SOLUTIONS

Math 413/513 Fall 2010 STUDENT NAME: ______

EXAM I – Linear Algebra I

Part I – In Class

You may use any theorems which were proved in class or in the textbook provided that you state them clearly. You may not use results which were proved as part of your homework unless you prove them again here.

Part I (60%) - In class: Oct 6, 2010 Part II (40%)- Take Home: Due ~~Monday~~ Wed, Oct ~~11~~ 13, 2010 at 4:30pm

- **Problem 1**
  For each of the following statements, give either a short proof (if the statement is true) or a counterexample (if the statement is false)

  – If S_1 and S_2 are subsets of a vector space V then

  $$\text{span}(S_1) \cap \text{span}(S_2) \subseteq \text{span}(S_1 \cap S_2)$$

$\boxed{\text{FALSE}}$: Take two bases S_1, S_2 for $\mathbb{R}^2$ such that $S_1 \cap S_2 = \phi$

Then $\text{span}(S_1) \cap \text{span}(S_2) = \mathbb{R}^2$ but $\text{span}(S_1 \cap S_2) = \text{span}\,\phi = \{0\}$.

[The opposite inclusion $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$ is always true!]

  – The set $\{A \in M_{n\times n}(\mathbb{F}) | A^2 = 0\}$ is a subspace of $M_{n\times n}(\mathbb{F})$.

$\boxed{\text{FALSE}}$ If A, B are such that $A^2=0$, $B^2=0$, it does not necessarily imply $(A+B)^2=0$. Indeed $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 = AB + BA$.

Example: $n=2$ $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A^2 = 0 = B^2$ but $(A+B)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I \neq$

  – The map $T : P(\mathbb{R}) \to P(\mathbb{R})$, which maps a polynomial $p(x)$ to $p(x^2)$, is linear.
  [e.g. $T(1 + 3x + x^2) = 1 + 3x^2 + (x^2)^2 = 1 + 3x^2 + x^4$] .

$\boxed{\text{TRUE}}$ For $p, g \in P(\mathbb{R}) \Rightarrow T(cp+g)(x) = (cp+g)(x^2) = cp(x^2) + g(x^2)$

$= cT(p)(x) + T(g)(x)$.

  – The vector spaces $\mathbb{F}^4$ and $M_{2\times 2}(\mathbb{F})$ are isomorphic.

$\boxed{\text{TRUE}}$ Both spaces have dimension $4 \Rightarrow$ the two spaces are isomorphic

An isomorphism is, for example

$\phi : \mathbb{F}^4 \to M_{2\times2}(\mathbb{F})$, $\Phi\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Φ is linear one-to-one onto !

- **Problem 2** Find a basis for the following subspace of $\mathbb{R}^5$:

  $$W = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_1 = a_3 = a_4 \text{ and } a_2 + a_5 = 0\}.$$

  What is the dimension of W?

Clearly, we can choose $a_4 = t$, $a_5 = s \Rightarrow a_1 = a_3 = t$ & $a_2 = -s$

$\Rightarrow W = \{(t, -s, t, t, s) \in \mathbb{R}^5 \mid t, s \in \mathbb{R}\}$

$= \{t(1,0,1,1,0) + s(0,-1,0,0,1) \mid t, s \in \mathbb{R}\} = \text{span}(\beta)$

where $\beta = \{(1,0,1,1,0), (0,-1,0,0,1)\}$ is a basis for W.

$\Rightarrow \dim W = 2$.

- **Problem 3**
  Consider the map $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ given by $T(f)(x) = \int_0^x f(t)\,dt$.
  Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, x, x^2, x^3\}$ be the standard ordered bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$.

  (a) Compute $[T]_\beta^\gamma$

  (b) What is the rank of T?

(a) $\left.\begin{array}{l} T(1) = x \\ T(x) = \frac{1}{2}x^2 \\ T(x^2) = \frac{1}{3}x^3 \end{array}\right\} \Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}_{4\times3}$

(b) $\text{rank}(T) = \dim R(T)$

But $R(T) = \{f \in P_3(\mathbb{R}) \mid f(x) = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\}$

$= \text{span}\{x, \frac{1}{2}x^2, \frac{1}{3}x^3\}$

$\Rightarrow \dim R(T) = 3 \Rightarrow \text{rank}(T) = 3$.

- **Problem 3**

Let $T : V \to W$ be a linear transformation between two vector spaces V and W.
(*Math 413 students may assume V and W are finite dimensional, whereas Math 513 students need to proceed without this assumption*)
(a) Show that if G is a generating set for V, then $T(G)$ is generating set for $R(T)$.

Let $G \subset V$ such that $\mathrm{span}(G) = V$. Let $w \in R(T) \Rightarrow \exists\, v \in V : Tv = w$. $\Big] \Rightarrow v = \sum_{i=1}^{n} a_i v_i$ for some n, $v_1, \dots, v_n \in G$

$\Rightarrow w = Tv = T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i T v_i \in \mathrm{span}(T(G))$

so $R(T) \subseteq \mathrm{span}\, T(G))$. Conversely $w \in \mathrm{span}\, T(G) \Rightarrow w = \sum_{i=1}^{n} a_i T v_i$

$\Rightarrow w = T\left(\sum a_i v_i\right) \in R(T)$ so $\mathrm{span}(T(G)) \subseteq R(T)$. In conclusion $\mathrm{span}(T(G)) = R(T)$

(b) Show that if L is a linearly independent set in $R(T)$, then there exist a linearly independent set $\bar{L}$ in V such that $T(\bar{L}) = L$.

Let L = lin ind set in $R(T)$. For each $w \in L$, there exists $v \in V$ such that $Tv = w$. The collection of such v's forms a set $\tilde{L}$ in V. To show $\tilde{L}$ is lin independent, consider

$a_1 v_1 + \dots + a_n v_n = 0$ [for $v_1, \dots, v_n \in \tilde{L}$, $a_1 \dots a_n \in F$] $\Rightarrow T(a_1 v_1 + \dots + a_n v_n) = 0$

$\Rightarrow a_1 T v_1 + \dots + a_n T v_n = 0 \Rightarrow a_1 w_1 + \dots + a_n w_n = 0 \overset{L = \text{lin ind}}{\Rightarrow} a_1 = \dots = a_n = 0$.

(c) Show that T is invertible if and only if, for every basis β in V, $T(\beta)$ is basis in W.

($\Rightarrow$) Let T = invertible $\Rightarrow R(T) = W$, $N(T) = \{0\}$

Let β = basis in $V \Rightarrow T(\beta)$ is generating set for $R(T) = W$ (by (a)) and it is also lin independent (by (b) applied to $T^{-1} : W \to V$)

so $T(\beta)$ is a basis.

($\Leftarrow$) Assume any basis β for V satisfies $T(\beta)$ is basis in W

Then we can construct an inverse to T by

$U : T(\beta) \longrightarrow \beta$, $UT(v) = v$ and $TU(w) = w$ for $v \in \beta$, $w \in T(\beta)$

and extend it by linearity (since $T(\beta)$ is a basis for W)

to $U : W \to V$, $UT(v) = v$ and $TU(w) = w$.

- **Problem 4**

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a - b \\ a + b \end{pmatrix}.$$

Find $[T]_\beta$, the matrix representation of T in the nonstandard basis $\beta = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\}$.

Method 1 (direct)

$$\left.\begin{aligned} T\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ T\begin{pmatrix} 2 \\ 0 \end{pmatrix} &= \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} \end{aligned}\right] \Rightarrow [T]_\beta = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$

Method 2 (using change of bases)

Let $\alpha = \{e_1, e_2\}$ standard ordered basis $\Rightarrow [T]_\alpha = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

The change of basis matrix from α to β $Q = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \Rightarrow Q^{-1} = \frac{1}{-2}\begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$\Rightarrow [T]_\beta = Q^{-1}[T]_\alpha Q = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$

- **Problem 5**

Let A be an $m \times n$ matrix with rank m. Prove than there exists an $n \times m$ matrix B such that $AB = I_m$. Does it imply that $BA = I_n$? Explain.
[Hint: Consider the left multiplication transformation $T = L_A$].

If $A_{m \times n}$ has rank $m \Rightarrow L_A : F^n \to F^m$ has $R(L_A) = F^m$

$\Rightarrow L_A$ is onto. It is enough to construct $L_B : F^m \to F^n$ such that $L_A L_B = I_{F^m}$. Indeed, define it on a basis for F^m

$L_B e_j = w_j$, where $w_j \in F^n$ in such a way that

$L_A w_j = e_j$ (this is possible since L_A is onto!)

On the basis vectors

$L_A L_B e_j = L_A w_j = e_j \Rightarrow L_A L_B = I_{F^m}$.

No, it does not imply $BA = I_n$ (unless $m = n$).

If $m < n \Rightarrow L_A$ cannot be one-to-one!

SOLUTIONS

Math 413/513 FALL 2010 STUDENT NAME:

EXAM I – Linear Algebra I

Part II – Take Home

DUE Wed, Oct 13 at 4:30pm

You may use any theorems which were proved in class or in the textbook provided that you state them clearly. You may not use results which were proved as part of your homework unless you prove them again here.

ABSOLUTELY NO TEAM WORK ALLOWED!!!

Math 513 students are required to answer all questions!
The starred (*) problems are optional (extra-credit) for Math 413 students.

- **Problem 1**
  Let V be a finite dimensional vector space over a field $\mathbb{F}$ with $\dim(V) = n$.

  (a) Show that the space of all linear transformations from V to V, $\mathcal{L}(V) = \{T : V \to V \,|\, T \text{ is linear}\}$, is isomorphic to $M_{n\times n}(\mathbb{F})$. [Hint: Use the matrix representation $[T]_\beta$ with respect to some fixed basis β.]

  (b) What is the dimension of $\mathcal{L}(V)$?

  (c) Given $T \in \mathcal{L}(V)$, define the trace of T to be $\text{trace}([T]_\beta)$, for some basis β for V. Show that this quantity is independent of the basis β, that is $\text{trace}([T]_\beta) = \text{trace}([T]_{\beta'})$ for any two bases β and β' for V. [See exercise 10 page 118 and exercise 13 page 97]

(a) Let β be ordered basis for V. Define $\Phi : \mathcal{L}(V) \to M_{n\times n}(\mathbb{F})$ by $\Phi(T) = [T]_\beta$.
By Theorem 2.8 (textbook), Φ is linear. To show Φ is isomorphism, it is sufficient to construct $\Psi : M_{n\times n}(\mathbb{F}) \to \mathcal{L}(V)$ such that $\Phi\Psi = I$, $\Psi\Phi = I$.
Indeed, for $A \in M_{n\times n}(\mathbb{F})$, $A = (a_{ij})$, define $\Psi(A)(v_j) = \sum_i a_{ij} v_i$ and then extend $\Psi(A)$ by linearity to V. One verifies $\Phi(\Psi(A)) = [\Psi(A)]_\beta = (a_{ij}) = A$ and $\Psi(\Phi(T)) = \Psi([T]_\beta) = T$, so Φ is isomorphism (and $\Psi = \Phi^{-1}$).

(b) Since $M_{n\times n}(\mathbb{F})$ has dimension n^2, it follows (by theorem 2.19) that $\dim \mathcal{L}(V) = n^2$.

(c) Since $A = [T]_\beta$ and $B = [T]_{\beta'}$ are similar ($B = Q^{-1}AQ$ for some Q invertible)
we show that $\text{trace}(B) = \text{trace}(A)$ for A, B similar matrices.
First, we have $\text{trace}(PQ) = \text{trace}(QP)$ for any two matrices P, Q, since
$\text{trace}(PQ) = \sum_i \left(\sum_j P_{ij} Q_{ji}\right) = \sum_j \left(\sum_i Q_{ji} P_{ij}\right) = \text{trace}(QP)$
Then, $\text{trace}(B) = \text{trace}(Q^{-1}AQ) = \text{trace}(QQ^{-1}A) = \text{trace}(A)$. Done.

- **Problem 2**
  (a) Let α, β_1, β_2 be three ordered bases for a vector space V. Denote Q_1 and Q_2 the change of basis matrix from α to β_1 and from α to β_2, respectively. Show that the change of basis from β_1 to β_2 is then given by the matrix $Q = Q_1^{-1}Q_2$.

  (b) Using part (a), find the change of basis matrix from the basis $\beta_1 = \{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\}$ to the basis $\beta_2 = \{\begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\3\\-2\end{pmatrix}\}$ of $\mathbb{R}^3$.

  (c) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given as the projection on $W_1 = \text{span}\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\}$ along $W_2 = \text{span}\{\begin{pmatrix}0\\1\\1\end{pmatrix}\}$ (see definition on page 76). Determine $T\begin{pmatrix}a\\b\\c\end{pmatrix}$ and verify that $T^2 = T$.

(a) Let $\alpha \xrightarrow{Q_1} \beta_1$, $\alpha \xrightarrow{Q_2} \beta_2$ be change of basis matrices.
Let $\beta_1 \xrightarrow{Q} \beta_2$. We note $Q_1 = [I_V]_{\beta_1}^{\alpha}$, $Q_2 = [I_V]_{\beta_2}^{\alpha}$
$\Rightarrow Q = [I_V]_{\beta_2}^{\beta_1} = [I_V I_V]_{\beta_2}^{\beta_1} = [I_V]_{\alpha}^{\beta_1}[I_V]_{\beta_2}^{\alpha} = Q_1^{-1}Q_2$.

(b) $Q_1 = \begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$, $Q_2 = \begin{bmatrix}2&0&1\\1&-1&3\\3&1&-2\end{bmatrix}$. We compute $Q_1^{-1} = \begin{bmatrix}1/2&1/2&-1/2\\1/2&-1/2&1/2\\-1/2&1/2&1/2\end{bmatrix}$

$\Rightarrow Q = Q_1^{-1}Q_2 = \begin{bmatrix}0&-1&3\\2&1&-2\\1&0&0\end{bmatrix}$.

(c) We observe $[T]_{\beta_1} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ since T is the projection on W_1 along W_2

Using (b) $[T]_\alpha = Q_1[T]_{\beta_1}Q_1^{-1} = \begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}1/2&1/2&-1/2\\1/2&-1/2&1/2\\-1/2&1/2&1/2\end{bmatrix} = \begin{bmatrix}1&0&0\\1/2&1/2&-1/2\\1/2&-1/2&1/2\end{bmatrix}$

So $T\begin{pmatrix}a\\b\\c\end{pmatrix} = [T]_\alpha\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}a\\ \frac{1}{2}(a+b-c)\\ \frac{1}{2}(a-b+c)\end{pmatrix}$

One verifies $[T]_\alpha^2 = \begin{bmatrix}1&0&0\\1/2&1/2&-1/2\\1/2&-1/2&1/2\end{bmatrix}\begin{bmatrix}1&0&0\\1/2&1/2&-1/2\\1/2&-1/2&1/2\end{bmatrix} = \begin{bmatrix}1&0&0\\1/2&1/2&-1/2\\1/2&-1/2&1/2\end{bmatrix} = [T]_\alpha$.

- **Problem 3***

Let V be a vector space (not necessarily finite dimensional) and $T, U : V \to V$ be linear transformations.

(a) Show that $\dim(N(UT)) \leq \dim(N(U)) + \dim(N(T))$.

(b) Show that $\dim(R(UT)) \leq \min\{\dim(R(U)), \dim(R(T))\}$.
When does equality occur in (a) and in (b)? (See Lemma 2 page 135 for (a))

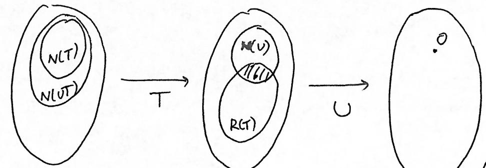
[Do not assume V is finite dimensional! Consider **all cases** when the dimensions in questions are finite or infinite. Inequalities such as $\infty \leq \infty + \infty$ etc are considered true.]

(c) For a given $T : V \to V$, consider the map $\Phi_T : \mathcal{L}(V) \to \mathcal{L}(V)$ given by $\Phi_T(U) = UT$. (See problem 1 for notation). Show that Φ_T is a linear transformation and describe its null space $N(\Phi_T)$ and range $R(\Phi_T)$. Compute its nullity and rank.

[Hint: you may try first some simple example, e.g. $V = \mathbb{R}^2$ and $T = L_A$ with $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$]

(d) Is there any difference if one considers instead at $\Psi_T : \mathcal{L}(V) \to \mathcal{L}(V)$ given by $\Psi_T(U) = TU$?

(a)



Without loss of generality, we assume $\dim N(U) < \infty$ and $\dim N(T) < \infty$ (otherwise (a) is automatically satisfied)

We know $N(UT) = T^{-1}(N(U)) \supset N(T)$. First, we show that $\dim N(UT) < \infty$ as well. Indeed, let $\beta = \{v_1, \dots, v_k\}$ a basis for $N(U) \cap R(T)$. Consider $\{u_1, \dots, u_k\} \subset N(UT)$ such that $Tu_i = v_i$, $i = \overline{1,k}$. In addition let $\{u_{k+1}, \dots, u_{k+p}\}$ be a basis for $N(T)$. We claim $\{u_1, \dots, u_k, u_{k+1}, \dots, u_{k+p}\}$ is generating set for $N(UT)$. Indeed,

let $w \in N(UT) \Rightarrow UTw = 0 \Rightarrow Tw \in N(U) \cap R(T) \Rightarrow$

$\Rightarrow Tw = a_1v_1 + \dots + a_kv_k \Rightarrow Tw = a_1Tu_1 + \dots + a_kTu_k$

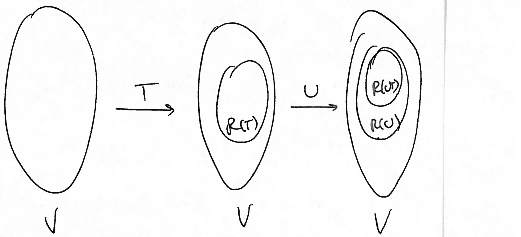
$\Rightarrow T(w - a_1u_1 - \dots - a_ku_k) = 0 \Rightarrow w - a_1u_1 - \dots - a_ku_k \in N(T)$

$\Rightarrow w - a_1u_1 - \dots - a_ku_k = a_{k+1}u_{k+1} + \dots + a_{k+p}u_{k+p} \Rightarrow w \in \operatorname{span}\{u_1, \dots, u_{k+p}\}$

So $N(UT)$ is finitely generated and $\dim N(UT) \leq \dim N(U) + \dim N(T)$

Note: Equality occurs if $R(T) \supset N(U)$. Indeed the dimension theorem applied to $T|_{N(UT)} : N(UT) \longrightarrow R(T) \cap N(U)$ gives $\dim N(UT) = \dim N(T) + \dim(R(T) \cap N(U))$

(b) $R(UT) \subset R(U)$



So $\dim R(UT) \leq \dim R(U)$.
(assuming, wlog, $\dim R(U) < \infty$).

To show $\dim R(UT) \leq \dim R(T)$

let $U|_{R(T)} : R(T) \to R(UT)$ linear

Then $\dim R(UT) \leq \dim R(T)$.
(again, assuming wlog $\dim R(T) < \infty$).

Equality in (b) occurs if $R(UT) = R(U)$ or $U|_{R(T)}$ is one-to-one.

(c) Let $T : V \to V$ and $\Phi_T : \mathcal{L}(V) \to \mathcal{L}(V)$, $\Phi_T(U) = UT$.

$\Phi_T(cU_1 + U_2) = (cU_1 + U_2)T = cU_1T + U_2T$ (by laws of composition)

$= c\Phi_T(U_1) + \Phi_T(U_2) \Rightarrow \Phi_T$ is linear.

$N(\Phi_T) = \{U \in \mathcal{L}(V) \mid UT = 0\} = \{U \in \mathcal{L}(V) \mid R(T) \subseteq N(U)\}$

$R(\Phi_T) = \{UT \mid U \in \mathcal{L}(V)\}$.

To compute $\dim N(\Phi_T)$ and $\dim R(\Phi_T)$, assume V = finite dimensional (or at least $N(T)$ = finite dimensional). $(r = \dim R(T))$

Note $U \in N(\Phi_T) \Leftrightarrow U|_{R(T)} = 0$. Let $\{w_1, \dots, w_r\}$ be a basis for $R(T)$ and complete it to a basis $\beta = \{w_1, \dots, w_r, w_{r+1}, \dots, w_n\}$ for V

We define (for $i = \overline{1,n}$, $j = \overline{r+1,n}$) $U^{ij} \in \mathcal{L}(V)$ such that $U^{ij}(w_j) = w_i$

$k \neq j$ $U^{ij}(w_k) = 0$

Then $\gamma = \{U^{ij} \mid i = \overline{1,n},\ j = \overline{r+1,n}\}$ forms a basis for $N(\Phi_T)$

so $\dim N(\Phi_T) = n(n-r) = \dim V \cdot \dim N(T)$.

Since $\dim \mathcal{L}(V) = \dim N(\Phi_T) + \dim R(\Phi_T)$ (dimension theorem)
it follows that $\dim R(\Phi_T) = n^2 - n(n-r) = nr = \dim V \cdot \dim R(T)$.