SOLUTIONS TO SAMPLE FINAL

MATH 4420/5420 - SPRING 2017

#1

(a) $\min \frac{1}{2} x^T Q x - c^T x$ subject to $Ax = b$.

$Q = Q^T > 0$
$A = m \times n$
$m < n$
rank $A = m$

Lagrangian $L(x, \lambda) = \frac{1}{2} x^T Q x - c^T x - \lambda^T (Ax - b)$

KKT:

$$\begin{cases} 0 = \nabla_x L = Qx - c - A^T \lambda \\ 0 = Ax - b \end{cases}$$

$\Rightarrow Qx = c + A^T\lambda \Rightarrow x = Q^{-1}(c + A^T\lambda)$

$Ax = b \Rightarrow AQ^{-1}(c + A^T\lambda) = b \Rightarrow AQ^{-1}c + AQ^{-1}A^T\lambda = b$

$\Rightarrow \underbrace{AQ^{-1}A^T}\lambda = b - AQ^{-1}c$

Now $AQ^{-1}A^T$ is positive definite (since $Q = Q^T > 0$)

Indeed $y^T(AQ^{-1}A^T)y = (y^TA)Q^{-1}(A^Ty) = (A^Ty)^TQ^{-1}A^Ty > 0$

whenever $A^Ty \neq 0$. Since A has max row rank (rank $= m < n$)

it follows that $A^Ty = 0 \Leftrightarrow y = 0 \Rightarrow y^T(AQ^{-1}A^T)y > 0$, $\forall y \neq 0$

$\Rightarrow AQ^{-1}A^T$ is positive definite, so in particular, invertible.

$\Rightarrow \lambda = (AQ^{-1}A^T)^{-1}(b - AQ^{-1}c)$.

Finally, the solution in closed form is

$x = Q^{-1}c + Q^{-1}A^T\lambda \Rightarrow \boxed{x = Q^{-1}c + Q^{-1}A^T(AQ^{-1}A^T)^{-1}(b - AQ^{-1}c)}$

(b) Using part (a), we identify $c=0$ & $Q=I$

and $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Then
$$x = A^T (AA^T)^{-1} b$$

Using Matlab, we get $x = \begin{bmatrix} 29/14 \\ 11/14 \\ -1/7 \end{bmatrix}$.

<u>Note</u>: One can solve this problem directly, via the Lagrange Multiplier method (which is the same as KKT)

e.g. solving $\nabla f = \lambda_1 \nabla c_1 + \lambda_2 \nabla c_2$, $\nabla c_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\nabla c_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$\begin{cases} x_1 = \lambda_1 + \lambda_2 \\ x_2 = \lambda_1 - \lambda_2 \\ x_3 = -\lambda_1 + 2\lambda_2 \end{cases}$$

together with
$$\begin{cases} x_1 + x_2 - x_3 = 3 \\ x_1 - x_2 + 2x_3 = 1 \end{cases}$$

The matrix computation above ~~make~~ provides a more systematic approach to solving this system.

#2 min $x_1 + x_2 + 3x_3$ subject to
$$x_1 - x_2 = 1$$
$$x_1 + x_3 \geq 2$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

In standard form

min $\underbrace{x_1 + x_2 + 3x_3 + 0 \cdot x_4}_{c^T x}$ subj to $\begin{cases} x_1 - x_2 = 1 \\ x_1 + x_3 - x_4 = 2 \end{cases}$ or $Ax = b$

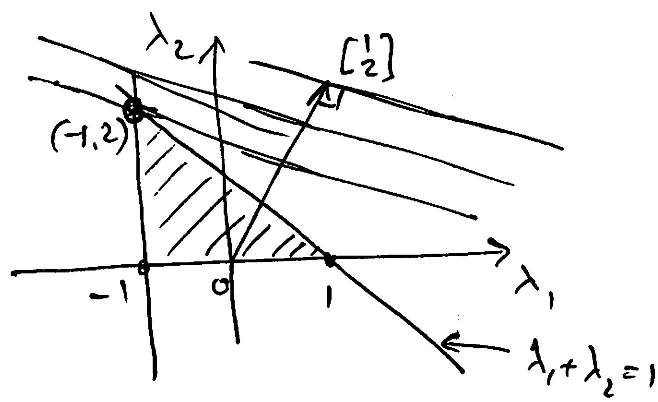
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

The dual problem is

$$\max \underbrace{\lambda_1 + 2\lambda_2}_{b^T\lambda} \quad \text{subject to} \quad \begin{cases} \lambda_1 + \lambda_2 \le 1 \\ -\lambda_1 \le 1 \\ \lambda_2 \le 3 \\ -\lambda_2 \le 0 \end{cases} \quad \text{or } A^T\lambda \le c$$

The feasible set for the dual is

$$\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 : \quad \lambda_1 + \lambda_2 \le 1, \ \lambda_1 \ge -1, \ 0 \le \lambda_2 \le 3$$



By the graphical method, the max point is $(\lambda_1, \lambda_2) = (-1, 2)$

(b) ~~By KKT~~: $L(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x$

KKT: $0 = \nabla_x L = c - A^T\lambda - s \Rightarrow s = c - A^T\lambda = \begin{bmatrix}1\\1\\3\\0\end{bmatrix} - \begin{bmatrix}1 & 1\\-1 & 0\\0 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$

$Ax = b$

$s^T x = 0$ $\quad \Rightarrow s = \begin{bmatrix}1\\1\\3\\0\end{bmatrix} - \begin{bmatrix}1\\1\\2\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\1\\2\end{bmatrix}$

$s \ge 0, \ x \ge 0$

$s_3 \ne 0, s_4 \ne 0$
$\Rightarrow x_3 = 0, \ x_4 = 0 \Rightarrow x_1 = 2, \ x_2 = 1 \Rightarrow \boxed{x^* = \begin{bmatrix}2\\1\\0\\0\end{bmatrix}}$

#3

max $3x_1 + 2x_2$ subject to $x_1 \le 4$

$x_2 \le 1$

$2x_1 + x_2 \le 5$

$x_1, x_2 \ge 0$

In standard form:

$$\min\ -3x_1 - 2x_2 \quad \text{subject to} \quad \begin{cases} x_1 + x_3 = 4 \\ x_2 + x_4 = 1 \\ 2x_1 + x_2 + x_5 = 5 \end{cases}$$

$x_1, x_2, x_3, x_4, x_5 \ge 0$

Simplex tableau:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 5 \\ \hline -3 & -2 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$

Choosing as initial basic variables x_3, x_4, x_5 (″4, ″1, ″5) means $x_1 = 0$, $x_2 = 0$ is the initial vertex.

Entering variable: x_1. Ratios $\begin{matrix} 4/1 \\ 5/2 \end{matrix} \Rightarrow$ leaving variable x_5

$\Rightarrow$ New basic variables $\{x_3, x_4, x_1\}$, Nonbasic variables $\{x_5, x_2\}$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{5}{2} \\ \hline -3 & -2 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[3R_3 + R_4 \to R_4]{-R_3 + R_1 \to R_1} \left[\begin{array}{ccccc|c} 0 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{5}{2} \\ \hline 0 & -\frac{1}{2} & 0 & 0 & \frac{3}{2} & \frac{15}{2} \end{array}\right]$$

Vertex: $x_5 = 0$, $x_2 = 0$, $x_1 = \frac{5}{2}$, $x_3 = \frac{3}{2}$, $x_4 = 1$

Entering variable x_2. Ratios: $\frac{3/2}{-1/2} = -3$, $\frac{1}{1} = 1$, $\frac{5/2}{1/2} = 5$

$\Downarrow$

Leaving variable x_4

$\Rightarrow$ New basic variables $\{x_3, x_2, x_1\}$. Nonbasic variables $\{x_5, x_4\}$

$$\left[\begin{array}{ccccc|c} 0 & -1/2 & 1 & 0 & -1/2 & 3/2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1/2 & 0 & 0 & 1/2 & 5/2 \\ \hline 0 & -1/2 & 0 & 0 & 3/2 & 15/2 \end{array}\right] \xrightarrow[\substack{-\frac{1}{2}R_2 + R_3 \to R_3 \\ \frac{1}{2}R_2 + R_4 \to R_4}]{\frac{1}{2}R_2 + R_1 \to R_1} \left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1/2 & -1/2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1/2 & 1/2 & 2 \\ \hline 0 & 0 & 0 & 1/2 & 3/2 & 8 \end{array}\right]$$

positive pricing

So optimization terminated and optimal vertex is

$$\underbrace{x_1 = 2, \ x_2 = 1}, \ x_3 = 2, \ x_4 = 0, \ x_5 = 0$$

with optimal value $\boxed{8}$.