Math 413/513 Fall 2010 **STUDENT NAME:** ______________________

FINAL EXAM – Linear Algebra I

Take Home - DUE Thursday, Dec 16, 2010 before 5:00pm
Return it to my office - ENG 271

You may use any theorems which were proved in class or in the textbook provided that you state them clearly. You may not use results which were proved as part of your homework unless you prove them again here.

ABSOLUTELY NO TEAM WORK ALLOWED!!!

Math 513 students are required to answer all questions!

The starred (*) problems are optional for Math 413 students. They will be considered extra credit.

- **Problem 1**
  Determine whether the following pairs of matrices (A, B) are similar or not. If yes, find a matrix Q such that $B = Q^{-1}AQ$.

  (a)
  $$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

  (b)
  $$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

- **Problem 2**

(a) Given a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, with QR decomposition given by the Gram-Schmidt process, and R having singular value decomposition $R = U\Sigma V^*$, determine the singular value decomposition of the matrix A.

(b) Find both (QR and SVD) decompositions for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$$

- **Problem 3**

Let V, W be two finite dimensional complex inner product spaces.

(a) If $U : V \to V$ is self-adjoint, show that all its eigenvalues are real,
(b) If $U : V \to V$ is skew-adjoint ($U^* = -U$), show that all its eigenvalues are imaginary.
(c) If $T : V \to W$ is *any* linear transformation, show that

$$rank(T^*T) = rank(TT^*) = rank(T).$$

(d)* If T is as in (c) , show that T^*T and TT^* have the same nonzero eigenvalues, that is, if λ is an eigenvalue for T^*T and $\lambda \neq 0$ then λ is also eigenvalue for TT^*. [Do not use the Singular Value Decomposition of T.]

- **Problem 4**

(i) If A and Q are unitary matrices, show that $U = Q^{-1}AQ$ is also unitary.

(ii) If a unitary matrix U is also upper triangular, show that U must be diagonal.

(iii) Using part (i) and (ii), show directy that any unitary operator T on a finite dimensional *complex* inner product space is diagonalizable.

(iv)* Does the conclusion in part (iii) remain valid if V is a *real* inner product space? Justify your answer.

- **Problem 5**

Let V be a finite dimensional inner product space.

(1) If $T : V \to V$ is an orthogonal projection, prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$. Give an example of a projection for which this inequality does not hold (for some $x \in V$).

(2)* Prove the converse of part (1): That is, suppose that $T : V \to V$ is a projection such that $\|T(x)\| \leq \|x\|$ for all $x \in V$, then prove that T must be an orthogonal projection. [Hint: Take $x = z - \frac{<z,y>}{\|y\|^2} y$, for $y \in N(T)$ and $z \in R(T)$.]

- **Problem 6**

Given the 4×4 matrix $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$,

(a) Find the Jordan canonical form J for A and a matrix Q such that $J = Q^{-1}AQ$. Verify the relation $J = Q^{-1}AQ$ (with or without computing Q^{-1}).

(b) Compute the exponential map $e^{tJ} = \sum_{k=0}^{\infty} \frac{(tJ)^k}{k!}$. [Hint: do it first for a single Jordan block only!]

(c) Compute the exponential map $e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$.

(d) Verify that the function $X(t) = e^{tA}$, $t \in \mathbb{R}$, found in (b), satisfies

$$\frac{d}{dt}X(t) = AX(t).$$

- **Extra Credit Problem**

(a) Let V be an inner product space, T a self-adjoint operator on V ($T^* = T$), U a skew-adjoint operator on V ($U^* = -U$) such that

$$UT - TU = I.$$

Show that

$$\|x\|^2 \leq 2\|Tx\|\|Ux\|, \qquad \text{for all } x \in V,$$

where the norm is induced by the inner product.
[Hint: start with $< x, x >=< x, (UT - TU)x >= ...$ etc]

(b) Let $V = \{f \in C^\infty(\mathbb{R}) | \int_{\mathbb{R}} (1 + x^2)|f(x)|^2 \, dx < \infty\}$ be endowed with

$$< f, g >= \int_{\mathbb{R}} f(x)\overline{g(x)} \, dx$$

Let $T : V \to V$ given by $(Tf)(x) = xf(x)$ and $U : V \to V$ given by $(Uf)(x) = \frac{d}{dx} f(x)$. Show that ALL the hypothesis of part (a) are satisfied. Conclude that the inequality

$$\|T\phi\|^2 \cdot \|U\phi\|^2 \geq \frac{1}{4}, \qquad \text{for all } \phi \in V, \quad \|\phi\|^2 = 1,$$

holds. This is known as the Heisenberg Uncertainty Principle, which states that it is impossible to get both the position error and the momentum error both very small at the same time.