(S+T) \ge \inf S + \inf T$.
This proves (**).

To prove (*), $\forall \varepsilon>0$, $\exists s \in S$ with $s < \inf S + \varepsilon/2$
$\exists t \in T$ with $t < \inf T + \varepsilon/2$

$\Rightarrow$ $s+t < \inf S + \inf T + \varepsilon/2 + \varepsilon/2 = \inf S + \inf T + \varepsilon$ $\Rightarrow$
$\Rightarrow$ $\inf S+T \le s+t < \inf S + \inf T + \varepsilon$. Since this is true for all $\varepsilon>0$.
$\Downarrow$
$\inf S+T \le \inf S + \inf T$

(b) Describe the set $S+T$, where $S=[0,1]\bigcap\mathbb{Q}$ and $T=[0,1]\setminus\mathbb{Q}$.

$s \in S \Rightarrow$ s is rational in $[0,1]$
$t \in T \Rightarrow$ t is irrational in $[0,1]$
$\Big\}\Rightarrow$ $s+t$ is irrational in $[0,2]$
$\Downarrow$
$S+T \subset [0,2]\setminus\mathbb{Q}$.

On the other hand, any irrational in $[0,2]$, say q, can be written as a sum of a rational (say $s=1$) and an irrational (say $t=q-1$) so that $q = 1+(q-1) \in S+T$)

We conclude
$\boxed{S+T=[0,2]\setminus\mathbb{Q}}$

6. (a) State the Bolzano-Weierstrass theorem.

Every bounded sequence has a convergent subsequence.

(b) Suppose (s_n) is a bounded sequence such that all *convergent* subsequences converge to the same number s. Prove that (s_n) must converge to s. [Hint: Assume that $s_n \nrightarrow s$, then apply Bolzano-Weierstrass to a conveniently chosen subsequence].

Assume the contrary, that s_n does not converge to s.
So $\exists \varepsilon > 0$, such that $|s_n - s| > \varepsilon$ for infinitely many n's
Let (s_{n_k}) be such a subsequence for which $|s_{n_k} - s| > \varepsilon$
By Bolzano Weierstrass applied to the set $\{s_{n_k}\}$ which is bounded, $\exists$ a (sub)subsequence $s_{n_{k_p}}$ which converges.
By hypothesis though $s_{n_{k_p}} \to s$. This contradicts $|s_{n_{k_p}} - s| > \varepsilon \ \forall p$.
Done!

(c) Determine whether the sequence

$$s_n = \frac{n}{n+1} \cos \frac{n\pi}{2}$$

is bounded or not. Does it have convergent subsequences? Is (s_n) convergent itself? Explain!

Since $\left(\cos \frac{n\pi}{2}\right)_{n \geq 0} = 1, 0, -1, 0, 1, 0, -1, 0, \ldots$

it means $|s_n| = \left|\frac{n}{n+1}\right| \leq 1 \Rightarrow s_n$ is bounded.

s_n does have convergent subsequences

(e.g. $s_1, s_3, s_5, \ldots \to 0$, since $s_{2k+1} = 0 \ \forall k$.)

also $s_0, s_4, s_8, \ldots, \to 1$, since $s_{4k} = \frac{4k}{4k+1} \to 1$ as $k \to \infty$

and $s_2, s_6, s_{10}, \ldots \to -1$ since $s_{4k+2} = -\frac{4k+2}{4k+3} \to -1$

Therefore, s_n is <u>not</u> convergent (as a whole).