SOLUTIONS TO SAMPLE EXAM 1

Math 413/513 - FALL 2010

- **Problem 1**

Let V be the subspace of R^4 generated by the vectors $v_1 = (1,1,0,0), v_2 = (0,1,1,0), v_3 = (0,0,1,1)$ and W generated by the vectors $w_1 = (1,0,1,0), w_2 = (0,2,1,1), w_3 = (1,2,1,2)$ in $\mathbb{R}^4$.

(a) Determine the dimensions of V and W.

Solution. $\dim(V) = 3$ since $V = \text{ span}\{v_1, v_2, v_3\}$ and the vectors v_1, v_2, v_3 are lin. independent. Indeed, $a_1v_1 + a_2v_2 + a_3v_3 = 0 \Rightarrow a_1(1,1,0,0) + a_2(0,1,1,0) + a_3(0,0,1,1) = (0,0,0,0) \Rightarrow (a_1, a_1 + a_2, a_2 + a_3, a_3) = (0,0,0,0) \Rightarrow a_1 = 0, a_2 = 0, a_3 = 0$.

Same argument for W: $\dim(W) = 3$ since w_1, w_2, w_3 are lin. ind. (a similar argument is needed!)

(b) Find a basis for the sum $V + W$.

Solution. Since $\{v_1, v_2, v_3\}$ is lin. ind. and $w_1 \notin \text{span}\{v_1, v_2, v_3\}$ (This needs proof!) then, by Theorem 1.7, $\{v_1, v_2, v_3, w_1\}$ is lin. ind. in $\mathbb{R}^4$. Hence $\text{span}\{v_1, v_2, v_3, w_1\} = \mathbb{R}^4$. Now $\{v_1, v_2, v_3, w_1\} \subset V + W$ means that $V + W = \mathbb{R}^4$. We conclude that $\{v_1, v_2, v_3, w_1\}$ is a basis for $V + W$.

(c) Find a basis for the intersection $V \cap W$. Verify that $\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W)$.

Solution. For a vector v to belong to $V \cap W$, it is necessary and sufficient to have v as a linear combination of both bases $\{v_1, v_2, v_3\}$ (for V) and $\{w_1, w_2, w_3\}$ for W:

$$v = a_1v_1 + a_2v_2 + a_3v_3 = b_1w_1 + b_2w_2 + b_3w_3, \quad \text{for } a_i, b_i \in \mathbb{R}, i = 1,2,3.$$

Hence a's and b's are solutions to the underdetermined linear system

$$a_1 \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + a_2 \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + a_3 \begin{pmatrix}0\\0\\1\\1\end{pmatrix} = b_1 \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + b_2 \begin{pmatrix}0\\2\\1\\1\end{pmatrix} + b_3 \begin{pmatrix}1\\2\\1\\2\end{pmatrix}$$

Performing reduced row echelon form on the matrix $\begin{pmatrix}1&0&0&1&0&1\\1&1&0&0&2&2\\0&1&1&1&1&1\\0&0&1&0&1&2\end{pmatrix}$ yields: $\begin{pmatrix}1&0&0&0&1&2\\0&1&0&0&1&0\\0&0&1&0&1&2\\0&0&0&1&-1&-1\end{pmatrix}$

hence the system above is equivalent to

$$a_1 \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + a_2 \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + a_3 \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = b_1 \begin{pmatrix}0\\0\\0\\1\end{pmatrix} + b_2 \begin{pmatrix}1\\1\\1\\-1\end{pmatrix} + b_3 \begin{pmatrix}2\\0\\2\\-1\end{pmatrix},$$

or $a_1 = b_2 + 2b_3, a_2 = b_2, a_3 = b_2 + 2b_3, 0 = b_1 - b_2 - b_3$. As a consequence, we can choose arbitrarily $b_2 = s$ and $b_3 = t$, then $a_1 = s + 2t, a_2 = s, a_3 = s + 2t, b_1 = s + t$.
Then every $v \in V \cap W$ can be expressed as $v = (s + 2t)v_1 + sv_2 + (s + 2t)v_3 = (s + t)w_1 + sw_2 + tw_3$. or $v = s(v_1 + v_2 + v_3) + t(2v_1 + 2v_3) = s(w_1 + w_2) + t(w_1 + w_3)$. It means that a basis for $V \cap W$ consists of the two vectors $v_1 + v_2 + v_3 = w_1 + w_2 = (1,2,2,1)$ and $2v_1 + 2v_3 = w_1 + w_3 = (2,2,2,2)$.

One verifies that $\dim(V + W) + \dim(V \cap W) = 4 + 2 = 6 = 3 + 3 = \dim(V) + \dim(W)$. Q.E.D.

- **Problem 2**
  Given a vector space V, show that when two finite dimensional subspaces W_1 and W_2 satisfy
  $$\dim(W_1 + W_2) = \dim(W_1 \cap W_2) + 1$$
  then either $W_1 \subset W_2$ or $W_2 \subset W_1$ and $|\dim(W_1) - \dim(W_2)| = 1$.

  Solution. Denote $n = \dim(W_1 + W_2)$. We distinguish two cases:

  Case 1. If $W_1 \subseteq W_2$ then obviously $W_1 \cap W_2 = W_1$ and $W_1 + W_2 = W_2$, $\dim(W_1) = n-1$ and $\dim(W_2) = n$.

  Case 2. If $W_1 \nsubseteq W_2$, then there exists $w \in W_1 \setminus W_2$. Let $\{v_1, v_2, \dots v_{n-1}\}$ be a basis for $W_1 \cap W_2$. We claim that $\{v_1, v_2, \dots v_{n-1}, w\}$ is then a basis in $W_1 + W_2$. Indeed, by Theorem 1.7, $\{v_1, v_2, \dots v_{n-1}, w\}$ is lin. independent and it contains exactly n elements, therefore it generates (is a basis for) $W_1 + W_2$. This means $W_1 = W_1 + W_2$, $\dim(W_1) = n$ and $\dim(W_2) = n - 1$. Q.E.D.

- **Problem 3**
  Consider a linear transformation $T : V \to V$ over a finite dimensional vector space V with $\dim(V) = n$. Let $v \in V$ be a given vector such that $T^n(v) = 0$, but $T^{n-1}(v) \neq 0$.

  (a) Prove that the set $\beta = \{v, T(v), T^2(v), \dots T^{n-1}(v)\}$ forms a basis for V.

  Solution. Since the set contains exactly $n = \dim(V)$ elements, to show it is a basis for V it is sufficient to show that it is a linear independent set in V. Assume
  $$a_0 v + a_1 T(v) + a_2 T^2(v) + \dots a_{n-1} T^{n-1}(v) = 0.$$
  Applying T^{n-1} to both sides and using linearity and the fact that $T^n(v) = 0$, we obtain $a_0 T^{n-1}(v) = 0$, and since $T^{n-1}(v) \neq 0$, we conclude $a_0 = 0$. Hence
  $$a_1 T(v) + a_2 T^2(v) + \dots a_{n-1} T^{n-1}(v) = 0.$$
  We continue by applying T^{n-2} to both sides, to obtain $a_1 T^{n-1}(v) = 0$, hence $a_1 = 0$, etc. After n iterations of this procedure, we obtain $a_0 = a_1 = a_2 = \dots = a_{n-1} = 0$. Q.E.D.

  (b) Find a basis for the null-space $N(T)$ and the range $R(T)$ of T and verify the dimension theorem.

  Solution. Since $T^n(v) = 0$ it means $T^{n-1}(v) \in N(T)$. Also, it is clear that $\{T(v), T^2(v), \dots, T^{n-1}(v)\} \subset R(T)$. Based on these observations, we claim that
  $$N(T) = \text{ span}\{T^{n-1}(v)\} \text{ and } R(T) = \text{ span}\{T(v), T^2(v), \dots, T^{n-1}(v)\}$$
  Indeed, by the inclusions above it means that $\dim(N(T) \geq 1$ and $\dim(R(T)) \geq n - 1$. But the dimension theorem guarantees that $\dim(N(T)) + \dim(R(T)) = n$, hence $\dim(N(T) = 1$ and $\dim(R(T)) = n - 1$. Thus the claim is proved.

  (c) Calculate $[T]_\beta$.

  Solution. It is easy to construct the matrix representation of T in the basis β, since $T(v) = 0 \cdot v + 1 \cdot T(v) + \dots + 0 \cdot T^{n-1}(v)$ etc. We obtain:
  $$[T]_\beta = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

- **Problem 4**
Let V be a finite dimensional vector space.

(a) Show that for any given subspace W_1 of V, there exists a subspace W_2 of V such that $W_1 \oplus W_2 = V$

Solution. Since V is finite dimensional ,say $\dim(V) = n$, so is W_1. Denote $\dim(W_1) = k$. Let $\beta_1 = \{v_1, v_2, \dots, v_k\}$. be a basis for W_1. Then we can complete β_1 to a basis $\beta = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V. We claim that $W_2 = \operatorname{span}\{v_{k+1}, \dots, v_n\}$ is a subspace in V satisfying $W_1 \oplus W_2 = V$. Indeed, one easily verifies that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$, using the properties of the basis β.

(b) Let $T : V \to V$ be the projection on W_1 along W_2, where W_1 and W_2 are as in part (a). Show that $T^2 = T$.

Solution. Let $v \in V = W_1 \oplus W_2$, with the unique representation $v = v_1 + v_2$, with $v_1 \in W_1, v_2 \in W_2$. Then $T^2(v) = T(T(v)) = T(v_1) = v_1$, the last equality since $v_1 = v_1 + 0$ is the unique representation of $v_1 \in W_1$ as a sum of elementsl in W_1 and W_2. Hence $T^2(v) = T(v)$. Since v was arbirary in V, it follows that $T^2 = T$.

(c) Conversely, show that any linear transformation $T : V \to V$ satisfying $T^2 = T$ is a projection on $R(T)$ along $N(T)$.

Solution.
Consider T a linear transformation satisfying $T^2 = T$. Denote $W_1 = R(T)$ and $W_2 = N(T)$. We claim that $N(T) \oplus R(T) = V$ and T is the projection onto W_1 along W_2.

Indeed, for $v \in V$, write $v = T(v) + v - T(v)$. We have $T(v) \in R(T)$ and $v - T(v) \in N(T)$, since $T(v - T(v)) = T(v) - T^2(v) = T(v) - T(v) = 0$. Hence $V = W_1 + W_2$.
To show $W_1 \cap W_2 = \{0\}$, let $v \in W_1 \cap W_2$. It implies that $v = T(w)$ for some $W \in V$ and $T(v) = 0$. Then $0 = T(v) = T^2(w) = T(w) = v$, hence $v = 0$.

Using the unique representation of any v in $W_1 \oplus W_2$ as $v = T(v) + (v - T(v))$, it is clear that $T(v) = T(v)$ implies that T acts on the direct sum as the projection on W_1 along W_2.

- **Problem 5**
Given a 2×2 matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the map $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$ given by left multiplication by P:

$$T(A) = PA, \qquad \textit{for } A \in \mathcal{M}_{2\times 2}(\mathbb{R}).$$

(a) Show that T is linear.

Solution. We easily verify $T(A_1 + A_2) = P(A_1 + A_2) = PA_1 + PA_2 = T(A_1) + T(A_2)$ and $T(cA) = PcA = cPA = cT(A)$ for $c \in \mathbb{R}$, by using the properties of matrix addition and multiplication.

(b) Determine the matrix representation of T in the standard ordered basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

Solution. We compute $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ etc and therefore obtain

$$[T] = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

- **Problem 6**
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as the reflection about the plane $x + y + 2z = 0$.
(a) Explain in a few words why T is a linear transformation.

Solution. Addition of vectors obeys the parallelogram rule, hence whether it is performed first between two vectors v, w and then reflected about the plane, or performed directly on the mirror of v and w yields the same result. Same with scalar multiplication.

(b) Find an expression for $T(a,b,c)$, for any $(a,b,c) \in \mathbb{R}^3$.
[Hint: Find a convenient basis β' in which $[T]_{\beta'}$ is easy to be computed, then use the change of bases formula to compute $[T]_\beta$, where β is the standard ordered basis in $\mathbb{R}^3$.]

Solution. We pick a convenient basis (with 3 vectors) in $\mathbb{R}^3$ so that two vectors belong to the plane $x+y+2z=0$, say $v_1=(2,0,-1)$ and $v_2=(0,2,-1)$ and a third vector is perpendicular to the plane, say $v_3=(1,1,2)$. Then $T(v_1)=v_1, T(v_2)=v_2$, and $T(v_3)=-v_3$. Therefore, in the ordered basis $\beta'=\{v_1,v_2,v_3\}$, T is represented by the matrix:

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Denoting my β the standard ordered basis in $\mathbb{R}^3$, we know the change of bases matrix from β to β' is $Q = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}$. We compute $Q^{-1} = \frac{1}{12}\begin{pmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ 2 & 2 & 4 \end{pmatrix}$ use the relation $[T]_{\beta'} = Q^{-1}[T]_\beta Q$ to obtain

$$[T]_\beta = Q[T]_{\beta'}Q^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & -1 \end{pmatrix}.$$

We conclude that $T(a,b,c) = \frac{1}{3}(2a-b-2c, -a+2b-2c, -2a-2b-c)$ is the expression for this transformation (in the standard coordinate system).

- **Problem 7***

(a) Given a vector space V, show that the union of an increasing sequence of subspaces of V is a subspace of V.

Solution. Let $W_1 \subseteq W_2 \subseteq W_3 \subseteq \ldots \subseteq W_k \subseteq W_{k+1} \subseteq \ldots$ be an increasing sequence of subspaces of V. Denote $W = \bigcup_{k \geq 1} W_k$. We show W is a subspace by showing that it is closed under addition and scalar multiplication. Indeed, let c be a scalar, $u, v \in W$, then there exists $k \geq 1$ such that $u, v \in W_k$. But since W_k is subspace, $u + cv \in W_k$ as well, hence $u + cv \in W$.

(b) Let V be the vector space of infinite sequences of real numbers that converge to 0 and T be the left shift operator on V (see exercise 21, page 76). Show that W, the subset of sequences that have only finitely many nonzero terms, satisfies

$$W = \bigcup_{k \geq 1} N(T^k)$$

and deduce that W is a subspace of V, which is also invariant under T. Determine a basis for W.

Solution. By the definition of T, $N(T)$ = set of all sequences that have all terms zero (except possibly the first term):

$$N(T) = \{\mathbf{a} = (a_1, 0, 0, \ldots) | a_1 \in \mathbb{R}\}.$$

Similarly,

$$N(T^2) = \{\mathbf{a} = (a_1, a_2, 0, \ldots) | a_1, a_2 \in \mathbb{R}\}.$$

and more generally

$$N(T^k) = \{\mathbf{a} = (a_j) | a_j \in \mathbb{R}, a_j = 0 \text{ whenever } j > k\}.$$

Clearly $N(T) \subset N(T^2) \subset \ldots$. (This is true in fact for ANY linear transformation $T : V \to V$), and, moreover W, being the set of all sequences that have only finitely many nonzero terms, equals their union. By part (a), W is a subspace of V.

W is invariant under T since the action of T does not increase the number of nonzero terms in a sequence (on the contrary, it may decrease the number of nonzero terms.) More precisely,

$$T(N(T^k)) = N(T^{k-1}), \text{ for all } k, \quad \Rightarrow T(W) = T(\bigcup_{k\geq 1} N(T^k)) = \bigcup_{k \geq 1} N(T^{k-1}) = W.$$

A basis for W consists of the infinite set $\{e_1, e_2, \ldots\}$, where e_k is the sequence that has zero everywhere except the k^{th} term, which is equal to 1: e.g. $e_1 = (1,0,0,\ldots), e_2 = (0,1,0,\ldots)$. (Show this is basis for W!)