Wed, Mar 4, 2009 **STUDENT NAME:** ______________________

EXAM I – MATH 443, SPRING 2009

READ EACH PROBLEM CAREFULLY! To get full credit, you must show all work! The exam has 5 problems on 5 pages. Make sure you turn in all pages! **NO GRAPHING CALCULATORS ALLOWED!**

- **Exercise 1** *[20pts]* Show that the differential system

$$X' = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} X,$$

has periodic (in time) solutions. Describe all initial conditions X_0 which correspond to periodic solutions $X(t)$, with $X(0) = X_0$.

- **Exercise 2** ***[20pts]*** Given the (nonlinear) differential equation

$$x' = x^2 - 2ax + b$$

(a) Determine the equilibria (in terms of the parameters a and b).
(b) Sketch the phase lines and direction fields for the values $(a, b) = (0, 4)$, $(2, 4)$ and $(4, 4)$.
(c) Identify the regions in the parameter plane (a, b) where the system has distinct behavior (that is, the systems corresponding to different regions are not conjugate). Explain your conclusion.

- **Exercise 3** *[20pts]*

(a) *Determine whether each of the following statements is TRUE or FALSE, (giving a brief explanation if true and or a counterexample if false*

Two 2×2 linear differential systems are conjugate if and only if they have (counting multiplicity):
(i) the same number of eigenvalues with negative real part ; TRUE FALSE
(ii) the same number of eigenvalues with positive real part ; TRUE FALSE
(iii) the same number of pure imaginary eigenvalues. TRUE FALSE

(b) Let A be a 3×3 matrix with eigenvalues -1 and $2 \pm i$ and B a 3×3 matrix with eigenvalues -3 and $1 \pm 2i$. Assume both are in canonical form. Sketch a typical solution for each of the systems $X' = AX$ and $Y' = BY$. Determine whether these systems are conjugate or not.

(c) Let A be a 4×4 matrix with eigenvalues -1, $2 \pm i$ and B a 4×4 matrix with eigenvalues -3, $1 \pm 2i$. List all possible combinations of canonical forms for A and B. Does the same conclusion, as in (b), hold?
[Hint: Using the homeomorphisms $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ which exist between conjugate 2×2 systems, such as in part (a), write down a homeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$ which acts as a conjugacy between the two 4×4 systems.]

- **Exercise 4** ***[20pts]*** Find the general (real-valued) solution of the system

$$X' = AX, \qquad \text{where } A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

- **Exercise 5** *[20pts]*

  (a) Show that the matrix $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is such that $B^k = 0$ for some $k > 0$.

  (b) Compute $\exp(tB)$.

  (c) For $\lambda \neq 0$, let $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \lambda I_3 + B$.

  Show that

$$\exp(tA) = e^{t\lambda} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

  (d) Write down $\exp(tC)$ (no need to show computations, just briefly explain your answer) for

$$C = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$