Math 413/513 Fall 2010 **STUDENT NAME:** ____________________________

EXAM I – Linear Algebra I

Part I – In Class

You may use any theorems which were proved in class or in the textbook provided that you state them clearly. You may not use results which were proved as part of your homework unless you prove them again here.

Part I (60%) - In class: Oct 6, 2010 Part II (40%)- Take Home: Due Monday, Oct 11, 2010 at 4:30pm

- **Problem 1**
  For each of the following statements, give either a short proof (if the statement is true) or a counterexample (if the statement is false)

  – If S_1 and S_2 are subsets of a vector space V then
  $$\mathrm{span}(S_1) \cap \mathrm{span}(S_2) \subseteq \mathrm{span}(S_1 \cap S_2)$$

  – The set $\{A \in M_{n\times n}(\mathbb{F}) | A^2 = 0\}$ is a subspace of $M_{n\times n}(\mathbb{F})$.

  – The map $T : P(\mathbb{R}) \to P(\mathbb{R})$, which maps a polynomial $p(x)$ to $p(x^2)$, is linear.
  [e.g. $T(1 + 3x + x^2) = 1 + 3x^2 + (x^2)^2 = 1 + 3x^2 + x^4$] .

  – The vector spaces $\mathbb{F}^4$ and $M_{2\times 2}(\mathbb{F})$ are isomorphic.

- **Problem 2** Find a basis for the following subspace of $\mathbb{R}^5$:

$$W = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_1 = a_3 = a_4 \text{ and } a_2 + a_5 = 0\}.$$

  What is the dimension of W?

- **Problem 3**
  Consider the map $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ given by $T(f)(x) = \int_0^x f(t)\,dt$.
  Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, x, x^2, x^3\}$ be the standard ordered bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$.

  (a) Compute $[T]_\beta^\gamma$

  (b) What is the rank of T?

- **Problem 3**

  Let $T : V \to W$ be a linear transformation between two vector spaces V and W.
  (*Math 413 students may assume V and W are finite dimensional, whereas Math 513 students need to proceed without this assumption*)
  (a) Show that if G is a generating set for V, then $T(G)$ is generating set for $R(T)$.

  (b) Show that if L is a linearly independent set in $R(T)$, then there exist a linearly independent set $\tilde{L}$ in V such that $T(\tilde{L}) = L$.

  (c) Show that T is invertible if and only if, for every basis β in V, $T(\beta)$ is basis in W.

- **Problem 4**

  Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a - b \\ a + b \end{pmatrix}.$$

  Find $[T]_\beta$, the matrix representation of T in the nonstandard basis $\beta = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\}$.

- **Problem 5**

  Let A be an $m \times n$ matrix with rank m. Prove than there exists an $n \times m$ matrix B such that $AB = I_m$. Does it imply that $BA = I_n$? Explain.
  [Hint: Consider the left multiplication transformation $T = L_A$].

Math 413/513 FALL 2010 **STUDENT NAME:** ______________________

EXAM I – Linear Algebra I

Part II – Take Home

DUE Wed, Oct 13 at 4:30pm

You may use any theorems which were proved in class or in the textbook provided that you state them clearly. You may not use results which were proved as part of your homework unless you prove them again here.

ABSOLUTELY NO TEAM WORK ALLOWED!!!

Math 513 students are required to answer all questions!
The starred (*) problems are optional (extra-credit) for Math 413 students.

- **Problem 1**
  Let V be a finite dimensional vector space over a field $\mathbb{F}$ with $\dim(V) = n$.

  (a) Show that the space of all linear transformations from V to V, $\mathcal{L}(V) = \{T : V \to V \,|\, T \text{ is linear}\}$, is isomorphic to $M_{n\times n}(\mathbb{F})$. [Hint: Use the matrix representation $[T]_\beta$ with respect to some fixed basis β.]

  (b) What is the dimension of $\mathcal{L}(V)$?

  (c) Given $T \in \mathcal{L}(V)$, define the trace of T to be $\text{trace}([T]_\beta)$, for some basis β for V. Show that this quantity is independent of the basis β, that is $\text{trace}([T]_\beta) = \text{trace}([T]_{\beta'})$ for any two bases β and β' for V. [See exercise 10 page 118 and exercise 13 page 97]

- **Problem 2**

(a) Let α, β_1, β_2 be three ordered bases for a vector space V. Denote Q_1 and Q_2 the change of basis matrix from α to β_1 and from α to β_2, respectively. Show that the change of basis from β_1 to β_2 is then given by the matrix $Q = Q_1^{-1}Q_2$.

(b) Using part (a), find the change of basis matrix from the basis $\beta_1 = \{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\}$ to the basis $\beta_2 = \{\begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\3\\-2\end{pmatrix}\}$ of $\mathbb{R}^3$.

(c) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given as the projection on $W_1 = \text{span}\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\}$ along $W_2 = \text{span}\{\begin{pmatrix}0\\1\\1\end{pmatrix}\}$ (see definition on page 76). Determine $T\begin{pmatrix}a\\b\\c\end{pmatrix}$ and verify that $T^2 = T$.

- **Problem 3***
  Let V be a vector space (not necessarily finite dimensional) and $T, U : V \to V$ be linear transformations.

  (a) Show that $\dim(N(UT)) \leq \dim(N(U)) + \dim(N(T))$.

  (b) Show that $\dim(R(UT)) \leq \min\{\dim(R(U)), \dim(R(T))\}$.
  When does equality occur in (a) and in (b)? (See Lemma 2 page 135 for (a))

  [Do not assume V is finite dimensional! Consider **all cases** when the dimensions in questions are finite or infinite. Inequalities such as $\infty \leq \infty + \infty$ etc are considered true.]

  (c) For a given $T : V \to V$, consider the map $\Phi_T : \mathcal{L}(V) \to \mathcal{L}(V)$ given by $\Phi_T(U) = UT$. (See problem 1 for notation). Show that Φ_T is a linear transformation and describe its null space $N(\Phi_T)$ and range $R(\Phi_T)$. Compute its nullity and rank.
  [Hint: you may try first some simple example, e.g. $V = \mathbb{R}^2$ and $T = L_A$ with $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$]

  (d) Is there any difference if one considers instead at $\Psi_T : \mathcal{L}(V) \to \mathcal{L}(V)$ given by $\Psi_T(U) = TU$?