Wed, April 15, 2009 **STUDENT NAME:** ______________________

EXAM II – MATH 4/543, SPRING 2009

Prof. Radu C. Cascaval

Due Mon, Apr 20

YOU MAY NOT USE ANY BREATHING SOURCE OF HELP
(e.g. HUMANS, which includes colleagues, friends, parents, grandparents, cousins, strangers)!!!

NO TEAM WORK OF ANY KIND WILL BE TOLERATED!!!!
Any attempt (voluntary or involuntary) to consult with others will be severely punished!!! No kidding! All parties involved will receive ZERO on the Exam and, in addition, disciplinary actions may be taken.

Please read each problem CAREFULLY. You may consult the lecture notes, homework solutions, textbook etc and also use computer to help in solving (parts) of the problems.

Turn in the solutions for each problem attached to the individual page with the statement of the problem, staple them in correct order (including this page). Make sure you have included all problems!!!

Exercise 1
Consider the 1-dimensional dynamical system governed by the equation

$$x' = f(x)$$

where f is a continuously differentiable function on x. Show that if $y \in \mathbb{R}$ is a ω-limit point for a particular solution $x(t) = \phi(t; x_0)$, then y is an equilibrium point.

Exercise 2
Consider the planar nonlinear system

$$x' = -y$$
$$y' = x - y + y^3$$

(a) Find the linearization around the origin and determine the stability of its equilibrium.
(b) Show that $L(x, y) = x^2 + y^2$ is a Liapunov function for the nonlinear system.
(c) Determine the 'size' of the basin of attraction of the origin (e.g. find a ball of radius R centered at the origin, which is included in the basin of attraction of the origin, with R as large as you possibly can).

Exercise 3

Given the 2nd order (scalar) nonlinear equation

$$x'' + x - \frac{1}{3}x^3 = 0.$$

(a) Write it as a system of 1st order equations and show the system is Hamiltonian, by explicitly computing a Hamiltonian function.

(b) Do the same thing for the more general equation

$$x'' + F(x) = 0,$$

where F is a smooth function on $\mathbb{R}$.

(c) Compare the phase portrait of the system in (a) with the phase portrait of the nonlinear pendulum

$$x'' + \sin x = 0$$

Exercise 4

Consider the dynamical system on $\mathbb{R}^2 \setminus \{(0,0)\}$

$$
\begin{aligned}
x' &= -\frac{y}{x^2+y^2} \quad [=: f(x,y)] \\
y' &= \frac{x}{x^2+y^2} \quad [=: g(x,y)]
\end{aligned}
$$

(a) Verify that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$.

(b) Prove that the system is NOT a gradient system.

(c) Show that the system is Hamiltonian, by explicitly computing a Hamiltonian function.

Exercise 5
Consider the differential system ($X = (x, y)^T \in \mathbb{R}^2$)

$$X' = AX + \lambda |X|^2 X$$

where $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, and α, β are real numbers, with $\alpha < 0$. Here $|X|^2 = x^2 + y^2$.

(a) Show that $L(x, y) = x^2 + y^2$ is a Liapunov function and determine the basin of attraction of the origin.

(b) Determine whether any bifurcation occurs with respect to the paramenter λ or not.

(c) Do the same conclusions hold if $\alpha = 0$? Explain.