SOLUTIONS TO SAMPLE EXAM 2

• **Problem 1**

(a) Show that the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \in \mathcal{M}_{3\times3}(\mathbf{R})$ is NOT diagonazilable over the field of reals.

(b) Show that the same matrix, viewed as $A \in \mathcal{M}_{3\times3}(\mathbf{C})$ IS diagonalizable over the complex field.

(c) Find an invertible matrix $Q \in \mathcal{M}_{3\times3}(\mathbf{C})$ such that $Q^{-1}AQ$ is diagonal.

(a) The characteristic polynomial of A is

$$f(t) = \det(A - tI) = \begin{vmatrix} -t & 0 & 1 \\ 1 & -t & -1 \\ 0 & 1 & 1-t \end{vmatrix} = -t\begin{vmatrix} -t & -1 \\ 1 & 1-t \end{vmatrix} + \begin{vmatrix} 1 & -t \\ 0 & 1 \end{vmatrix} =$$

$$= -t(-t(1-t)+1)+1 = t^2(1-t)+1-t$$

$\Rightarrow f(t) = (t^2+1)(1-t)$. Does not split over $\mathbb{F} = \mathbb{R}$.

$\Rightarrow$ A cannot be diagonalized over $M_{3\times3}(\mathbb{R})$.

(b) Same characteristic polynomial splits over $\mathbb{C}$: $f(t) = (t-i)(t+i)(1-t)$ and hence, $A \in M_{3\times3}(\mathbb{C})$ has 3 distinct eigenvalues $1, \pm i$ $\Rightarrow$

$\Rightarrow$ A is diagonalizable (cf. Corollary page 261)

(c) One finds the following eigenvectors for $A \in M_{3\times3}(\mathbb{C})$:

$\lambda_1 = 1 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\lambda_2 = i \Rightarrow v_2 = \begin{bmatrix} 1 \\ -1-i \\ i \end{bmatrix}$, $\lambda_3 = -i \Rightarrow v_3 = \begin{bmatrix} 1 \\ -1+i \\ -i \end{bmatrix}$

$$\Rightarrow Q = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1-i & -1+i \\ 1 & i & -i \end{bmatrix}$$

[The fact that v_3 and v_2 are complex conjugates isn't a surprise, is it?]

• **Problem 2**

Show that if $T: V \to V$ is diagonalizable, then $V = N(T) \oplus R(T)$.

V = finite dimensional

$T\ V \to V$ diagonalizable $\Rightarrow \exists$ a basis $\beta = \{v_1, \dots, v_n\}$ (of eigenvectors for T)

such that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

<u>Note</u>: $\{\lambda_1, \dots, \lambda_n\}$ may not be distinct.

<u>Case 1</u> If all eigenvalues $\lambda \neq 0$, then $N(T) = \{0\}$ and

$R(T) = \text{span}\{Tv_1, \dots, Tv_n\}$
$= \text{span}\{\lambda_1 v_1, \dots, \lambda_n v_n\}$
$= \text{span}\{v_1, \dots, v_n\} = V$

hence $N(T) \oplus R(T) = 0 \oplus V = V$.

<u>Case 2</u> If 0 is an eigenvalue, say $\lambda_1 = \dots = \lambda_m = 0$ and $\lambda_{m+1} \neq 0, \dots, \lambda_n \neq 0$ with multiplicity $m \geq 1$

Then $N(T) = \text{span}\{v_1, \dots, v_m\}$ and $R(T) = \text{span}\{Tv_1, \dots, Tv_n\}$
$= \text{span}\{v_{m+1}, \dots, v_n\}$

$\Rightarrow N(T) \oplus R(T) = \text{span}\{v_1, \dots, v_n\} = V$.

$\blacksquare$

- **Problem 3**
Let $T : V \to V$ be linear such that $T^k = 0$ for some positive integer k (such a T is called **nilpotent**).
(a) Show that 0 is an eigenvalue of T and no other eigenvalues exists for T.
(b) Using the Cayley-Hamilton theorem, show that $T^n = 0$, where $n = \dim(V)$.
(c) Show that T is not diagonalizable (unless $T = 0$)

(a) Let p the smallest integer such that $T^p = 0$. Hence $T^{p-1} \neq 0$
$\Rightarrow \exists\, x \in V$ such that $T^{p-1}x \neq 0$.
Then $T(T^{p-1}x) = T^p x = 0 \Rightarrow T^{p-1}x$ is an eigenvector for T corresponding to eigenvalue 0.

If $\lambda \neq 0$ were an eigenvalue and $v \neq 0$ a corresponding eigenvector, we would have $T^p v = \lambda^p v = 0 \Rightarrow$ contradiction!
Hence 0 is the ONLY eigenvalue of T.

(b) Cayley-Hamilton theorem implies $f_T(T) = 0$, where $f_T(t) = \det(T - tI)$ is the characteristic polynomial of T.
$= a_0 + a_1 t + \dots + a_n t^n$, $a_n = (-1)^n$

$\Rightarrow$ (*) $\boxed{a_0 I + a_1 T + \dots + a_n T^n = 0}$. We claim $p \leq n$. Indeed
Assume $p > n$. Then we can apply T^{p-1} to (*) to obtain

$$a_0 T^{p-1} + \underbrace{a_1 T^p}_{0} + \dots + \underbrace{a_n T^{n+p-1}}_{0} = 0 \Rightarrow a_0 T^{p-1} = 0 \Rightarrow a_0 = 0 \quad (\text{since } T^{p-1} \neq 0)$$

Recursively, we multiply (*) by T^{p-2} to conclude $a_1 = 0$,
Then T^{p-3} to get $a_2 = 0$ etc ... T^{p-n+1} to get $a_n = 0$.
Contradiction! Hence $p \leq n \Rightarrow T^n = T^p T^{n-p} = 0\, T^{n-p} = 0$.

(c) If T were diagonalizable, since 0 is the only eigenvalue, it would mean that there is a basis for V such that $[T]_\beta = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} = 0 \Rightarrow T = 0$.

- **Problem 4**
Let V be an inner product vector space over a scalar field F.
(a) Show that $| < x, y > | = \|x\| \cdot \|y\|$ if and only if one of the vectors x or y is a scalar multiple of the other.
(b) Similarly, show that $\|x + y\| = \|x\| + \|y\|$ if and only if $x = 0$ or $y = \lambda x$, for some $\lambda \in \mathbb{R}_+$
[See exercise 15, page 337]

(a) ($\Leftarrow$) is easy: If $y = \lambda x$ for some $\lambda \in F$, then
$|\langle x, y\rangle| = |\langle x, \lambda x\rangle| = |\lambda|\, \|x\|^2 = \|x\|\, \|\lambda x\|$.
($\Rightarrow$) Assume $|\langle x, y\rangle| = \|x\|\,\|y\|$. Need to show that there exists λ such that $x = \lambda y$. (The case $y = 0$ is trivial).

$$(*) \quad \boxed{\|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y\rangle = \|x\|^2 - 2\mathrm{Re}(\bar{\lambda}\langle x, y\rangle) + |\lambda|^2 \|y\|^2}$$

Choose $\lambda = c\langle x, y\rangle$, with $c \in \mathbb{R} \Rightarrow$

$$\|x - \lambda y\|^2 = \|x\|^2 - 2c|\langle x, y\rangle|^2 + c^2 |\langle x, y\rangle|^2 \|y\|^2$$
$$= \|x\|^2 - 2c\|x\|^2\|y\|^2 + c^2 \|x\|^2 \|y\|^4 = \|x\|^2 (1 - c\|y\|^2)^2$$

So it's enough to set $c = \frac{1}{\|y\|^2}$ or equivalently $\boxed{\lambda = \frac{\langle x, y\rangle}{\|y\|^2}}$.

(b) ($\Leftarrow$) If $x = 0$, $\|0 + y\| = \|0\| + \|y\|$ trivially
If $y = \lambda x$, $\lambda \in \mathbb{R}_+$, then $\|x + y\| = \|x + \lambda x\| = |1 + \lambda|\, \|x\| = (1+\lambda)\|x\|$
$= \|x\| + \lambda\|x\| = \|x\| + \|\lambda x\| = \|x\| + \|y\|$.

($\Rightarrow$) Assume $\|x + y\| = \|x\| + \|y\| \Rightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\cdot\|y\|$
$\Rightarrow 2\mathrm{Re}\langle x, y\rangle = 2\|x\|\cdot\|y\| \Rightarrow \mathrm{Re}\langle x, y\rangle = \|x\|\,\|y\|$

Again, using (*), choose $\lambda \in \mathbb{R} \Rightarrow$

$$\|x - \lambda y\|^2 = \|x\|^2 - 2\lambda\, \mathrm{Re}\langle x, y\rangle + \lambda^2 \|y\|^2$$
$$= \|x\|^2 - 2\lambda \|x\|\,\|y\| + \lambda^2 \|y\|^2 = (\|x\| - \lambda\|y\|)^2$$

So it is enough to pick $\boxed{\lambda = \frac{\|x\|}{\|y\|}}$ (the case $y = 0$ is trivial).

- **Problem 5**
Verify that the following vectors are orthogonal with respect to the standard inner product in $\mathbb{R}^4$, and then extend these to form an orthogonal basis in $\mathbb{R}^4$:

$$v_1 = \begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \quad v_2 = \begin{pmatrix}-2\\0\\1\\0\end{pmatrix}.$$

$\langle v_1, v_2 \rangle = 1(-2)+0\,0+2\cdot 1+1\cdot 0 = -2+2=0 \Rightarrow v_1 \perp v_2$.

We can extend $\{v_1, v_2\}$ to a basis of $\mathbb{R}^4$ using the replacement theorem and taking $\alpha = \{e_1, e_2, e_3, e_4\}$ the standard basis. Start with e_1: $\{v_1, v_2, e_1\}$ is lin. ind. (check), then add e_2: $\{v_1, v_2, e_1, e_2\}$ is linearly independent.

(indeed, $Q = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is invertible since $\det Q \neq 0$ check)

We now employ Gram-Schmidt to the ordered basis $\{e_1, e_2, v_1, v_2\}$

$w_1 = e_1$, $w_2 = e_2$ ($e_2 \perp e_1$). Let ($\|w_1\| = \|w_2\| = 1$)

$$W_3 = V_1 - \frac{\langle V_1, W_1\rangle}{\|W_1\|^2} W_1 - \frac{\langle V_1, W_2\rangle}{\|W_2\|^2} W_2 = \begin{pmatrix}1\\0\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow w_3 = \begin{pmatrix}0\\0\\2\\1\end{pmatrix}. \quad (\|w_3\| = \sqrt{5})$$

$$W_4 = V_2 - \frac{\langle V_2, W_1\rangle}{\|W_1\|^2} W_1 - \frac{\langle V_2, W_2\rangle}{\|W_2\|^2} W_2 - \frac{\langle V_2, W_3\rangle}{\|W_3\|^2} W_3$$

$$= \begin{pmatrix}-2\\0\\1\\0\end{pmatrix} - \begin{pmatrix}-2\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\0\end{pmatrix} - \frac{2}{5}\begin{pmatrix}0\\0\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\frac{1}{5}\\-\frac{2}{5}\end{pmatrix}.$$

$$\Rightarrow \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\\frac{1}{5}\\-\frac{2}{5}\end{pmatrix} \right\}$$ is orthogonal basis in $\mathbb{R}^4$

- **Problem 6**
Let $V = C^\infty[-1,1]$ be the space of complex valued functions, together wih the inner product $< f, g >= \int_{-1}^{1} f(t)\overline{g(t)}\,dt$.

(a) Let W_1 be the subspaces of even functions ($f(-t) = f(t)$) and W_2 the subspace of odd functions ($f(-t) = -f(t)$). Show that $W_2 = W_1^\perp$ and $W_1 \oplus W_2 = V$. What is the orthogonal projection of a function $f \in V$ onto W_1?

(b) Consider the linear transformations $Tf(t) = if'(t)$ and $Uf(t) = tf(t)$, restricted to the subspace W_0 of all functions satisfying $f(-1) = f(1)$. Show that both T and U are self-adjoint (that is $T^* = T$ and $U^* = U$).

(a) First, it is easy to check $W_1 \perp W_2$

$f \in W_1,\ g \in W_2 \Rightarrow \langle f, g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt = 0$ since $f\bar{g}$ is odd function on $[-1,1]$.

This means $W_2 \subset W_1^\perp$. To show $W_1^\perp \subset W_2$, we first show $V = W_1 \oplus W_2$. Indeed, if $h \in V$, $h(t) = \underbrace{\frac{h(t)+h(-t)}{2}}_{\in W_1} + \underbrace{\frac{h(t)-h(-t)}{2}}_{\in W_2}$

$\Rightarrow V = W_1 + W_2$. Also, $W_1 \cap W_2 = \{0\}$ since the only function which is both even and odd is zero $f(t) = -f(-t) = -f(t)$

Hence $\boxed{V = W_1 \oplus W_2}$

Let $h \in W_1^\perp \Rightarrow h = f + g$ with $f \in W_1$, $g \in W_2 \Rightarrow$

$\Rightarrow 0 = \langle h, f\rangle = \langle f, f\rangle + \langle g, f\rangle = \|f\|^2 \Rightarrow f = 0 \Rightarrow h = g \in W_2$

Hence $W_1^\perp \subset W_2 \Rightarrow \boxed{W_1^\perp = W_2}$.

If $f \in V$, the orthogonal projection onto W_1 is $\frac{f(t)+f(-t)}{2}$.

(b) First, note that $W_0 = \{f \in V,\ f(-1) = f(1)\}$ is invariant under both T and U.

$$\langle Tf, g\rangle = \int_{-1}^{1} if'(t)\,\overline{g(t)}\,dt = i\, f(t)\overline{g(t)}\Big|_{-1}^{1} - i\int_{-1}^{1} f(t)\,\overline{g'(t)}\,dt =$$

$$= 0 + \int_{-1}^{1} f(t)\,\overline{i g'(t)}\,dt = \langle f, Tg\rangle, \quad \forall f, g \in V \Rightarrow$$

$\Rightarrow T$ is self adjoint. ($T = T^*$)

Similarly, $U = U^*$ (check this by integration of $\langle Uf, g\rangle = \int_{-1}^{1} t f(t)\overline{g(t)}\,dt = \langle f, Ug\rangle$.

- **Problem 7**
  Show that, for two linear transformations S and T from a finite dimensional inner product space V to itself, $R(S) \perp R(T)$ if and only if $S^*T = 0$.

$(\Rightarrow)$ Assume $R(S) \perp R(T) \Rightarrow \forall x, y \in V$,

$$0 = \langle Sx, Ty \rangle = \langle x, S^*Ty \rangle \Rightarrow S^*Ty = 0 \Rightarrow \underline{\underline{S^*T = 0}}.$$

$(\Leftarrow)$ Assume $S^*T = 0 \Rightarrow S^*Ty = 0, \ \forall y \in V \Rightarrow$

$$\Rightarrow 0 = \langle x, S^*Ty \rangle = \langle Sx, Ty \rangle, \ \forall x \in V, y \in V$$

$$\Rightarrow R(S) \perp R(T).$$

(Note that $R(S) \perp R(T)$ means $R(S) \subset R(T)^{\perp}$, NOT $R(S) = R(T)^{\perp}$)

- **Problem 8***
  Let $\mathcal{V} = \mathcal{M}_{n\times n}(\mathbb{C})$ and for $A, B \in \mathcal{V}$, define $< A, B >= \text{ tr}(AB^*)$.
  (a) Show that $< \cdot, \cdot >$ is an inner product on $\mathcal{V}$.
  (b) Show that if A is normal $(A^*A = AA^*)$, then

$$\|A\|^2 = \text{ tr}(AA^*) = \sum_{i=1}^{n} |\lambda_i|^2$$

where λ^i are the (not necessarily distinct) eigenvalues of A.
(c) Find the orthogonal complement of the subspace of $\mathcal{V}$ consisting of all diagonal matrices.

(a) $\langle A_1 + A_2, B \rangle = \langle A_1, B \rangle + \langle A_2, B \rangle$

$\langle cA, B \rangle = c\langle A, B \rangle$

$\overline{\langle A, B \rangle} = \langle B, A \rangle$, $\langle A, A \rangle > 0$ for $A \neq 0$

can be easily verified.

e.g. $A = (a_{ij})_{i,j=\overline{1,n}} \Rightarrow tr(AA^*) = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \geq 0$.

(b). If A = normal $(\mathbb{F} = \mathbb{C}) \Rightarrow A$ has an orthonormal basis of eigenvectors $\Rightarrow$ $A = Q^{-1}DQ$, where Q is unitary matrix $(Q^* = Q^{-1})$

$$\Rightarrow AA^* = Q^{-1}D\underbrace{QQ^*}D^*(Q^{-1})^* = Q^{-1}DD^*(Q^*)^{-1} = Q^{-1}DD^*Q$$

$$\Rightarrow tr(AA^*) = tr(DD^*) = \sum_{i=1}^{n} |\lambda_i|^2 .$$

(c) Let $W = \{ B \in M_{n\times n}(\mathbb{C}) \ , \ B = \text{diagonal} \}$

$A \in W^{\perp} \Leftrightarrow \langle A, B \rangle = 0$ for all $B \in W$

$\Rightarrow tr(AB^*) = 0$ for all $B \in W$

B = diagonal $\Rightarrow 0 = tr(AB^*) = d_1 a_{11} + d_2 a_{22} \cdots + d_n a_{nn}$

for any $d_i \in \mathbb{C}$ $(i = \overline{1, n})$. $\Rightarrow a_{11} = a_{22} = \cdots = a_{nn} = 0 \Rightarrow$

$\Rightarrow W^{\perp} = \{ A \in M_{n\times n}(\mathbb{C}) \mid \text{diagonal entries of } A \text{ are zero!} \}$