Definition V, W two vector spaces.
We say V is isomorphic to W if there exists an invertible linear transformation $T: V \to W$. Such a linear transformation is called an isomorphism between V and W

Theorem 2.19 V, W be finite dimensional.
Then V is isomorphic to W iff $\dim V = \dim W$

Consequence Any n-dimensional vector space is isomorphic to $\mathbb{F}^n$

Proof of 2.19 "$\Rightarrow$" V isomorphic to W $\Rightarrow$ There exists $T: V \to W$ invertible

for any basis β in V, $T(\beta)$ is a basis for W
[β = lin ind $\Rightarrow$ $T(\beta)$ lin ind since T is 1-1 so $N(T) = 0$]
β = span V $\Rightarrow$ $T(\beta)$ spans $R(T) = W$ since T is onto

Moreover $\underset{\dim V}{|\beta|} = \underset{\dim W}{|T(\beta)|}$ since T is 1-1 so $\dim V = \dim W$

"$\Leftarrow$" If $\dim V = \dim W = n$, let $\{v_1, \dots, v_n\}$ be a (any) basis of V and $\{w_1, \dots, w_n\}$ basis of W

Construct $T: V \to W$ by $T(v_j) = w_j$ and then extend it to V by linearity.

One shows T is one-to-one ($N(T)=0$) and T is onto ($R(T)=W$) or, alternately, one constructs a linear tr. $U: W \to V$ such that $UT = I_V$, $TU = I_W$ by $U(w_j) = v_j$ and extend it by linearity. This means T is invertible, hence an isomorphism. ∎

Let V = n-dimensional vector space, $\beta = \{v_1, \dots, v_n\}$ ordered basis in V.

$\phi_\beta : V \to \mathbb{F}^n$, $\phi_\beta(v_i) = e_i$ and extend by linearity; $\{e_1, \dots, e_n\}$ standard ordered basis in V

is an isomorphism $\Rightarrow$ $\phi_\beta(v) = [v]_\beta$ (coordinate vector!) is called the standard representation of V with respect to β

V, W two finite dim vector spaces $\dim V = n$, $\dim W = m$

β = basis for V

γ = basis for W

Then

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ \mathbb{F}^n \arrow[r, dashed, "L_A"] & \mathbb{F}^m \end{tikzcd}$$

$A = [T]_\beta^\gamma$

$T = \phi_\gamma^{-1} L_A \phi_\beta$ or $\phi_\gamma T = L_A \phi_\beta$

Change of Bases

Matrix representation of a linear transformation $T: V \to V$ depends on the basis chosen. How does this matrix change when we change the basis?

Say β is a basis for V, $\beta = \{v_1, \dots, v_n\}$.

Recall $[T]_\beta = (a_{ij})$ where $Tv_j = \sum_{i=1}^{n} a_{ij} v_i$, $j = 1, \dots, n$

Now let β' a different basis for V $\beta' = \{v_1', \dots, v_n'\}$

<u>Theorem</u>: Given two bases β, β' for V, there exists an invertible matrix $Q \in M_{n \times n}$ such that

$$[T]_{\beta'} = Q^{-1} [T]_\beta Q$$

<u>Proof</u>: Let $\beta = \{v_1, \dots, v_n\}$ and $\beta' = \{v_1', \dots, v_n'\}$

Then for each $j = \overline{1,n}$, $v_j' = \sum_{i=1}^{n} v_i a_{ij}$

or in more original notation $[v_1' \dots v_n'] = [v_1 \dots v_n] \begin{bmatrix} a_{11} & a_{1j} & a_{1n} \\ \vdots & \cdots \vdots & \cdots \vdots \\ a_{n1} & a_{nj} & a_{nn} \end{bmatrix}$

"row vector" row vector

We define $Q = [a_{ij}]$

Notice that $Q = [I_V]_\beta^{\beta'}$ and is invertible since $I_V : V \to V$ is the identity map

Now we prove $Q[T]_{\beta'} = [T]_\beta Q$.

Indeed $Q[T]_{\beta'} = [I_V]_\beta^{\beta'} [T]_{\beta'} = [I_V T]_\beta^{\beta'} = [T I_V]_\beta^{\beta'} = [T]_\beta [I_V]_\beta^{\beta'} = [T]_\beta Q$

Since Q^{-1} exist $\Rightarrow$ $\boxed{[T]_{\beta'} = Q^{-1}[T]_\beta Q}$

$Q: \beta \to \beta'$ is called the change of bases matrix from β to β'.

Ex: $V = \mathbb{R}^2$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a-b \\ a+3b \end{pmatrix}$

$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \Rightarrow [T]_\alpha = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$

$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \Rightarrow [T]_\beta = ?$

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} a+b=2 \\ a-b=4 \end{cases} \Rightarrow \begin{matrix} a=3 \\ b=-1 \end{matrix}$$

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} c+d=4 \\ c-d=-2 \end{cases} \Rightarrow \begin{matrix} c=1 \\ d=3 \end{matrix}$$

So $[T]_\beta = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} \neq [T]_\alpha$

$\beta' = \left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$, $[T]_{\beta'} = ?$

$$T\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \end{pmatrix} = a\begin{pmatrix} 2 \\ 4 \end{pmatrix} + b\begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} a=4 \\ b=-2 \end{matrix}$$

$$T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = c\begin{pmatrix} 2 \\ 4 \end{pmatrix} + d\begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} c=1 \\ d=2 \end{matrix}$$

So $[T]_{\beta'} = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$

Now we show that $[T]_{\beta'} = Q^{-1}[T]_\beta Q$, where Q is the change of bases matrix from β to β'.

$\beta' = \left\{ \begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix} \right\} \Rightarrow \begin{pmatrix}2\\4\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\-1\end{pmatrix} \rightarrow \begin{matrix} a+b=2 \\ a-b=4 \end{matrix}$

$a = 3 \Rightarrow b = -1$

$\begin{pmatrix}3\\1\end{pmatrix} = c\begin{pmatrix}1\\1\end{pmatrix} + d\begin{pmatrix}1\\-1\end{pmatrix} \Rightarrow \begin{cases} c+d=3 \\ c-d=1 \end{cases}$

$\Rightarrow c = 2$, $d = 1$

$$\Rightarrow \boxed{Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}}$$

Now verify: $Q[T]_{\beta'} = [T]_\beta Q$

$$Q[T]_{\beta'} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ -6 & 1 \end{pmatrix}$$

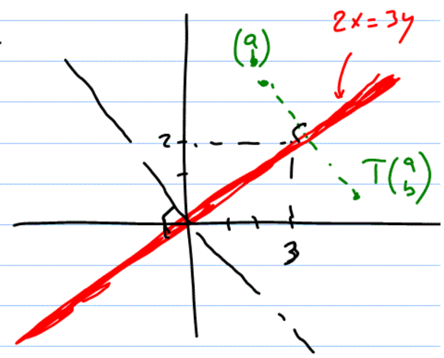
$$[T]_\beta Q = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ -6 & 1 \end{pmatrix}$$

match!

Ex: (Reflection about a line)

Consider the line $2x = 3y$ in $\mathbb{R}^2$ and define $T: \mathbb{R}^2 \to \mathbb{R}^2$ as in the figure

$T\begin{pmatrix}a\\b\end{pmatrix}$ is the reflection of the point $\begin{pmatrix}a\\b\end{pmatrix}$ relative to the line $2x = 3y$



First, T is linear since this reflection preserves vector addition (given by the parallelogram rule) and scalar multiplication.

To find explicitly $T\begin{pmatrix}a\\b\end{pmatrix}$, we consider the basis

$\beta = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\}$. In this basis $\left.\begin{array}{l} T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ T\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \end{array}\right\} \Rightarrow [T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Now let $\beta' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \Rightarrow [T]_{\beta'} = Q^{-1}[T]_\beta Q$

where $Q : \beta' \to \beta \Rightarrow Q = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$ $\qquad Q^{-1} = \frac{1}{13}\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$

So $$[T]_{\beta'} = \frac{1}{13}\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 3 & -2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}$$

So $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{5a-12b}{13} \\ \frac{-12a-5b}{13} \end{pmatrix}$.

Note: In the textbook, Q is called the change of coordinate matrix that changes β'-coordinates into β-coordinates. Here is why!

$$v \in V \Rightarrow v = a_1' v_1' + \dots + a_n' v_n' = [v_1' \dots v_n'] \begin{bmatrix} a_1' \\ \vdots \\ a_n' \end{bmatrix} = [v_1 \dots v_n] Q \begin{bmatrix} a_1' \\ \vdots \\ a_n' \end{bmatrix}$$

$$v = a_1 v_1 + \dots + a_n v_n = [v_1 \dots v_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$\Rightarrow \boxed{Q \begin{bmatrix} a_1' \\ \vdots \\ a_n' \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}$ or, in short $Q[v]^{\beta'} = [v]^{\beta}$.

(see Theorem 2.22)