$T: \mathbb{F}^n \to \mathbb{F}^n$, $\alpha =$ standard ordered basis for $\mathbb{F}^n$

$$[T]_\alpha = A \implies T = L_A$$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a-b \\ a+3b \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} \implies A = [T]_\alpha$

$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \implies [T]_\beta = Q^{-1}[T]_\alpha Q$ where Q is the change of basis matrix from α to β

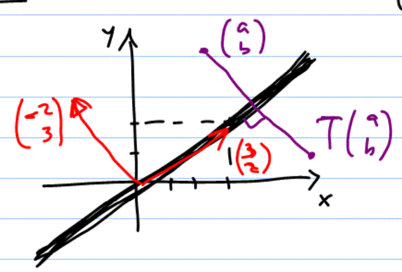
$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \begin{matrix} a = 1 \\ b = 1 \end{matrix}$

similar for

$\implies Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ columns of Q are exactly the vectors in β

$Q^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \implies [T]_\beta = Q^{-1}[T]_\alpha Q = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T\begin{pmatrix} a \\ b \end{pmatrix}$ is reflection of $\begin{pmatrix} a \\ b \end{pmatrix}$ about the line $3y = 2x$



$2x = 3y$

What is $T\begin{pmatrix} a \\ b \end{pmatrix}$?

Convenient basis for T

$\beta = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\} \implies [T]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Since $T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$T\begin{pmatrix} -2 \\ 3 \end{pmatrix} = -\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Since $[T]_\beta = Q^{-1}[T]_\alpha Q$

$\implies [T]_\alpha = Q[T]_\beta Q^{-1}$

where $\alpha \xrightarrow{Q} \beta$ $\quad Q = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$

$$Q^{-1} = \frac{1}{13}\begin{pmatrix}3&2\\-2&3\end{pmatrix} \Rightarrow \quad [T]_\alpha = \begin{pmatrix}3&-2\\2&3\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{1}{13}\begin{pmatrix}3&2\\-2&3\end{pmatrix} = \frac{1}{13}\begin{pmatrix}3&2\\2&-3\end{pmatrix}\begin{pmatrix}3&2\\-2&3\end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix}5&12\\12&-5\end{pmatrix}$$

So, for example, the reflection of $\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ is

$$T\begin{pmatrix}1\\2\end{pmatrix} = \frac{1}{13}\begin{pmatrix}5&12\\12&-5\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \frac{1}{13}\begin{pmatrix}29\\2\end{pmatrix}!$$

In general, if $\beta \xrightarrow{Q} \beta'$ (change of basis matrix from β to β')

$$[v_1' \dots v_n'] = [v_1 \dots v_n]\begin{bmatrix} Q \end{bmatrix}$$

How do coordinates of a vector $v \in V$ change when we

change the basis from β (old) to β' (new)?

$v \in V$, $[v]_\beta = \begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix} \Leftrightarrow v = a_1v_1 + \dots + a_nv_n$

$[v]_{\beta'} = \begin{pmatrix}a_1'\\ \vdots\\ a_n'\end{pmatrix} \Leftrightarrow v = a_1'v_1' + \dots + a_n'v_n'$

$$v = [v_1' \dots v_n']\begin{bmatrix}a_1'\\ \vdots\\ a_n'\end{bmatrix} = [v_1 \dots v_n]\, Q\begin{bmatrix}a_1'\\ \vdots\\ a_n'\end{bmatrix}$$

$$= [v_1 \dots v_n]\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}$$

$$\Longrightarrow \quad Q\begin{bmatrix}a_1'\\ \vdots\\ a_n'\end{bmatrix} = \begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix} \quad \text{or} \quad \boxed{Q[v]_{\beta'} = [v]_\beta}$$

Application: Let $\{(x,y) \mid 2x^2-4xy+5y^2=1\}$ be a conic in the plane

Draw it! (ellipse)

One can rewrite $1 = 2x^2-4xy+5y^2 = (x\ \ y)\overset{A}{\begin{pmatrix}2 & -2\\ -2 & 5\end{pmatrix}}\begin{pmatrix}x\\ y\end{pmatrix} = X^T A X$

If we make a change of basis from $\beta = \{(\begin{smallmatrix}1\\0\end{smallmatrix}), (\begin{smallmatrix}0\\1\end{smallmatrix})\}$ to β' with Q such that $Q^{-1} = Q^T$, then

$$1 = X^T A X = (QX')^T A\, QX' = (X')^T \underbrace{Q^T A Q}\, X'$$
$$= (X')^T \underbrace{Q^{-1} A Q}\, X$$

So if there is a change of basis such that $Q^{-1}AQ$ is as

simple as possible (diagonal, as we'll see in the next two chapters) then $1 = (X')^T D X' = d_{11}(x')^2 + d_{22}(y')^2$!!!

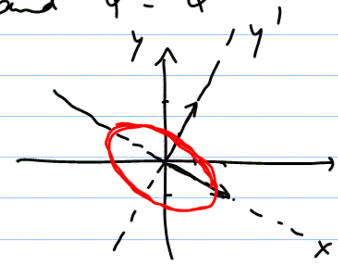
In the case $A = \begin{pmatrix}2 & -2\\ -2 & 5\end{pmatrix}$ there is such a change of basis

$$\beta' = \left\{ \begin{pmatrix}\frac{2}{\sqrt5}\\ -\frac{1}{\sqrt5}\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt5}\\ \frac{2}{\sqrt5}\end{pmatrix} \right\}$$ (don't ask why now!)

$\Rightarrow Q = \begin{pmatrix}\frac{2}{\sqrt5} & \frac{1}{\sqrt5}\\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5}\end{pmatrix}$ or $\frac{1}{\sqrt5}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}$ and $Q^{-1} = Q^T$

and $Q^{-1}AQ = \begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}$!!

$\Rightarrow \quad 1 = (x')^2 + 6(y')^2 \quad \Rightarrow$



Note: Given two basis β, β', the change of basis matrix Q (from β to β') can be computed using gaussian elimination

$$[\beta \,\vdots\, \beta'] \longrightarrow [I \,\vdots\, Q]$$

Ex: Reflect across $x+y+2z=0$ $\quad T: \mathbb{R}^3 \to \mathbb{R}^3$

$$\beta' = \{v_1', v_2', v_3'\} \qquad \underbrace{v_1' = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ v_2' = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{\text{in the plane } x+y+2z=0}, \ \underbrace{v_3' = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}_{\text{normal to the plane}}$$

$\beta = \{e_1, e_2, e_3\}$

$$T(v_1') = v_1', \ T(v_2') = v_2', \ T(v_3') = -v_3' \Rightarrow [T]_{\beta'} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$Q = \text{change of basis matrix } \beta \to \beta' \Rightarrow Q = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\Rightarrow [T]_{\beta'} = Q^{-1}[T]_\beta Q \Rightarrow [T]_\beta = Q[T]_{\beta'}Q^{-1} \quad \ldots$$

In $\mathbb{F}^n$, β, β' two basis: To compute $\beta \xrightarrow{Q} \beta'$

let α = standard ordered basis

$$\left.\begin{matrix} \alpha \xrightarrow{Q_1} \beta \\ \alpha \xrightarrow{Q_2} \beta' \end{matrix}\right] \Rightarrow \beta \xrightarrow{Q_1^{-1}} \alpha \xrightarrow{Q_2} \beta' \Rightarrow Q = Q_1^{-1}Q_2 \ !$$

(and not $Q_2 Q_1^{-1}$!)

Why?

$\boxed{3.2}$

Fact: If $P =$ invertible transformation, V, W, Z finite dim.

$$\operatorname{rank}(PT) = \operatorname{rank}(T)$$

$$V \xrightarrow{T} W \xrightarrow{P} Z$$

Indeed $P|_{R(T)} : R(T) \to R(PT)$ is one-to-one (since P is) and onto, since $R\left(P|_{R(T)}\right) = R(PT)$

$$\Rightarrow \dim\left(R(T)\right) = \dim\left(R(PT)\right)$$

Fact: If $Q =$ invertible transformation

$$\Rightarrow \operatorname{rank}(TQ) = \operatorname{rank} T$$

More generally.

$$\operatorname{rank}(UT) \leq \min\{\operatorname{rank}(U), \operatorname{rank}(T)\}$$

$$\operatorname{rank}(U+T) \leq \operatorname{rank}(U) + \operatorname{rank}(T)$$

$$W_1 = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_5 \end{pmatrix} \in \mathbb{F}^5 \;\middle|\; a_1 - a_3 - a_4 = 0 \right\}$$

$$\Rightarrow a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} a_3 + a_4 \\ 0 \\ a_3 \\ a_4 \\ 0 \end{pmatrix} = a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$L_A : \mathbb{F}^5 \to \mathbb{F}^1, \quad A = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow W_1 = N(L_A) \qquad \dim N(L_A) = 5 - \dim R(L_A) = 5 - 1 = 4 \;!$$