Math 413/513 Fall 2010 **STUDENT NAME: ____________________________**

EXAM II – Linear Algebra I

Take Home - DUE Mon, Nov 22, 2010 at 4:30pm

You may use any theorems which were proved in class or in the textbook provided that you state them clearly. You may not use results which were proved as part of your homework unless you prove them again here.

ABSOLUTELY NO TEAM WORK ALLOWED!!!

Math 513 students are required to answer all questions!

The starred (*) problems are optional for Math 413 students. They will be considered extra credit.

- **Problem 1**

  Given a linear transformation T on a finite dimensional vector space V, satisfying $T^2 = T$,

  (1) Show that the only possible eigenvalues of T are $\lambda = 0$ and $\lambda = 1$, and that $N(T)$ and $R(T)$ are the only possible eigenspaces.

  (2) Show that T is diagonalizable.
  [Hint: Use that fact that the relation $T^2 = T$ implies $N(T) \oplus R(T) = V$ and T is the projection on $R(T)$ along $N(T)$.]

- **Problem 2**

  Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$.

  (a) Using Cayley-Hamilton theorem for the matrix A, express A^n as linear combination of $I(= I_2)$ and A.

  (b) Determine whether A is diagonalizable or not. If yes, find an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

  (c*) Compute the exponential matrix e^A using the Cayley-Hamilton method AND using the representation in (b)

  (d*) When does it make sense to talk about $\log(A)$, given a matrix A?

- **Problem 3**
  Given $V = \mathbb{R}^3$, with the (standard) Euclidean inner product,
  (a) Determine an orthonormal basis for the subspace $W = \{(x, y, z) \,|\, x - y + 3z = 0\}$
  (b) Find the orthogonal projection of the vector $u = (2, 3, 1)$ on the subspace W as in (a).
  [Do not use the cross product!!!]

- **Problem 4**

  Let $V = C([-1,1])$ be the space of continuous functions on the interval $[-1,1]$ over the field of reals.

  (a) Show that the following is an inner product on V:

  $$< f, g >= \int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1-x^2}}\,dx$$

  (b) Find an orthonormal basis for the subspace W of V, consisting of polynomials of degree at most 2. $[W = \mathbb{P}_2(\mathbb{R}) \bigcap V.]$ [Hint: Start with the set $\{1, x, x^2\}$ and apply Gram Schmidt orthogonalization process.]

- **Problem 5**

  Let $V = \mathbb{P}_2([-1,1], \mathbb{C})$ be the vector space of polynomials of degree at most 2 with complex coefficients, with the 'standard' inner product $< f, g >= \int_{-1}^{1} f(t)\overline{g(t)}\, dt$.

  Find the adjoint transformation T^* of the linear transformation $T : V \to V$, defined by

$$T(f) = if' + 2f.$$

- **Problem 6**

(a) Let x, y be linearly independent vectors in an inner product space $V, < \cdot, \cdot >$ such that $\|x\| = \|y\| = 1$. Prove that $\|(1-t)x + ty\| < 1$, for all $t \in (0, 1)$.

(b) Let $\|x\| = |x_1| + |x_2|$ for $x = (x_1, x_2) \in \mathbb{R}^2$, be the taxi-cab norm on $\mathbb{R}^2$ (see general definition of a 'norm' on page 339). Use (a) to show that there is no inner product $< \cdot, \cdot >$ on $\mathbb{R}^2$ such that $\|x\|^2 =< x, x >$.

- **Problem 7***

  Let V be a finite dimensional vector space and T, U linear transformations which commute, i.e. $TU = UT$, and whose characteristic polynomials split.

  (a) If λ is an eigenvalue for T, with corresponding eigenspace E_λ, show that E_λ is U-invariant.

  (b) Show that T and U must have at least one common eigenvector (in V).

  (c) If we assume, in addition, that T and U are both diagonalizable, show that T and U are simultaneous diagonalizable, that is there exists a common basis of eigenvectors for both T and U.
  [Hint: If $E_1, \ldots, E_k$ denote the distinct eigenspaces of T, and $F_1, \ldots, F_l$ denote the distinct eigenspaces of U, such that $E_1 \oplus \ldots \oplus E_k = V = F_1 \oplus \ldots \oplus F_l$, show that $\sum_j W \cap F_j = W$ for each $W = E_i, i = 1, \ldots k$.]