SAMPLE EXAM I – Linear Algebra I
Math 413/513 FALL 2010

You may use any theorems which were proved in class or in the textbook provided that you state them clearly. You may not use results which were proved as part of your homework unless you prove them again here.

- **Problem 1**
Let V be the subspace of R^4 generated by the vectors $v_1 = (1, 1, 0, 0), v_2 = (0, 1, 1, 0), v_3 = (0, 0, 1, 1)$ and W generated by the vectors $w_1 = (1, 0, 1, 0), w_2 = (0, 2, 1, 1), w_3 = (1, 2, 1, 2)$ in $\mathbb{R}^4$.

(a) Determine the dimensions of V and W.
(b) Find a basis for the sum $V + W$.
(c) Find a basis for the intersection $V \cap W$. Verify that $\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W)$.

- **Problem 2**
Given a vector space V, show that when two finite dimensional subspaces W_1 and W_2 satisfy
$$\dim(W_1 + W_2) = \dim(W_1 \cap W_2) + 1$$
then either $W_1 \subset W_2$ or $W_2 \subset W_1$ and $|\dim(W_1) - \dim(W_2)| = 1$.

- **Problem 3**
Consider a linear transformation $T : V \to V$ over a finite dimensional vector space V with $\dim(V) = n$. Let $v \in V$ be a given vector such that $T^n(v) = 0$, but $T^{n-1}(v) \neq 0$.

(a) Prove that the set $\beta = \{v, T(v), T^2(v), \dots T^{n-1}(v)\}$ forms a basis for V.
(b) Find a basis for the null-space $N(T)$ and the range $R(T)$ of T and verify the dimension theorem.
(c) Calculate $[T]_\beta$.

- **Problem 4**
Let V be a finite dimensional vector space.

(a) Show that for any given subspace W_1 of V, there exists a subspace W_2 of V such that $W_1 \oplus W_2 = V$
(b) Let $T : V \to V$ be the projection on W_1 along W_2, where W_1 and W_2 are as in part (a). Show that $T^2 = T$.
(c) Conversely, show that any linear transformation $T : V \to V$ satisfying $T^2 = T$ is a projection on $R(T)$ along $N(T)$.

- **Problem 5**
Given a 2×2 matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the map $T : \mathcal{M}_{2\times2}(\mathbb{R}) \to \mathcal{M}_{2\times2}(\mathbb{R})$ given by left multiplication by P:
$$T(A) = PA, \quad \text{for } A \in \mathcal{M}_{2\times2}(\mathbb{R}).$$
(a) Show that T is linear.
(b) Determine the matrix representation of T in the standard ordered basis for $\mathcal{M}_{2\times2}(\mathbb{R})$
$$\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

- **Problem 6**
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as the reflection about the plane $x + y + 2z = 0$.

(a) Explain in a few words why T is a linear transformation.
(b) Find an expression for $T(a, b, c)$, for any $(a, b, c) \in \mathbb{R}^3$. [Hint: Find a convenient basis β' in which $[T]_{\beta'}$ is easy to be computed, then use the change of bases formula to compute $[T]_\beta$, where β is the standard ordered basis in $\mathbb{R}^3$.]

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- **Problem 7***

  (a) Given a vector space V, show that the union of an increasing sequence of subspaces of V is a subspace of V.

  (b) Let V be the vector space of infinite sequences of real numbers that converge to 0 and T be the left shift operator on V (see exercise 21, page 76). Show that W, the subset of sequences that have only finitely many nonzero terms, satisfies

$$W = \bigcup_{k \geq 1} N(T^k)$$

  and deduce that W is a subspace of V, which is also invariant under T. Determine a basis for W.