V = vector space (over a field F)

Def: A subset S in V is called <u>linearly dependent</u> if there exist a <u>finite number</u> of distinct vectors $u_1, u_2, \dots, u_n \in S$ and scalars $a_1, a_2, \dots, a_n$, <u>not all zero</u>, such that

$$a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0$$

We say that this is a <u>nontrivial</u> representation of the zero vector as linear combination of vectors in S.

Def: A subset S is called <u>linearly independent</u> if it is not linearly dependent.

More precisely, S is linearly independent if

$$\left.\begin{array}{c} a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0 \\ u_i \in S, \ i = \overline{1,n} \end{array}\right\} \text{ implies } a_1 = a_2 = \dots = a_n = 0$$

Ex: The set $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 8 \\ 16 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix} \right\}$ in $\mathbb{R}^3$

is linearly dependent since we can find a nontrivial linear comb of these five vectors which equals 0:

$$a_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + a_3 \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + a_4 \begin{pmatrix} -3 \\ 8 \\ 16 \end{pmatrix} + a_5 \begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since

$$\begin{bmatrix} 1 & 0 & 2 & -3 & 2 \\ 2 & 2 & 0 & 8 & 6 \\ 1 & 3 & -3 & 16 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & -2 & 3 \end{bmatrix}$$

so the homogeneous system has nonzero solutions, e.g.

$a_1 = -1,\ a_2 = -3,\ a_3 = 2,\ a_4 = 1,\ a_5 = 0$ (not unique solution)

<u>Ex:</u> The set $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is linearly independent (in $\mathbb{R}^4$)

Indeed, $a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{array}$

<u>Theorem 1.6</u> Let V = vector space, $S_1 \subseteq S_2 \subseteq V$.
If S_1 is linearly dependent, then S_2 is linearly dependent

<u>Theorem 1.7</u> Let V = vector space $S_1 \subseteq S_2 \subseteq V$
If S_2 is linearly independent, then S_1 is linearly independent

<u>Note:</u> Lin dependence and lin independence are <u>"hereditary"</u> in the sense given above.

The important question is: can one 'enlarge' a lin. ind set and still have lin ind set? Also, can one <u>"shrink"</u> a lin dependent set and still be lin dependent ?

Answer is no, in general.

Theorem 1.7 Let S be lin independent subset of V and $v \in V$ which is not in S. Then $S \cup \{v\}$ is linearly dependent iff $v \in \operatorname{span}(S)$

Proof: ($\Rightarrow$) Assume $S \cup \{v\}$ is lin dependent and show $v \in \operatorname{span}(S)$

Since $S \cup \{v\}$ is lin dep $\Rightarrow \exists\, u_1, \dots, u_n$ and $v \in S \cup \{v\}$ and scalars $a_1, \dots, a_n, a_{n+1}$, not all zero, such that

$$a_1 u_1 + \dots + a_n u_n + a_{n+1} v = 0$$

we claim that $a_{n+1} \neq 0$ (otherwise $a_1 u_1 + \dots + a_n u_n = 0 \xRightarrow{S \text{ lin ind}}$ $\Rightarrow a_1 = \dots = a_n = 0$ Contradiction)

$\Rightarrow$ we can write v as

$$v = -a_1 a_{n+1}^{-1} u_1 - \dots - a_n a_{n+1}^{-1} u_n \in \operatorname{span}(V)$$

($\Leftarrow$) Assume $v \in \operatorname{span}(V) \Rightarrow v = b_1 u_1 + \dots + b_n u_n,\ u_i \in S$

$\Rightarrow \quad b_1 u_1 + \dots + b_n u_n + (-1) v = 0 \quad \Rightarrow S \cup \{v\}$ lin dep

$\blacksquare$

Note:

$\{v\}$ is linearly independent iff $v \neq 0$.

Indeed, $av = 0$, $a \neq 0 \Rightarrow v = 0$.

Note: Any set S containing 0 is linearly dependent since $1 \cdot 0 = 0$ is a nontrivial linear combination.

Note: Theorem 1.5 also shows that a linearly independent set which is generating V is "maximal" among lin ind sets, in the sense that it cannot be enlarged while still being linearly independent

Vice versa, a generating set which is linearly independent is minimal among generating sets, in the sense that it cannot be "shrunk" while still being generating set

Both these statements lead to the notion of a basis for V (which is any linearly independent and generating set for V)